Applications to $\aleph_0$-categorical theories:

17 (IV-13) THEOREM (Ryll-Nardzewski, Svenonius, Engeler, late 1950's): Suppose $\mathcal{L}$ countable and T a complete $\mathcal{L}$-theory. Then T is $\aleph_0$-categorical iff for each $n \geq 0$, T has no non-principal n-types.

SHARPER VERSION: For each n, consider $\mathbb{B}_n(T) =$ = (Boolean algebra of) equivalence classes of formulas $\Phi(v_1, \dots, v_n)$, where $\Phi \equiv \Psi \iff$ $\iff T \vdash \forall v_1 \dots v_n [\Phi(v_1, \dots, v_n) \leftrightarrow \Psi(v_1, \dots, v_n)]$. All types over T are principal iff $\mathbb{B}_n(T)$ is finite.

EXAMPLE: $\mathcal{L}$ has $<$, and $T = Th(\mathbb{Q}, <)$ = theory of dense linear orders without endpoints. Then T is $\aleph_0$-categorical (Cantor). In Cantor's proof, insist that f respects order at each stage. Types over T:

1-types: there is only one, determined by $(v_1 = v_1)$.
Let $p(v_1)$ and $q(v_1)$ be 1-types. Realize them in countable models $\mathcal{M}_1$ and $\mathcal{M}_2$ by a_1 and a_2. Cantor showed that there exists an $f: \mathcal{M}_1 \cong \mathcal{M}_2$, $f(a_1) = a_2$, so $p = Type_{\mathcal{M}_1}(a_1) = Type_{\mathcal{M}_2}(a_2) = q$.

2-types: It follows from Cantor's proof that theese are the only possibilities:
(a) generated by $(v_1 = v_2)$;
(b) " " $(v_1 < v_2)$;
(c) " " $(v_2 < v_1)$.

NEW FORMULATION: T is $\aleph_0$-categorical iff there are only finitely many n-types, for each $n \geq 0$.

REMARK: In the proof of (IV-12) one actually shows that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are countable atomic models, and $t_1, \dots, t_k \in \mathcal{M}_1$, $u_1, \dots, u_k \in \mathcal{M}_2$, with $Type_{\mathcal{M}_1}(t_1, \dots, t_k) = Type_{\mathcal{M}_2}(u_1, \dots, u_k)$, then there exists an isomorphism $f: \mathcal{M}_1 \cong \mathcal{M}_2$, with $f(t_i) = u_i$, $i = 1, \dots, k$.

(V) HOMOGENEITY, UNIVERSALITY AND ω-SATURATED MODELS

(V-1) DEFINITION: An $\mathcal{L}$-structure $\mathcal{M}$ is called *ω-homogeneous* iff the following holds: "if $t_1, \dots, t_k, u_1, \dots, u_k \in \mathcal{M}$ with $\mathrm{Type}_{\mathcal{M}}(t_1, \dots, t_k) = \mathrm{Type}_{\mathcal{M}}(u_1, \dots, u_k)$, and if $t_{k+1} \in \mathcal{M}$, then there exists $u_{k+1} \in \mathcal{M}$ with $\mathrm{Type}_{\mathcal{M}}(t_1, \dots, t_{k+1}) = \mathrm{Type}_{\mathcal{M}}(u_1, \dots, u_{k+1})$."

By inspection of the proof of (IV-12), one gets:

18 (V-2) LEMMA: If $\mathcal{M}$ is atomic, then $\mathcal{M}$ is ω-homogeneous.

NOTE: The converse is not true. Arbitrary theories do not have atomic models, but always have countable ω-homogeneous models.

19 (V-3) THEOREM: Let $\mathcal{L}$ be countable and T be a complete $\mathcal{L}$-theory. Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are countable ω-homogeneous models of T, realizing the same types. Then $\mathcal{M}_1 \cong \mathcal{M}_2$.

12/11/86: PROOF:

This is a modification of that for atomic models.

Crucial point: if $a_1, \dots, a_n \in \mathcal{M}_1$ and f is a map from $\{a_1, \dots, a_n\}$ into $\mathcal{M}_2$, such that $\mathrm{Type}_{\mathcal{M}_1}(a_1, \dots, a_n) = \mathrm{Type}_{\mathcal{M}_2}(f(a_1), \dots, f(a_n))$, and if $a_{n+1} \in \mathcal{M}_1$, then f extends to $g: \{a_1, \dots, a_{n+1}\} \to \mathcal{M}_2$ with $\mathrm{Type}_{\mathcal{M}_1}(a_1, \dots, a_{n+1}) = \mathrm{Type}_{\mathcal{M}_2}(g(a_1), \dots, g(a_{n+1}))$.

Let $c_1, \dots, c_{n+1} \in \mathcal{M}_2$, with $\mathrm{Type}_{\mathcal{M}_1}(a_1, \dots, a_{n+1}) = \mathrm{Type}_{\mathcal{M}_2}(c_1, \dots, c_{n+1})$. It is clear that $\mathrm{Type}_{\mathcal{M}_2}(c_1, \dots, c_n) = \mathrm{Type}_{\mathcal{M}_1}(a_1, \dots, a_n) = \mathrm{Type}_{\mathcal{M}_2}(f(a_1), \dots, f(a_n))$. By definition of ω-homogeneity, one can find $b_{n+1} \in \mathcal{M}_2$ such that $\mathrm{Type}_{\mathcal{M}_1}(a_1, \dots, a_{n+1}) = \mathrm{Type}_{\mathcal{M}_2}(c_1, \dots, c_{n+1}) = \mathrm{Type}_{\mathcal{M}_2}(f(a_1), \dots, f(a_n), b_{n+1})$. $\square$

PROBLEMS: ① Does T have an ω-homogeneous countable model? (Yes, if T has an atomic model, by preceding).

② When does T have an ω-homogeneous model $\mathcal{M}$ realizing all n-types, for all n? (Such $\mathcal{M}$ must be unique by (V-3).)

Obvious constraint: There are only countable many n-types, for each n.

NON-EXAMPLE: Let T be the theory of $\mathbb{R}$ as a field with an order, and constants $0, 1$. There are $2^{\aleph_0}$ 1-types over T. In fact, if $r, s \in \mathbb{R}$, $r \neq s$, then $\mathrm{Type}_{\mathbb{R}}(r) \neq \mathrm{Type}_{\mathbb{R}}(s)$. Suppose $0 < r < s$. Choose $m, n \in \mathbb{N}$, $n \neq 0$, and $r < m/n < s$. Then $\mathrm{Type}(r)$ contains $(n \cdot x < m)$ which does not belong to $\mathrm{Type}_{\mathbb{R}}(s)$.

(V-4) THEOREM: Let $\mathcal{M}$ be a countable model of T. Then there exists $\mathcal{N}$, $\mathcal{M} \prec \mathcal{N}$, $\mathcal{N}$ countable and ω-homogeneous.

PROOF:

Crucial part: Suppose $\mathcal{M}$ is not ω-homogeneous. So, there exists $a_1, \dots, a_{n+1}, b_1, \dots, b_n \in \mathcal{M}$ with $\mathrm{Type}(a_1, \dots, a_n) = \mathrm{Type}(b_1, \dots, b_n)$, but there is no $b_{n+1} \in \mathcal{M}$ with

$\circledast$: $\mathrm{Type}(a_1, \dots, a_{n+1}) = \mathrm{Type}(b_1, \dots, b_{n+1})$.

To remove this counterexample, one needs a $b_{n+1} \in \mathcal{N}$ satisfying $\circledast$.

CLAIM: There exist $\mathcal{N} \succ \mathcal{M}$ and $b_{n+1} \in \mathcal{N}$ satisfying $\circledast$.

Add to $\mathcal{L}$ constants $\bar{a}_1, \dots, \bar{a}_{n+1}, \bar{b}_1, \dots, \bar{b}_n$ corresponding to the a_i's and b_i's. Consider $\Sigma(v)$ consisting of all formulas $(\Phi(\bar{b}_1, \dots, \bar{b}_n, v) \leftrightarrow \Phi(\bar{a}_1, \dots, \bar{a}_{n+1}))$. Then Σ is finitely satisfiable, for if the bit given by $\Phi_1, \dots, \Phi_k$ is not satisfiable, then $T \vdash (\forall v) \neg (\bigwedge_1^k \Phi_i(\vec{a}) \leftrightarrow \Phi_i(\vec{b}, v))$. W.l.o.g., $\mathcal{M} \models \bigwedge_1^{\ell} \Phi_i(\vec{a}) \wedge \bigwedge_{\ell+1}^k \neg \Phi_i(\vec{a})$, so $\mathcal{M} \models (\forall v)(\neg \Phi_1 \vee \dots \vee \neg \Phi_\ell \vee \Phi_{\ell+1} \vee \dots \vee \Phi_k)(\vec{b}, v)$. But the same is true for $a_1, \dots, a_n$ (in the place of $\vec{b} = (b_1 \dots b_n)$), contradiction, since a_{n+1} gives suitable v.

How to ~~finish~~ complete the proof?

STAGE 0: Remove the first failure of homogeneity in $\mathcal{M}$, going to $\mathcal{N} = \mathcal{M}_0 \succ \mathcal{M}$.

STAGE 1: Remove the second failure in $\mathcal{M}$, if persistent in $\mathcal{M}_0$, in $\mathcal{M}_1 \succ \mathcal{M}_0$.

⋮

STAGE ω: Let $\mathcal{M}_\omega = \bigcup_{n\geq 0} \mathcal{M}_n$. In this stage one has removed all failures in $\mathcal{M}$.

⋮

STAGE $\omega+\omega$: Same procedure as above, getting $\mathcal{M}_{\omega+\omega} \succ \mathcal{M}_\omega \succ \mathcal{M}$, with all failures in $\mathcal{M}_\omega$ removed.

STAGE ω^2: This is the final stage, giving one $\mathcal{M}_{\omega^2} \succ \mathcal{M}$, ω-homogeneous. Note that Tarski's theorem (III-8) is crucial ~~here~~ in this proof.

□

(V-5) DEFINITION: Let $\mathcal{M} \models T$, T a complete theory. $\mathcal{M}$ is ω-saturated if the following holds: suppose $a_1, \dots, a_n \in \mathcal{M}$, and $p(v_1, \dots, v_{n+1})$ is an $(n+1)$-type extending $\mathrm{Type}_{\mathcal{M}}(a_1, \dots, a_n)$. Then there exists $a_{n+1} \in \mathcal{M}$ such that $a_1, \dots, a_{n+1}$ realize p.

Note that $\mathcal{M}$ realizes all 1-types (take $n=0$); then all 2-types, and so on, by induction.

21 (V-6) LEMMA: If $\mathcal{M}$ is ω-saturated, then $\mathcal{M}$ is ω-homogeneous.

14/11/86: PROOF:

Assume $\mathcal{M}$ is ω-saturated.

Let $p' = \mathrm{Type}_{\mathcal{M}}(a_1, \dots, a_n)$, q be an $(n+1)$-type extending p'. Then there exists $a_{n+1} \in \mathcal{M}$, such that $q = \mathrm{Type}(a_1, \dots, a_{n+1})$.

Suppose $\mathrm{Type}(a_1, \dots, a_n) = \mathrm{Type}(b_1, \dots, b_n)$ for some $b_1, \dots, b_n \in \mathcal{M}$. Consider $\Sigma(v)$ as before (i.e. add to $\mathcal{L}$ constants $\bar{a}_1, \dots, \bar{a}_{n+1}$, $\bar{b}_1, \dots, \bar{b}_n$, $\Sigma(v) = \{\Phi(\vec{\bar{a}}) \leftrightarrow \Phi(\bar{b}, v) : \text{all } \Phi(v_1 \dots v_{n+1})\}$). Then $\Sigma(v)$ is consistent.

Replace $a_1, \dots, a_{n+1}$ by variables $v_1, \dots, v_{n+1}$, $b_1, \dots, b_n$ by $v_{n+2}, \dots, v_{2n+1}$ and v by v_{2n+2}. Consider ~~Ma~~ $\mathrm{Type}(a_1, \dots, a_{n+1}, b_1, \dots, b_n) = p(v_1, \dots, v_{2n+1})$. Take $q(v_1, \dots, v_{2n+2})$ any type extending p, and containing the formulas θ: $\Phi(v_1, \dots, v_{n+1}) \leftrightarrow \Phi(v_{n+2}, \dots, v_{2n+2})$.

One has many such q, by Zorn's lemma, since $p \cup$ set of all θ is consistent.

Solve this by $a_1, \dots, a_{n+1}, b_1, \dots, b_n$, and b_{n+1} for v_{2n+2}.

$\square$

<u>NOTE</u>: One now has: (i) ω-saturated $\Rightarrow$ all types are realized; (ii) ω-saturated $\Rightarrow$ ω-homogeneous. So,

22 (V-7) <u>LEMMA</u>: There is at most one countable ω-saturated model for a complete theory T.

23 (V-8) <u>THEOREM</u>: Suppose $\mathcal{M}$ is an ω-saturated model of the complete theory T. Let $\mathcal{N}$ be a countable model of T. Then there exists an elementary embedding $\mathcal{N} \hookrightarrow \mathcal{M}$.

<u>PROOF</u>:

Here it will be used a forth argument.

Enumerate $\mathcal{N}$ as $n_0, n_1, \dots$. Define, by recursion in stages, maps $f_k : \{n_0, \dots, n_k\} \to \mathcal{M}$ such that f_{k+1} extends f_k, and $\mathrm{Type}_{\mathcal{N}}(n_0, \dots, n_k) = \mathrm{Type}_{\mathcal{M}}(f_k(n_0), \dots, f_k(n_k))$. Put f equals the union (or limit) of the f_k.

Suppose f_k is constructed satisfying the above condition, and consider $\mathrm{Type}_{\mathcal{N}}(n_0, \dots, n_{k+1}) = p$. $\mathcal{M}$ realizes p and $\mathrm{Type}_{\mathcal{N}}(n_0, \dots, n_k) \subseteq p$, so there is an $m \in \mathcal{M}$ such that $p = \mathrm{Type}_{\mathcal{M}}(f(n_0), \dots, f(n_k), m)$. Put $f_{k+1}(n_{k+1}) = m$.

$\square$

For the converse one has:

(V-9) <u>DEFINITION</u>: Say a countable model $\mathcal{M}$ of a complete theory T is <u>countably universal</u> (or, in some texts, <u>ω_1-universal</u>) if for every countable $\mathcal{N} \models T$, there exists an elementary $f : \mathcal{N} \to \mathcal{M}$.

24 (V-10) <u>THEOREM</u>: Let $\mathcal{M}$ be countable. Then $\mathcal{M}$ is ω-saturated iff $\mathcal{M}$ is countably universal and ω-homogeneous.

<u>PROOF</u>:

The "only if" part has already been done in (V-6) and (V-8).

For the converse, let $a_1, \dots, a_n \in \mathcal{M}$, $\Sigma = \text{Type}(a_1, \dots, a_n)$, and p be an $(n+1)$-type extending Σ. One has to find $a_{n+1} \in \mathcal{M}$ such that $a_1, \dots, a_{n+1}$ realize p.

Construct a (countable) model $\mathcal{N}$ with $b_1, \dots, b_{n+1} \in \mathcal{N}$ realizing p. Then there exists an elementary map $f : \mathcal{N} \hookrightarrow \mathcal{M}$, and $\text{Type}_{\mathcal{N}}(b_1, \dots, b_{n+1}) = \text{Type}_{\mathcal{M}}(f(b_1), \dots, f(b_{n+1})) = q$, since $\mathcal{M}$ is countably universal.

Since $\mathcal{M}$ is ω-homogeneous, and $q \supseteq \Sigma$, there exists an $a_{n+1} \in \mathcal{M}$ such that $a_1, \dots, a_{n+1}$ realize p.

$\square$

Now one deals with the problem of existence for (countable) ω-saturated and for atomic models.

<u>NOTE</u>: obvious constraint for <u>ω-saturated</u>: for such a model to exist, there must be only countabily many n-types, each n.

25 (V-11) <u>THEOREM</u>: Suppose there exist only countabily many n-types over T, each n. Then there exist a countable ω-saturated model.

<u>PROOF</u>:

List all n-types as $p_{0,n}(v_1, \dots, v_n), p_{1,n}, \dots$

Add to $\mathcal{L}$ constants $c_{j,n,i}$ ($j, n \in \mathbb{N}$, and $1 \leq i \leq n$), and consider T plus all $p_{j,n}(c_{j,n,1}, \dots, c_{j,n,n})$. This is clearly finitely satisfiable.

Let $\mathcal{N}$ be a model of T realizing all the types, $\mathcal{N}$ countable. Now get $\mathcal{M} \succ \mathcal{N}$, $\mathcal{M}$ ω-homogeneous.

It is straightforward to check $\mathcal{M}$ is ω-saturated.

$\square$

(V-12) <u>REMARK</u>: If $\mathcal{M} \models T$ and $\mathcal{N}$ is a countable atomic model, then there exists an elementary embedding $\mathcal{N} \to \mathcal{M}$.

To see this, first enumerate $\mathcal{N}$ as $\{n_0, n_1, \dots\}$. Define $f_k : \{n_0, \dots, n_k\} \to \mathcal{M}$ so that (as above for ω-saturated $\mathcal{M}$) f_k preserves type, and require f_{k+1} extends f_k as before. Since $\mathcal{N}$ realizes only principal types, $\mathcal{M}$ realizes all types of $\mathcal{N}$. One gets f_0 by preceding remark. Suppose now f_k is constructed. Let $p = \text{Type}(n_0, \dots, n_k)$ and $q = \text{Type}(n_0, \dots, n_{k+1})$ (notice that both are principal). Then $\mathcal{M}$ realizes q. Suppose q is controlled by a formula $\Phi(v_0, \dots, v_{k+1})$. One has $(\exists v_{k+1})\Phi(v_0, \dots, v_{k+1}) \in p = \text{Type}_{\mathcal{M}}(f_k(n_0), \dots, f_k(n_k))$. So, there exists $\alpha \in \mathcal{M}$, $\mathcal{M} \models \Phi(f_k(n_0), \dots, f_k(n_k), \alpha)$ which forces $q(f_k(n_0), \dots, f_k(n_k), \alpha)$. Take $f_{k+1}(n_{k+1}) = \alpha$. □

Now, for <u>atomic models</u> to exist, there should be « many » principal types.

<u>Possible obstruction</u>: suppose there is a formula $\Phi(v_1, \dots, v_n)$ consistent with T, such that Φ belongs to a non-principal type. Then:

26 (V-13) <u>LEMMA</u>: T has no atomic model.

<u>PROOF</u>:

One has $T \vdash (\exists \vec{v})\Phi(\vec{v})$. Let $\mathcal{M}$ be atomic. Choose $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ in $\mathcal{M}$ such that $\mathcal{M} \models \Phi(\vec{\alpha})$. Let $p = \text{Type}(\vec{\alpha})$. Then p is principal and $\Phi \in p$, contradiction. □

27 (V-14) <u>THEOREM</u>: T has an atomic model iff for every consistent $\Phi(v_1, \dots, v_n)$ there exists a principal type p with $\Phi \in p$. (One says that isolated types are dense.)

19/11/86 <u>PROOF</u>:

One will construct an atomic $\mathcal{M}$ using the method of constants.

Add to $\mathcal{L}$ a set of constants $C = \{c_0, c_1, \dots\}$. Construct in $\mathcal{L}(C)$ a maximal consistent $\Sigma \supseteq T$, with C as a set of witnesses, and $\mathcal{M}$ will be the model canonically built from Σ.

What is required to get $\mathcal{M}$ atomic.

It suffices to ensure: for each n-tuple $c_{i_1}, \dots, c_{i_n}$, there is a formula $\Phi(v_1, \dots, v_n)$ such that: (1) $\Phi(c_{i_1}, \dots, c_{i_n}) \in \Sigma$; (2) Φ determines a principal type over T.

Σ is constructed in ω stages:

At any finite stage one adds to T only a finite set of $\mathcal{L}(C)$-sentences, with three kinds of requirements:

(a) heading for maximal consistency, as usual;

(b) heading for witnessing, as usual;

(c) for each n-tuple $c_{i_1}, \dots, c_{i_n}$, find a suitable Ψ and put $\Psi(c_{i_1}, \dots, c_{i_n})$ in Σ (there are only countable many of this requirement).

Why does (c) succeed?

Suppose one got to T plus $\Theta_1(c_{j_1}, \dots, c_{j_r}), \dots, \Theta_h(c_{j_1}, \dots, c_{j_r})$. Write this as T plus $\Theta(c_{i_1}, \dots, c_{i_n}, c_{i_{n+1}}, \dots, c_{i_{n+\ell}})$ (no sugestion that all of these constants occur).

This commits one to $(\exists v_{n+1} \cdots v_{n+\ell})\Theta(\vec{c}, \vec{v}) \in$ $\in \mathrm{Type}_m([c_{i_1}], \dots, [c_{i_m}])$.

Select some formula $\Gamma(v_1, \dots, v_n)$ so that Γ determines a principal n-type p and $(\exists \vec{v})\Theta(\vec{c}, \vec{v}) \in p$.

So, next stage of Σ is old Σ plus $\Gamma(c_{i_1}, \dots, c_{i_n})$.

This is obviously consistent. $\square$

Now one will see the connection between ω-saturated countable models and atomic models.

28 (V-15) <u>LEMMA</u>: Suppose the isolated types are not dense. Then there exist $2^{\aleph}$ n-types, some n.

<u>PROOF</u>:

Select a $\Phi(v_1, \dots, v_n)$ so that Φ is consistent with T, but not included in any principal type.

Associate to every finite sequence of 0's and 1's a finite set of formulas Σ_s in free variables $v_1, \dots, v_n$, so that:

(1) if t extends s, $\Sigma_s \subseteq \Sigma_t$;

(2) $\Phi \in \Sigma_s$, for all s;

(3) Σ_s is consistent with T;

(4) $\Sigma_{\langle s,0\rangle}$ is inconsistent with $\Sigma_{\langle s,1\rangle}$.

Note then that if one defines Σ_β for an ω-sequence of 0's and 1's by $\Sigma_\beta = \bigcup_s \Sigma_s$ (the "union" of all initial segments s of β), then each Σ_β is consistent (for every finite subset is), and for $\beta_1 \neq \beta_2$, $\Sigma_{\beta_1} \cup \Sigma_{\beta_2}$ is inconsistent.

This will give $2^{\aleph_0}$ n-types.

Σ_s is constructed by induction.

For $\Sigma_{\langle 0 \rangle}$ and $\Sigma_{\langle 1 \rangle}$ choose $\Psi(v_1, \dots, v_n)$ so that $\Phi \wedge \Psi$ and $\Phi \wedge \neg\Psi$ are consistent. If there is no such Ψ, then Φ determines a principal type. Take $\Sigma_{\langle 0 \rangle} = \{\Phi, \Psi\}$ and $\Sigma_1 = \{\Phi, \neg\Psi\}$.

Now, given s, let s' be the truncation of s got by cutting off last entry (i.e., s is either $\langle s', 0 \rangle$ or $\langle s', 1 \rangle$).

$\Sigma_{s'}$ is finite and contains Φ, so it does not determine a principal type. Choose Ψ so that $\{\Sigma_{s'}, \Psi\}$ and $\{\Sigma_{s'}, \neg\Psi\}$ are consistent. Define $\Sigma_{\langle s', 0 \rangle} = \{\Sigma_{s'}, \Psi\}$ and $\Sigma_{\langle s', 1 \rangle} = \{\Sigma_{s'}, \neg\Psi\}$. $\square$

(V-16) <u>COROLLARY</u>: If T has a countable ω-saturated model, then T has an atomic model.

<u>PROOF</u>:

T has less than $2^{\aleph_0}$ n-types, each n. So, isolated types are dense. $\square$

<u>NOTE</u>: T may have atomic model but no ω-saturated one.

<u>EXAMPLE</u>: $\mathcal{L}$ has $+, \cdot, 0, 1, <$, and $T = Th(\mathbb{R})$. Let $\mathfrak{M}$ be the field of real algebraic numbers. $\mathfrak{M}$ is an atomic model of T.

The definitions of ω-homogeneity, countabily universal and ω-saturated models may be generalized to the uncountable. In what follows, κ is an infinite cardinal.

(V-17) <u>DEFINITION</u>: If A is a subset of an $\mathcal{L}$-structure $\mathfrak{M}$, one can form $\mathcal{L}(A)$ by adding names for elements of A. Then $\mathcal{M}$ can be made an $\mathcal{L}(A)$-structure. Say an $\mathcal{L}$-structure $\mathcal{M}$ is <u>κ-saturated</u> if whenever $A \subseteq \mathcal{M}$, card(A) $< \kappa$, and Σ is an $\mathcal{L}(A)$-type, then Σ is realized in $\mathcal{M}$. (Check the sense of this for $\kappa = \omega$.)

(V-18) <u>DEFINITION</u>: $\mathcal{M}$ is <u>κ-homogeneous</u> if whenever $A \subseteq \mathcal{M}$, card(A) $< \kappa$, and $f: A \to \mathcal{M}$ is a map such that f respects types, and if $\alpha \in \mathcal{M}$, then f extends to a type respecting $g: A \cup \{\alpha\} \to \mathcal{M}$.

(V-19) <u>DEFINITION</u>: $\mathcal{M}$ is a <u>κ-universal</u> model of the theory T (complete) if every $\mathcal{N} \models T$, card($\mathcal{N}$) $< \kappa$, embeds elementarily in $\mathcal{M}$.

<u>NOTE</u>: In earlier definition of countably universal, it is done in terms of satisfaction of n-types. Any model of Th($\mathcal{M}$), of cardinal less than κ, is elementary embeddable in $\mathcal{M}$.

21/11/86 (V-20) <u>REMARK</u>: For $\mathcal{L}$ countable, one can prove:
(i) $\mathcal{M}$ is κ-saturated $\Leftrightarrow$ $\mathcal{M}$ is κ-homogeneous and κ^+-universal;
(ii) if $\kappa > \aleph_0$, $\mathcal{M}$ is κ-saturated $\Leftrightarrow$ $\mathcal{M}$ is κ-homogeneous and κ-universal.

(V-21) <u>DEFINITION</u>: $\mathcal{M}$ is <u>saturated</u> if $\mathcal{M}$ is κ-saturated, where κ = card($\mathcal{M}$).

<u>NOTE</u>: $\mathcal{M}$ cannot be κ^+-saturated. Consider $\{v \neq \overline{m} : m \in \mathcal{M}\}$.

(V-22) <u>REMARK</u>: The existence of uncountable saturated models depends on set-theoretic assumptions.

<u>Specimen</u>: (a) There are complete theories T which has a saturated model of power $\aleph_1$ iff $\aleph_1 = 2^{\aleph_0}$ (C.H.)
(b) (Assume CH.) There is a saturated model of T of cardinal $\aleph_1$, where T is any complete theory with infinite model.

(V-23) EXAMPLE: Let T be the theory of dense linear orders with no endpoints.
Constraints on an $\aleph_1$-saturated dense linear order Λ.
Let $A, B \subseteq \Lambda$, $|A|=|B|=\aleph_0$. Suppose $A<B$; then the type $A<x<B$ is finitely satisfiable, and so satisfiable. One may consider two other constraints, $A<x$, and $x<B$.

In fact, saturated models are generalizations of the Hausdorff's η_α-sets (i.e., dense linear orders Λ without endpoints such that for all $A, B \subseteq \Lambda$, $|A|, |B| < \omega_\alpha$, and $A<B$, then there exists a $c \in \Lambda$, $A<c<B$).

In the above example, Λ is an η_1-set. One has that:
(a) any two η_1-sets of power ω_1 are isomorphic (generalization of Cantor's theorem for countable case);
(b) any η_1-set has cardinal $\geq 2^{\aleph_0}$ (one can embed $\mathbb{R}$ in it).
This corresponds to general theorem that there is at most one saturated model of T of power K.

(VI) DEFINABILITY

(VI-1) DEFINITION: Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Suppose $R \subseteq \mathcal{M}^k$. Say that R is definable (definable with parameters in $\mathcal{M}$) iff there is an $\mathcal{L}$-formula $\Phi(v_1, \ldots, v_k)$ (or an $\mathcal{L}$-formula $\Phi(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+r})$ and elements $a_1, \ldots, a_r \in \mathcal{M}$) such that $R = \{\vec{x} : \Phi(\vec{x})\}$ (or $R = \{\vec{x} : \Phi(\vec{x}, \vec{a})\}$). Say a function is definable (in either sense) if its graph is.

(32)

Basic constraint:

29 (VI-2) LEMMA: Suppose $\sigma \in \mathrm{Aut}(\mathcal{M})$. Let $R \subseteq \mathcal{M}^k$, and let $R^\sigma = \{(x_1, \dots, x_k) : (\sigma(x_1), \dots, \sigma(x_k)) \in R\}$. If R is definable (without parameters) then $R^\sigma = R$.

PROOF:
Trivial induction on complexity of defining formula.

$\square$

REMARK: The obvious version with parameters $\{a_i\}_1^r$ requires $\sigma(a_i) = a_i$, $i = 1, \dots, r$.

(VI-3) "PADUA'S METHOD": To show a notion is not definable from some parameters, it suffices to construct an automorphism fixing the primitives, but moving the notion.

EXAMPLE: One cannot define $+$ from $\cdot$ on $\mathbb{N}$. Let $\mathcal{M}$ be this set with "$\cdot$"; one wants a permutation σ of $\mathbb{N}$ such that:
(i) $\sigma(x \cdot y) = \sigma(x)\sigma(y)$, all $x, y \in \mathbb{N}$;
(ii) $\exists a, b$, $\sigma(a+b) \neq \sigma(a) + \sigma(b)$; e.g., "switch 2 and 3", if $x = 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} \dots$, put $\sigma(x) = 2^{\alpha_2} 3^{\alpha_1} \cdot 5^{\alpha_3} \dots$; thus $\sigma(2+2) = \sigma(4) = \sigma(2^2) = 3^2 = 9 \neq 6 = \sigma(2) + \sigma(2)$.

(VI-4) CONVERSE TO PADUA'S METHOD: GLOBAL APPROACH:
Suppose $\mathcal{L}_1 \subseteq \mathcal{L}_2$. Suppose $\mathcal{M}$ is an $\mathcal{L}_2$-structure, and suppose $R \subseteq \mathcal{M}^k$, $\mathcal{L}_2$-definable but not $\mathcal{L}_1$-definable. One aims for $\mathcal{M} \prec_{\mathcal{L}_2} \mathcal{N}$. Suppose Φ defines R on $\mathcal{M}$. Then Φ defines $\overline{R}$ on $\mathcal{N}$, $R \subseteq \overline{R}$. One aims to get an $\mathcal{L}_1$-automorphism σ of $\mathcal{N}$ such that $\overline{R}^\sigma \neq \overline{R}$. This would show that R is not $\mathcal{L}_1$-definable. Suppose R is defined by an $\mathcal{L}_1$-formula Ψ. Then $\mathcal{M} \models \Phi \leftrightarrow \Psi$, so $\mathcal{N} \models \Phi \leftrightarrow \Psi$. So Ψ defines $\overline{R}$ on $\mathcal{N}$; but relation defined by Ψ is invariant under σ, since σ respects all $\mathcal{L}_1$-structure. One wants to show that this gives a necessary and sufficient condition for $\mathcal{L}_1$-definability.

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30 (VI-5) THEOREM: Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}, R, \overline{R}$ be as above. Suppose R is not $\mathcal{L}_1$-definable. Then there exists $\mathcal{M} \prec_{\mathcal{L}_2} \mathcal{N}$, and $\sigma \in \mathrm{Aut}_{\mathcal{L}_1}(\mathcal{N})$

such that $\bar{R}^\sigma \neq \bar{R}$.

PROOF:

Fix an $\mathcal{L}_2$-formula $\Phi(v_1, \dots, v_k)$ defining R, and extend $\mathcal{L}_2$ to $\mathcal{L}_2(\mathcal{M})$.

STEP 1: One shows that the following is consistent: $Th_{\mathcal{L}_2(\mathcal{M})}(\mathcal{M}) + \Phi(v_1, \dots, v_k) \leftrightarrow \neg\Phi(v_{k+1}, \dots, v_{2k}) +$ + " $(v_1, \dots, v_k)$ and $(v_{k+1}, \dots, v_{2k})$ have the same $\mathcal{L}_1$-type" + all $\theta(v_1, \dots, v_k) \leftrightarrow \theta(v_{k+1}, v_{2k})$, θ in $\mathcal{L}_1$.

Suppose not. So, there is a single θ such that $Th_{\mathcal{L}_2(\mathcal{M})}(\mathcal{M}) + \Phi(v_1, \dots, v_k) \leftrightarrow \neg\Phi(v_{k+1}, \dots, v_{2k}) \vdash \theta(v_1, \dots, v_k) \leftrightarrow \neg\theta(v_{k+1}, \dots, v_{2k})$. Write $\bar{x}$ for $(v_1, \dots, v_k)$ and $\bar{y}$ for $(v_{k+1}, \dots, v_{2k})$. One has
$\circledast$ $T + \Phi(\bar{x}) \leftrightarrow \neg\Phi(\bar{y}) \vdash \theta(\bar{x}) \leftrightarrow \neg\theta(\bar{y})$.

Suppose $\Phi \wedge \theta(\bar{x})$ is consistent with T. Then $\neg\Phi \wedge \theta(\bar{y})$ is inconsistent.

Suppose both $\Phi \wedge \theta(\bar{x})$ and $\Phi \wedge \neg\theta(\bar{x})$ are consistent. Then, by $\circledast$, both $\neg\Phi \wedge \theta(\bar{y})$ and $\neg\Phi \wedge \neg\theta(\bar{y})$ are inconsistent.

So $\neg\Phi(\bar{y})$ is inconsistent. In this case, $\Phi(\bar{x})$ is equivalent to $x_1 = x_1$, an $\mathcal{L}_1$-formula, so the set defined by Φ is $\mathcal{L}_1$-definable, contradiction.

One concludes that if Φ is consistent (and if not, it is defined by $x_1 \neq x_1$), exactly one of $\Phi \wedge \theta(\bar{y})$ or $\Phi \wedge \neg\theta(\bar{y})$ is consistent, and also exactly one of $\neg\Phi \wedge \neg\theta$ or $\neg\Phi \wedge \theta$ is consistent.

In first case, $T \vdash \Phi \leftrightarrow \theta$, and in the second case, $T \vdash \Phi \leftrightarrow \neg\theta$, contradiction. This proves step 1.

STEP 2: Assume step 1. Take a model $\mathcal{M}_1$ of the above set, with $\bar{a} = (a_1, \dots, a_k)$ interpreting $v_1, \dots, v_k$ and $\bar{b} = (b_1, \dots, b_k)$ interpreting $v_{k+1}, \dots, v_{2k}$. One has $\mathcal{M} \prec_{\mathcal{L}_2} \mathcal{M}_1$, since $\mathcal{M}_1 \models Th_{\mathcal{L}_2(\mathcal{M})}(\mathcal{M})$.

<u>One wants</u>: $\mathcal{M}_1 \prec_{\mathcal{L}_2} \mathcal{M}_2$, with $\mathcal{M}_2$ $\mathcal{L}_1$-ω-homogeneous, for then, if everything is countable, one gets a $\sigma \in Aut_{\mathcal{L}_1}(\mathcal{M}_2)$, $\sigma(\bar{a}) = \bar{b}$, but clearly since $\Phi(\bar{a}) \leftrightarrow \neg\Phi(\bar{b})$, σ does not respect Φ (and so, $\bar{R}$). This would finish the proof for countable $\mathcal{M}$.

The proof of step 2 follows from the next lemma.

$\square$

31 (VI-6) LEMMA: Assume $\mathcal{L}_1 \subset \mathcal{L}_2$ countable, $\mathfrak{M}_2$ an $\mathcal{L}_2$-structure. Then there exists a countable $\mathfrak{N}$, $\mathfrak{M} \prec_{\mathcal{L}_2} \mathfrak{N}$, such that $\mathfrak{N}$, qua $\mathcal{L}_1$-structure, is ω-homogeneous.

PROOF: (Recall the proof given earlier - in (V-4) - for $\mathcal{L}_1 = \mathcal{L}_2$.)

It clearly suffices (via iteration) to show that, if $a_1, \dots, a_{n+1}, b_1, \dots, b_n \in \mathfrak{M}$, with $\mathrm{Type}_{\mathcal{L}_1}(a_1, \dots, a_n) = \mathrm{Type}_{\mathcal{L}_1}(b_1, \dots, b_n)$, then there exists $\mathfrak{M} \prec_{\mathcal{L}_2} \mathfrak{N}$, $\mathfrak{N}$ countable, and $\mathfrak{N}$ has an element b_{n+1} such that $\mathrm{Type}_{\mathcal{L}_1}(a_1, \dots, a_{n+1}) = \mathrm{Type}_{\mathcal{L}_1}(b_1, \dots, b_{n+1})$.

This reduces to the consistency question: consider $\mathrm{Th}_{\mathcal{L}_2(\mathfrak{M})}(\mathfrak{M}) + \theta(b_1, \dots, b_n, v) \leftrightarrow \theta(a_1, \dots, a_{n+1})$, θ an $\mathcal{L}_1$-formula.

This is consistent (exactly as in the earlier proof). $\square$

28/11/86: Now one is going to consider Beth's theorem, which connects implicit and explicit definability.

(VI-7) DEFINITION: Let Σ be a set of $\mathcal{L}_1$-sentences, $\mathcal{L}_2 \supseteq \mathcal{L}_1$ a language obtained from $\mathcal{L}_1$ by adding a single n-ary relation symbol R. Extend Σ to a set Γ in $\mathcal{L}_2$. Write Γ as $\Gamma(R)$. (Example: Σ is the theory of the field $\mathbb{R}$, R a symbol for order, Γ are axioms about order.) Consider the following situation:

$$\Sigma + \Gamma(R) + \Gamma(R_1) \vdash \forall \bar{x}\,(R(\bar{x}) \leftrightarrow R_1(\bar{x}))\ ;$$

where R_1 is a new n-ary relation symbol (it says: for all models of Σ, there is at most one interpretation of R making $\Gamma(R)$ true). Then one says that $\Gamma(R)$ implicitly defines the corresponding relation.

EXAMPLE: On any model of $\mathrm{Th}(\mathbb{R})$, there is only one order respecting $+$ and $\cdot$. In this case, order is definable, via $x > y \leftrightarrow \exists z\,(z \neq 0 \wedge x - y = z^2)$.

32 (VI-8) THEOREM (Beth): In the above situation, R is explicitly definable, i.e., there is an $\mathcal{L}_1$-formula $\Phi(\bar{x})$ such that $\Gamma(R) \vdash \forall \bar{x}\, [R(\bar{x}) \leftrightarrow \Phi(\bar{x})]$.

PROOF:

Fix a countable model $\mathfrak{M}$ of $\Gamma(R)$.

Suppose R (i.e., its interpretation) is not $\mathcal{L}_1$-definable. Then, by (VI-5), there exists $\mathfrak{M} \prec_{\mathcal{L}_2} \mathfrak{N}$, and $\sigma \in \mathrm{Aut}(\mathfrak{N})$, such that $\bar{R}^\sigma \neq \bar{R}$. Note that $\mathfrak{N} \models \Gamma(\bar{R})$ and $\mathfrak{N} \models \Gamma(\bar{R}^\sigma)$, so; contradiction.

So R is defined in $\mathfrak{M}$ by $\Phi(\bar{x})$, say. Since $\mathfrak{M}$ was an arbitrary countable model, R is defined by Φ in all models of $\Gamma(R)$. □

33 (VI-9) THEOREM (of Joint Consistency, of Robinson): Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ be three languages, $\mathcal{L}_1 = \mathcal{L}_2 \cap \mathcal{L}_3$, and T be a complete $\mathcal{L}_1$-theory. Let T_2 and T_3 be respective to $\mathcal{L}_2$ and $\mathcal{L}_3$ extending T (consistently). Then $T_2 \cup T_3$ is consistent.

PROOF:

One begins with (countable) models $\mathfrak{M}_2 \models T_2$ and $\mathfrak{M}_3 \models T_3$, and wants to find $\mathfrak{M}_2 \prec_{\mathcal{L}_2} \mathfrak{M}_{2,\omega} \overset{\sigma}{\cong}_{\mathcal{L}_1} \mathfrak{M}_{3,\omega} \succ_{\mathcal{L}_3} \mathfrak{M}_3$.

This allows one to enrich $\mathfrak{M}_{2,\omega}$ to a model of $T_2 \cup T_3$. One has to interpret $\mathcal{L}_3 \setminus \mathcal{L}_2$ symbols by taking its interpretation in $\mathfrak{M}_{3,\omega}$ and transfering to $\mathfrak{M}_{2,\omega}$, using σ^{-1}.

One has to ensure that $\mathfrak{M}_{2,\omega}$ and $\mathfrak{M}_{3,\omega}$, qua $\mathcal{L}_1$-structures, are ω-homogeneous and realize same types.

This requires only the following minor modification of lemma (VI-6). There began with an $\mathcal{L}_2$-structure $\mathfrak{M}$ and got $\mathfrak{M} \prec_{\mathcal{L}_2} \mathfrak{N}$, with $\mathfrak{N}$ $\mathcal{L}_1$-ω-homogeneous. Now one require in addition that $\mathfrak{N}$ realizes some prescribed countable set of $\mathcal{L}_1$-types (over $\mathrm{Th}_{\mathcal{L}_1(\mathfrak{M})}(\mathfrak{M})$).

Required modification: let types be $p_0, p_1, p_2, \ldots$; for convenience, assume these are 1-types. Go to $\mathcal{L}_2(\mathfrak{M})$, and also c_i to realize p_i. Consider $\mathrm{Th}_{\mathcal{L}_2(\mathfrak{M})}(\mathfrak{M}) + p_i(c_i)$, all $i \geq 0$. It is a simple exercise to show that this is consistent. Let $\mathfrak{M}'$ be a model of it, $\mathfrak{M} \prec_{\mathcal{L}_2} \mathfrak{M}'$, and all

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p_i's realized. Now apply lemma (VI-6) to $\mathfrak{M}'$.

To finish the proof, consider the following:

$$\begin{tikzcd}[row sep=small]
 & \mathfrak{M}_2 & & \mathfrak{M}_3 & \\
 & \wedge_{\mathcal{L}_2} & & \wedge_{\mathcal{L}_3} & \\
(\omega\text{-homog}) & \mathfrak{M}_{2,1} \arrow[rr, dashed] & & \mathfrak{M}_{3,1} \arrow[ddll, dashed] & (\omega\text{-homog. and realizes all } \mathcal{L}_1\text{-types realized by } \mathfrak{M}_{2,1}) \\
 & \wedge_{\mathcal{L}_2} & & \wedge_{\mathcal{L}_3} & \\
 & \mathfrak{M}_{2,2} \arrow[rr, dashed] & & \mathfrak{M}_{3,2} & \\
 & \vdots & \vdots & \vdots & \\
(\text{limit}) & \wedge_{\mathcal{L}_2} & & \wedge_{\mathcal{L}_3} & \\
 & \mathfrak{M}_{2,\omega} & & \mathfrak{M}_{3,\omega} &
\end{tikzcd}$$

$\square$

34 **(VI-10) THEOREM (Craig):** Let $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ be as in (VI-9). Suppose that σ_2 and σ_3 are respectively $\mathcal{L}_2$- and $\mathcal{L}_3$-sentences. Suppose Σ is a consistent set of $\mathcal{L}_1$-sentences. Suppose $\Sigma \vdash \sigma_2 \to \sigma_3$. Then there exists a σ_1 from $\mathcal{L}_1$ such that $\Sigma \vdash \sigma_2 \to \sigma_1$ and $\Sigma \vdash \sigma_1 \to \sigma_3$.

NOTE: The only reason this is not immediate consequence of (VI-9) is that Σ is not assumed complete.

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PROOF:

CASE 1: Suppose Σ is complete. One can apply (VI-9). Since $\Sigma \vdash \sigma_2 \to \sigma_3$, $\Sigma, \sigma_2, \neg\sigma_3$ is inconsistent. So either (i) Σ, σ_2 is inconsistent, or (ii) $\Sigma, \neg\sigma_3$ is inconsistent.

In (i), $\Sigma \vdash \neg\sigma_2$; take σ_1 to be any contradiction in $\mathcal{L}_1$ (e.g. $(\exists x)(x \neq x)$). Then $\Sigma \vdash \neg\sigma_1 \to \neg\sigma_2$, so $\Sigma \vdash \sigma_2 \to \sigma_1$, and also $\Sigma \vdash \sigma_1 \to \sigma_3$.

In (ii), $\Sigma \vdash \sigma_3$; take σ_1 a validity (e.g. $\forall x(x=x)$).

CASE 2: General case. One reduces to case 1 by a maximality argument.

One begins with $\Sigma \vdash \sigma_2 \to \sigma_3$ and assumes no interpolant.

Strategy: go to a consistent $\Sigma \cup \{\delta\}$ or $\Sigma \cup \{\neg\delta\}$ for δ in $\mathcal{L}_1$, still with no interpolant.

Suppose one has interpolant α for $\Sigma \cup \{\delta\}$ and interpolant β for $\Sigma \cup \{\neg\delta\}$. So, $\Sigma \cup \{\delta\} \vdash \sigma_2 \to \alpha,\ \alpha \to \sigma_3$, and $\Sigma \cup \{\neg\delta\} \vdash \sigma_2 \to \beta,\ \beta \to \sigma_3$. But $\Sigma \cup \{\delta\} \vdash \sigma_2 \to \alpha \Rightarrow \Sigma \vdash \sigma_2 \to (\delta \to \alpha)$, and $\Sigma \cup \{\neg\delta\} \vdash \sigma_2 \to \beta \Rightarrow \Sigma \vdash \sigma_2 \to (\neg\delta \to \beta)$. So $\Sigma \vdash \sigma_2 \to \gamma$, where γ is $(\delta \to \alpha) \wedge (\neg\delta \to \beta)$. It is easy to see that $\Sigma \vdash \gamma \to \sigma_3$, contradiction.

Complete proof by expanding a Σ without interpolant to a complete Σ' without interpolant.

□

(VII) ULTRAPRODUCTS

Limited goal: to give a "new" proof of the compactness theorem, which is more complete, in the following sense:

Given Σ such that each of its finite subsets has a model. Select for each finite $\Sigma_0 \subseteq \Sigma$ a model $\mathfrak{M}_{\Sigma_0}$ of Σ_0. Construct out of the $\mathfrak{M}_{\Sigma_0}$'s a model $\mathfrak{M}$ of Σ. In this case $\mathfrak{M}$ will be an ultraproduct of the $\mathfrak{M}_{\Sigma_0}$'s, where the ultraproduct lives on the set of all finite subsets of Σ.

(VII-1) Basic facts on filters: a filter D on an index set I is a collection of subsets of I such that: (i) $X, Y \in D \Rightarrow X \cap Y \in D$; (ii) $X \in D \wedge X \subseteq Y \Rightarrow Y \in D$.

NOTE: if $\emptyset \in D$, then $D = \mathcal{P}(I)$.

EXAMPLE: pick $t_0 \in I$; let $D = \{X : t_0 \in X\}$; this is called a principal filter.

An ultrafilter is a maximal proper filter. This is equivalent to $\forall X \subseteq I (X \in D \vee I \setminus X \in D) \wedge D \neq \mathcal{P}(I)$.

Essential non-constructible element: if

I is infinite, there is a non-principal ultrafilter on I.

<u>Extension principle</u>: if F is a set of subsets of I with the f.i.p. (finite intersection property — i.e., if $X_1, \ldots, X_m \in F$, then $X_1 \cap \ldots \cap X_m \neq \emptyset$), then F can be extended to an ultrafilter.

(VII-2) <u>DEFINITION</u>: Let $\mathfrak{M}_i$ ($i \in I$) be a family of $\mathcal{L}$-structures, and D a filter on I. Consider $\prod_{i\in I} \mathfrak{M}_i = \{f : f : I \to \bigcup \mathfrak{M}_i, \forall i \in I, f(i) \in \mathfrak{M}_i\}$. From D one defines an equivalence relation $\equiv_D$ on $\prod \mathfrak{M}_i$, by $f \equiv_D g \Leftrightarrow$ $\Leftrightarrow \{i : f(i) = g(i)\} \in D$. One calls $\prod \mathfrak{M}_i / D$ ($= \prod \mathfrak{M}_i / \equiv_D$) the <u>reduced</u> <u>product</u> of $\mathfrak{M}_i$. If D is an ultrafilter, this is called an ultraproduct.

One writes $[f]_D$ (or simply $[f]$) for the equivalence class of f. The underlying set of $\prod \mathfrak{M}_i / D$ is $\{[f] : f \in \prod \mathfrak{M}_i\}$.

(VII-3) <u>Imposing $\mathcal{L}$-structure on $\prod \mathfrak{M}_i / D$</u>:

(i) interpretation of a <u>constant symbol</u> c: let γ_i be the interpretation of c in $\mathfrak{M}_i$, and $\gamma \in \prod \mathfrak{M}_i$, $\gamma(i) = \gamma_i$; interpret c in $\prod \mathfrak{M}_i / D$ by $[\gamma]$;

(ii) let F be an n-ary <u>function symbol</u>: let $[f_1], \ldots, [f_n] \in \prod \mathfrak{M}_i / D$; one has to give a sense to $F([f_1], \ldots, [f_n])$ as $[f]$; pointwise $F(f_1(i), \ldots, f_n(i))$ is defined in $\mathfrak{M}_i$; define f by $f(i) = F(f_1(i), \ldots, f_n(i))$, and then put $F([f_1], \ldots, [f_n]) = [f]$; (<u>EXERC</u>: show this is well-defined.)

(iii) let R be an n-ary <u>relation symbol</u>; define $R([f_1], \ldots, [f_n])$ holds iff $\{i : \mathfrak{M}_i \models R(f_1(i), \ldots, f_n(i))\} \in D$. (<u>EXERC</u>: Show this is well-defined.)

35 (VII-4) <u>THEOREM</u>: (<u>Łos</u>, <u>1955</u>). Let $\mathfrak{M}_i$ ($i \in I$) be a family of $\mathcal{L}$-structures, and D an <u>ultrafilter</u> on I. Let $\Phi(v_1, \ldots, v_n)$ be any $\mathcal{L}$-formula, and let $[f_1], \ldots, [f_n] \in \prod \mathfrak{M}_i / D$.

Then $\prod \mathfrak{M}_i/D \models \Phi([f_1],\dots,[f_n])$ iff $\{i : \mathfrak{M}_i \models \Phi(f_1(i),\dots,f_n(i))\} \in D$.

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CORALLARY: If Φ is a sentence, $\prod \mathfrak{M}_i/D \models \Phi$ iff $\{i : \mathfrak{M}_i \models \Phi\} \in D$.

(NOTE: One does not prove this directly. One must go through the formula version.)

PROOF OF (VII-4):

By induction on the complexity of Φ. One denotes $\prod \mathfrak{M}_i/D$ by $\mathfrak{M}$.

(1) Φ is atomic: two cases:

(1.1) $\Phi(v_1,\dots,v_n)$ is $R(v_1,\dots,v_n)$, where R is an n-ary relation symbol. True by definition.

(1.2) $\Phi(\bar{v})$ is $\tau(\bar{v}) = \mu(\bar{v})$, where τ and μ are terms. This is done by induction on the complexity of terms. The basic cases again are true by definition.

(2) Φ is built from Φ_1 and Φ_2 by $\wedge$ (or $\vee$). One has: $\mathfrak{M} \models \Phi([f_1],\dots,[f_n]) \iff \mathfrak{M} \models \Phi_1(\overline{[f]})$ and (or) $\mathfrak{M} \models \Phi_2(\overline{[f]}) \iff$ (by hypothesis) $\{i : \mathfrak{M}_i \models \Phi_1(\bar{f}(i))\} \in D$ and (or) $\{i : \mathfrak{M}_i \models \Phi_2(\bar{f}(i))\} \in D \iff$ (in "$\vee$" case, use that D is ultrafilter) $\{i : \mathfrak{M}_i \models \Phi(\overline{[f]})\} \in D$.

(3) Φ is built from Φ_1 by $\neg\Phi_1$. One has: $\mathfrak{M} \models \Phi(\overline{[f]}) \iff \mathfrak{M} \models \neg\Phi_1(\overline{[f]}) \iff \mathfrak{M} \not\models \Phi_1(\overline{[f]}) \iff$ (by hypothesis, and D is ultrafilter) $\{i : \mathfrak{M}_i \models \Phi_1(\bar{f}(i))\} \notin D$ $\iff \{i : \mathfrak{M}_i \models \neg\Phi_1(\bar{f}(i))\} \in D \iff \{i : \mathfrak{M}_i \models \Phi(\bar{f}(i))\} \in D$.

(4) Φ is $\exists v_{n+1} \Psi(\bar{v}, v_{n+1})$. One has: $\mathfrak{M} \models \Phi(\overline{[f]}) \iff \exists f_{n+1}$ s.t. $\mathfrak{M} \models \Psi(\overline{[f]}, [f_{n+1}]) \iff$ (by induction) $\exists f_{n+1}$ s.t. $\{i : \mathfrak{M}_i \models \Psi(\bar{f}(i), f_{n+1}(i))\} \in D$: ① $\overset{\circledast}{\iff}$ $\overset{\circledast}{\iff}$ $\{i \in I : \mathfrak{M}_i \models \Phi(\bar{f}(i))\} \in D$: ②. To see last equivalence, first suppose ①. Choose f_{n+1} to work. Then $\{i : \mathfrak{M}_i \models \Psi(\bar{f}(i), f_{n+1}(i))\} \subseteq \{i : \mathfrak{M}_i \models \Phi(\bar{f}(i))\}$. Since the first set is in D, the second set is in D. Now suppose ②. Let $X = \{i : \mathfrak{M}_i \models \exists v_{n+1} \Psi(\bar{f}(i), v_{n+1})\} \in$ $\in D$. Define (assume A.C.) $f_{n+1} : I \to \bigcup \mathfrak{M}_i$ to satisfy:

$$\begin{cases} \text{(i) if } i \in X, \ \mathfrak{M}_i \models \Psi(\bar{f}(i), f_{n+1}(i)); \\ \text{(ii) if } i \notin X, \ f_{n+1}(i) \in \mathfrak{M}_i \text{ arbitrary.} \end{cases}$$

Then f_{n+1} makes ① work.

$\square$

(VII-5) "New" proof of compactness (Tarski):

Let Σ be a set of $\mathcal{L}$-sentences such that every finite subset i of Σ has a model $\mathcal{M}_i$. Let I be the set of all finite subsets of Σ. One shows that there exists an ultrafilter D on I such that $\prod \mathcal{M}_i/D \models \Sigma$. It suffices to show that if j is a finite subset of Σ, then $\prod \mathcal{M}_i/D \models j$. By Łos, this amounts to showing $\{i : \mathcal{M}_i \models j\} \in D$. So one has to get each such (one for each j) into D.

By "extension principle" (VII-1), it suffices to show (as far as existence of such a D is concerned) that the family $(\{i : \mathcal{M}_i \models j\})_{j \in I}$ has f.i.p..

Suppose not. Then the intersection of the sets corresponding to $j_1, \dots, j_n$, say, is empty. Let $i = j_1 \cup \dots \cup j_n$. Then $\mathcal{M}_i \models j_1, \dots, j_n$, contradiction.

$\square$

<u>NOTE</u>: Everything is concentrated in the "extension principle".

(VII-6) <u>The diagonal map</u>: Assume all $\mathcal{M}_i = \mathcal{M}$. Write $\mathcal{M}^I/D$ for $\prod \mathcal{M}_i/D$ (this is called an <u>ultrapower</u> of $\mathcal{M}$). There is a natural map $\mathcal{M} \xrightarrow{\Delta} \mathcal{M}^I/D$, $m \mapsto [f]$, where $f(i) = m$ for all $i \in I$. (Δ is called the <u>diagonal map</u> from $\mathcal{M}$ to $\mathcal{M}^I/D$.) Δ is clearly 1-1.

36 (VII-7) <u>THEOREM</u>: Δ is elementary.

<u>PROOF</u>:

Suppose $\mathcal{M}^I/D \models \Phi(\Delta(m_1), \dots, \Delta(m_n))$. Then, by Łos, $\{i \in I : \mathcal{M} \models \Phi(\Delta(m_1)(i), \dots, \Delta(m_n)(i))\} = X \in D$. ($X$ is either ϕ or I).

So, for all i, $\mathcal{M} \models \Phi(\overline{m})$, i.e., $\mathcal{M} \models \Phi(\overline{m})$.

$\square$

When is Δ not onto? (i.e., when is $\mathcal{M}^I/D$ bigger than $\mathcal{M}$?)

Does there exist $[f]$ s.t. $[f] \neq \Delta(m)$, for any $m \in \mathcal{M}$?

The answer to this question is

not if $\mathcal{M}$ is finite (and thus $|\mathcal{M}^I/D| = |\mathcal{M}|$, since there is a sentence fixing the cardinal of $\mathcal{M}$), or if D is principal.

<u>EXERC</u>: Let $\mathcal{M}$ and I be countable, D be a non-principal ultrafilter on I. Then Δ is not onto. (W.l.o.g., $\mathcal{M} = I$. Let $f: \mathcal{M} \to \mathcal{M}$ be the identity map — $f(x) = x$, $\forall x \in \mathcal{M}$ —, for any $m \in \mathcal{M}$, $f(i) = \Delta m(i) \iff i \in \{m\} \notin D$ — since D is non-principal.)

<u>NOTE</u>: In this case, $\mathcal{M}^I/D$ has cardinality $2^{\aleph_0}$ and is $\aleph_1$-saturated, if $\mathcal{L}$ is countable.

(VII-8) <u>THEOREM</u> (Keisler - Shelah): $Th(\mathcal{M}) = Th(\mathcal{N})$ iff there exist I and D such that $\mathcal{M}^I/D \cong \mathcal{N}^I/D$.

<u>EXERC</u>: $Th(\mathcal{M}) = Th(\mathcal{N})$ iff there exist $\mathcal{M}_1 \succ \mathcal{M}$ and $\mathcal{N}_1 \succ \mathcal{N}$ such that $\mathcal{M} \cong \mathcal{N}$.

THE END