MODEL THEORY

(I) BASIC NOTIONS

Let $\mathcal{L}$ denote a first-order language, with a set of constant symbols $C = \{c_\alpha : \alpha \in A\}$, a set of variables $V = \{v_n : n \geq 0\}$, a set of n-ary function symbols $F_n = \{f_i : i \in I\}$ (for each $n \geq 1$) and a set of n-ary relation symbols $R_n = \{R_\lambda : \lambda \in \Lambda\}$ (for each $n \geq 1$).

(I.1) DEFINITION: The terms of $\mathcal{L}$ are as follows:
(a) each $c_\alpha \in C$ is a term;
(b) each $v_n \in V$ is a term;
(c) if $\tau_1, \ldots, \tau_n$ are terms and $f \in F_n$ (a n-ary function symbol), then $f(\tau_1, \ldots, \tau_n)$ is a term;
(d) every term of $\mathcal{L}$ can be deduced in a finite number of steps from (a), (b) & (c).

For instance, if $\mathcal{L}$ has $C = \{0, 1\}$, $F_2 = \{+, \cdot\}$, $R_2 = \{<\}$ (or, in a shorter way, $\mathcal{L}$ has $\{0, 1, +, \cdot, <\}$), then $0, 1, v_0, v_1, v_2, v_1 + v_2, v_3 + 0 + 1 \cdot v_2 \cdot v_4$ are terms of $\mathcal{L}$.

(I.2) DEFINITION: The formulas of $\mathcal{L}$ are as follows:
(a) if τ_1, τ_2 are terms, $\tau_1 = \tau_2$ is a formula;
(b) if $\tau_1, \ldots, \tau_n$ are terms, and $R \in R_n$ (a n-ary relation symbol), then $R(\tau_1, \ldots, \tau_n)$ is a formula. The formulas defined in (a) and (b) are said atomic formulas.
(c) if $\varphi, \varphi_1, \varphi_2$ are formulas, then $\neg\varphi$, $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$, $(\exists v)\varphi$, $(\forall v)\varphi$ (where v is a free variable, i.e., does not appear in any quantifier of φ) are formulas;
(d) the formulas of $\mathcal{L}$ can be deduced in

a finite number of steps from (a), (b) and (c).

One says that a formula φ is a <u>sentence</u> of $\mathcal{L}$ (or an <u>$\mathcal{L}$-sentence</u>) if all variables that occur in φ are <u>bounded</u> (i.e., not free). A <u>theory</u> T of $\mathcal{L}$ is a consistent set of sentences. A <u>complete theory</u> is a theory T such that for all sentence φ in $\mathcal{L}$, either $\varphi \in T$ or $\neg\varphi \in T$.

An <u>$\mathcal{L}$-structure</u> $\mathfrak{M}$ consists of a set $\mathfrak{M}$ (abusing of notation), and for each $f \in F_n$, corresponds a function $\hat{f} : \mathfrak{M}^n \to \mathfrak{M}$, and to each $R \in R_n$ corresponds a relation $\hat{R} \subseteq \mathfrak{M}^n$. ~~For an $\mathcal{L}$-structure $\mathfrak{M}$ one has~~

~~(I-3) DEFINITION: Let Fla($\mathcal{L}$) denote the set of $\mathcal{L}$-formulas. One defines the function v : Fla($\mathcal{L}$) $\to \{0,1\}$, by (a)~~

Now one wants to define interpretation of terms and formulas in $\mathfrak{M}$.

(I-3) <u>DEFINITION</u>: <u>Interpretation of terms of $\mathcal{L}$ in $\mathfrak{M}$</u>:

(a) if $c \in C$, $\hat{c}$ is an element of $\mathfrak{M}$;
(b) if $f \in F_n$, $\hat{f}$ is a function $\mathfrak{M}^n \xrightarrow{f} \mathfrak{M}$;
(c) if one has defined $\hat{\tau}_1, \dots, \hat{\tau}_n$ for terms $\tau_1, \dots, \tau_n$ and $f \in F_n$, then one defines $(f(\tau_1, \dots, \tau_n))^\wedge$ as $\hat{f}(\hat{\tau}_1, \dots, \hat{\tau}_n)$.

(I-4) <u>DEFINITION</u>: <u>Interpretation of formulas</u>.
Let Fla($\mathcal{L}$) denote the set of $\mathcal{L}$-formulas. One defines v : Fla($\mathcal{L}$) $\to \{0,1\}$, by:
~~[illegible]~~

(a) if $\varphi = \varphi(v_0, \dots, v_k)$ is of the form $\tau_1 = \tau_2$, and $m_0, \dots, m_k \in \mathfrak{M}$, then $v(\varphi(m_0 \dots m_k)) = 1$ iff $\hat{\tau}_1(\vec{m}) = \hat{\tau}_2(\vec{m})$;
(b) if $\varphi = \varphi(v_0, \dots, v_k)$ is of the form $R(\tau_0, \dots, \tau_n)$ then $v(\varphi(\vec{m})) = 1$ iff $(\hat{\tau}_0(\vec{m}), \dots, \hat{\tau}_n(\vec{m})) \in \hat{R}$;
(c) if one has defined $v(\varphi)$, $v(\varphi_1)$ and $v(\varphi_2)$, then one defines $v(\neg\varphi) = 1 - v(\varphi)$,

$v(\varphi_1 \wedge \varphi_2) = \min(v(\varphi_1), v(\varphi_2))$, $v(\varphi_1 \vee \varphi_2) = \max(v(\varphi_1), v(\varphi_2))$ and if $\varphi = \varphi(v, v_0, \dots, v_k)$, then $v(\exists v\, \varphi(v, \vec{m})) = \sup\{v(\varphi(m', \vec{m})) : m' \in \mathfrak{M}\}$ and $v(\forall v\, \varphi(v, \vec{m})) = \inf\{v(\varphi(m', \vec{m})) : m' \in \mathfrak{M}\}$.

If $m_0, \dots, m_k \in \mathfrak{M}$, then one writes $\mathfrak{M} \models \varphi(\vec{m})$ (read "$\mathfrak{M}$ satisfies φ in $\vec{m} = (m_0, \dots, m_k)$") if $v(\varphi(\vec{m})) = 1$. If φ is a sentence, then one writes $\mathfrak{M} \models \varphi$ if $v(\varphi) = 1$.

If T is a theory in $\mathcal{L}$, one says $\mathfrak{M}$ is a model of T iff for all $\varphi \in T$, $\mathfrak{M} \models \varphi$ (or $\mathfrak{M} \models T$).

One says that a set Σ of $\mathcal{L}$-sentences is <u>finitely satisfiable</u> if for all finite $\Sigma_0 \subseteq \Sigma$, there is a model ~~for~~ of Σ_0.

(I-5) <u>DEFINITION</u>: Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\mathcal{L}$-structures, and $f : \mathfrak{N} \to \mathfrak{M}$ be a function:

(a) f is said <u>isomorphism (of $\mathcal{L}$-structures)</u> if f is bijective and for all atomic $\varphi(v_0, \dots, v_k)$, and all $n_0, \dots, n_k \in \mathfrak{N}$, $\mathfrak{N} \models \varphi(\vec{n})$ iff $\mathfrak{M} \models \varphi(f(\vec{n}))$

(b) f is an <u>injective map</u>, if f is a 1-1 map of the underlying set of $\mathfrak{M}$ to that of $\mathfrak{N}$, and f respects $\mathcal{L}$-structure, that is:

(1) f sends interpretation of a constant a in $\mathfrak{M}$ to that of a in $\mathfrak{N}$;

(2) if $g : \mathfrak{M}^k \to \mathfrak{M}$ is the interpretation of a k-ary function symbol, and $h : \mathfrak{N}^k \to \mathfrak{N}$ is its interpretation in $\mathfrak{N}$, then $f(g(x_1, \dots, x_k)) = h(f(x_1), \dots, f(x_k))$;

(3) if R is the interpretation of a k-ary relation symbol in $\mathfrak{M}$, and S is its interpretation in $\mathfrak{N}$, then, for all $x_1, \dots, x_k \in \mathfrak{M}$, $R(x_1, \dots, x_k) \leftrightarrow S(f(x_1), \dots, f(x_k))$.

(c) suppose $f : \mathfrak{M} \to \mathfrak{N}$ is an injective map. Then f is <u>elementary</u> if for all $\mathcal{L}$-formulas $\Phi(v_1, \dots, v_n)$ and all $x_1, \dots, x_n \in \mathfrak{M}$, then $\mathfrak{M} \models \Phi(x_1, \dots, x_n) \leftrightarrow \mathfrak{N} \models \Phi(f(x_1), \dots, f(x_n))$. In this case one says that $\mathfrak{M}$ is <u>elementarily equivalent</u> to $\mathfrak{N}$, <u>$\mathfrak{M} \equiv \mathfrak{N}$</u>.

④

(I-6) DEFINITION: $\mathcal{M} \prec \mathcal{N}$, $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, or, $\mathcal{N}$ is an elementary extension of $\mathcal{M}$, if $\mathcal{M}$ is a substructure of $\mathcal{N}$ (that is, $\mathcal{M} \subseteq \mathcal{N}$, and interpretation of constants, function and relation symbols are induced from $\mathcal{N}$) and the inclusion map $i: \mathcal{M} \hookrightarrow \mathcal{N}$ is elementary.

(II) COMPACTNESS

15/10/86 In this section it will be proved the following result:

1 (II-1) THEOREM (COMPACTNESS): Suppose $\mathcal{L}$ is a first-order language and Σ is a set of $\mathcal{L}$-sentences. Then, the following are equivalent:
(i) there exists $\mathfrak{M}$ such that $\mathfrak{M} \models \Sigma$;
(ii) for each finite subset $\Sigma_0 \subseteq \Sigma$, there is a model $\mathfrak{M}_0$ for Σ_0 (that is, Σ is finitely satisfiable).

This theorem is due to Malcev (1936). Gödel's work was for countable $\mathcal{L}$ only. Gödel completeness: "Σ has a model iff Σ is consistent (relative to one of the natural axiom systems for predicate calculus)".

2 (II-2) LEMMA: Any increasing union of finitely satisfiable sets is finitely satisfiable.

This will be used in Zorn's lemma arguments to get maximal finitely satisfiable subsets.

NOTE: Replace "finitely satisfiable" by "consistent" and lemma (II-2) is still true.

EXAMPLE: Let $\mathfrak{M}$ be an $\mathcal{L}$-structure. Define $Th(\mathfrak{M})$ by $Th(\mathfrak{M}) = \{\Phi : \mathfrak{M} \models \Phi\}$ (Φ sentence).
Then $Th(\mathfrak{M})$ is finitely satisfiable, and in fact is maximal with this property: For any Φ, either Φ or $\neg\Phi$ is in $Th(\mathfrak{M})$, whence maximality.

(II-3) DEFINITION: Let Σ be a set of $\mathcal{L}$-sentences, and C be a set of $\mathcal{L}$-constants. Then C is a set of witnesses for Σ if for all $\Phi \in \Sigma$ of the form $(\exists v)\Psi(v)$, there is a $c \in C$ such that the formula $(\exists v)\Psi(v) \longrightarrow \Psi(c)$ is in Σ.

EXAMPLE: Let $\mathfrak{M}$ be an $\mathcal{L}$-structure. Add to $\mathcal{L}$ new distinct constants $\overline{m}$ for each $m \in \mathfrak{M}$.

⑥

This gives a new language $\mathcal{L}(\mathfrak{M})$; convert $\mathfrak{M}$ into an $\mathcal{L}(\mathfrak{M})$-structure $\mathfrak{M}_{\mathfrak{m}}$, by interpreting $\overline{m}$ by m. And let $\Sigma = Th_{\mathcal{L}(\mathfrak{M})}(\mathfrak{M}_{\mathfrak{m}})$, and $C = \{\overline{m} : m \in \mathfrak{M}\}$. Then C is a set of witnesses for Σ.

3 (II-4)(TRUTH) LEMMA: Let Σ be a maximal finitely satisfiable set of $\mathcal{L}$-sentences, and C be a set of witnesses for Σ. Then there is an $\mathcal{L}$-structure $\mathfrak{M}$ such that $\Sigma = Th_{\mathcal{L}(\mathfrak{M})}(\mathfrak{M})$.

PROOF: $\mathfrak{M}$ will be constructed as the set of (equivalence classes of) witnesses.

Define $\cong$ on C by $c \cong d$ iff $(c=d) \in \Sigma$.

CLAIM: $\cong$ is an equivalence relation (clear!).

One has to:

(i) interpret all constants, function symbols, and relation symbols;

(ii) prove "TRUTH LEMMA": $\mathfrak{M} \models \Phi([c_1], \dots, [c_n])$ iff $\Phi(c_1, \dots, c_n) \in \Sigma$, where $\mathfrak{M} = \{[c] : c \in C\}$.

STEP 1: Interpretation of:

(A) constants: let a be an $\mathcal{L}$-constant. Now, by maximality of Σ, $(\exists v (v=a)) \in \Sigma$. So, for some $c \in C$, $(\exists v (v=a) \to (c=a)) \in \Sigma$, since C is a set of witnesses for Σ, and thus $(c=a) \in \Sigma$. Interpret a by $[c]$. By maximality again, the result does not depend on the choice of c.

(B) function symbols: let f be n-ary; interpret f by the n-ary function on $\mathfrak{M}$, $\hat{f}$, where $\hat{f}([c_1], \dots, [c_n]) = [c]$, where $(f(c_1, \dots, c_n) = c) \in \Sigma$.

(C) relation symbols: let R be n-ary; interpret R by $\hat{R}$, where $\hat{R}([c_1], \dots, [c_n])$ holds iff $R(c_1, \dots, c_n) \in \Sigma$.

17/10/86 STEP 2: Now prove: for all $\mathcal{L}$-formula $\Phi(v_1, \dots, v_n)$ and all $c_1, \dots, c_n \in C$, $\mathfrak{M} \models \Phi([c_1], \dots, [c_n])$ iff $\Phi(c_1, \dots, c_n) \in \Sigma$.

This is proved by induction on the complexity of formulas.

CASE 1: if Φ is atomic, by step 1, it is clear.

CASE 2: if $\Phi = \Phi_1 \vee \Phi_2$ and one has proved step 2 for Φ_1 and Φ_2. Then it is clear that $\mathfrak{M} \models \Phi_1 \vee \Phi_2 ([c_1], \dots, [c_n])$ iff $\mathfrak{M} \models \Phi_1([c_1], \dots, [c_n])$ or

$\mathfrak{M} \models \Phi_2([c_1], \dots, [c_n])$ iff $\Phi_1(\bar{c})$ or $\Phi_2(\bar{c})$ is in Σ iff $\Phi_1 \vee \Phi_2(\bar{c}) \in \Sigma$ (where $\bar{c} = (c_1 \dots c_n)$).

CASE 3: if Φ is $\neg\Psi$, then $\mathfrak{M} \models \Phi([c_1], \dots, [c_n])$ iff $\mathfrak{M} \not\models \Psi([c_1], \dots, [c_n])$ iff $\Psi(c_1, \dots, c_n) \notin \Sigma$ iff $\Phi(\bar{c}) \in \Sigma$.

CASE 4: if Φ is $\exists v\, \Psi(v, v_1, \dots, v_n)$, then $\mathfrak{M} \models \Phi(\overline{[c_i]})$ iff there exists a $[c] \in \mathfrak{M}$ such that $\mathfrak{M} \models \Psi([c], \overline{[c_i]})$, iff $\Psi(c, \bar{c}) \in \Sigma$, iff $\exists v \Psi(v, \bar{c}) \in \Sigma$.

□

PROOF OF COMPACTNESS THEOREM (II-1):

Start with a finitely satisfiable set Σ in $\mathcal{L}$. One wants to construct a model of Σ.

Strategy: Extend $\mathcal{L}$ to $\mathcal{L}'$ with Σ' a set of $\mathcal{L}'$-sentences, $\Sigma \subseteq \Sigma'$, Σ' maximal finitely satisfiable and with a witness set in $\mathcal{L}'$. Use the above to get a model for Σ', giving automatically a model for Σ.

Sketch: one begins with $\Sigma_0 = \Sigma$ in $\mathcal{L}_0 = \mathcal{L}$, and to get Σ_{i+1} and $\mathcal{L}_{i+1}$, add to $\mathcal{L}_i$ a constant c for each existencial formula $(\exists v)\Phi \in \Sigma_i$, and add to Σ_i the formula $(\exists v)\Phi(v) \to \Phi(c)$. Let $\Sigma_\omega = \bigcup_{i \geq 0} \Sigma_i$ in $\mathcal{L}_\omega = \bigcup_{i \geq 0} \mathcal{L}_i$. Then $\Sigma' = \Sigma_\omega$ and $\mathcal{L}' = \mathcal{L}_\omega$ are as required.

□

(III) CARDINALITY AND LÖWENHEIM-SKOLEM THEOREMS

One wants to show that if a set Σ of sentences of $\mathcal{L}$ has an infinite model, then Σ has a model of "arbitrary" infinite cardinal.

(III-1) DEFINITION: The cardinality of $\mathcal{L}$, card($\mathcal{L}$) or $|\mathcal{L}|$, is the cardinal of the set of $\mathcal{L}$-formulas.

STRATEGY: Let Σ be in $\mathcal{L}$; move up to a new language with extra constants, one for each formula $(\exists v)\Phi(v)$ in Σ; one wants to add to Σ $(\exists v)\Phi(v) \to \Phi(c)$ for suitable new c. Repeat process infinitely often.

4 (III-2) LEMMA: Let Σ be a finitely satisfiable set of $\mathcal{L}$-sentences. Let $\kappa = \mathrm{card}(\mathcal{L})$. Add to $\mathcal{L}$ a set C of κ distinct new constants. Write $\mathcal{L}(C)$ for the new language. Then there exists Σ' in $\mathcal{L}(C)$, $\Sigma \subseteq \Sigma'$, Σ' maximal finitely satisfiable, and C as a witness set for Σ'.

PROOF:

Enumerate C as $\{c_\lambda : \lambda < \kappa\}$, a well-order of type κ. Enumerate the set of all $\mathcal{L}(C)$-sentences as $\{\sigma_\lambda : \lambda < \kappa\}$.

Σ' is constructed by recursion, meeting κ many requirements:

(1) for each formula $\exists v \Psi(v)$, there is some $c \in C$ with $\exists v \Psi(v) \to \Psi(c) \in \Sigma'$;

(2) Σ' is maximal finitely satisfiable; in particular, for each σ_λ, one of $\sigma_\lambda, \neg\sigma_\lambda$ is in Σ'.

Σ' is constructed as an increasing union $\bigcup_{\lambda<\kappa} \Sigma_\lambda$ so that:

(i) $\Sigma \subseteq \Sigma_\lambda$, $\forall \lambda < \kappa$;

(ii) at limits λ, $\Sigma_\lambda = \bigcup_{\mu<\lambda} \Sigma_\mu$

(iii) Σ_λ is a set of $\mathcal{L}(C_\lambda)$-sentences, where $C_\lambda \subseteq C$, and $|C_\lambda| < \kappa$ (to ensure that there are plenty of constants c left on each stage).

(iv) Σ_λ is finitely satisfiable.

$\Sigma_{\lambda+1}$ is constructed to take account of requirements given above.

(a) either σ_λ or $\neg\sigma_\lambda$ goes to $\Sigma_{\lambda+1}$

(b) if σ_λ is $\exists v \Psi(v)$, then for some constant c, $\exists v \Psi(v) \to \Psi(c)$ goes to $\Sigma_{\lambda+1}$. Insist that c is new (does not occur in Σ_λ).

STAGE 0: $\Sigma_0 = \Sigma$

SUCCESSOR STAGE: $\lambda = \delta + 1$: Σ_δ is finitely satisfiable, so it is possible to add one of σ_δ or $\neg\sigma_\delta$ to Σ_δ to preserve finitely satisfaction.

First add one to get $\Sigma_\delta \cup \{(\neg)\sigma_\lambda\}$.

Now see if σ_δ is $\exists v \Psi(v)$. Choose minimum $\mu < \kappa$ such that c_μ does not belong to the language of $\Sigma_\delta \cup \{(\neg)\sigma_\delta\}$. Add $(\exists v)\Psi(v) \to \Psi(c_\mu)$ to $\Sigma_\delta \cup \{(\neg)\sigma_\delta\}$ to get $\Sigma_{\delta+1}$. Usual arguments show this is finitely satisfiable. $\square$

5 (III-3) THEOREM (UPWARD LÖWENHEIM-SKOLEM): Let Σ be a set of $\mathcal{L}$-sentences with an infinite model. Let $\kappa \geq |\mathcal{L}|$ be any infinite cardinal. Then Σ has a model of cardinal κ.

EXERCISE: The above theorem (III-3) implies AC (= axiom of choice).

22/10/86 One has Σ finitely satisfiable and constructed a model $\mathcal{M}$ of Σ.

PROBLEM: What is the structure of the class $\text{Mod}(\Sigma) = \{\mathcal{N} : \mathcal{N} \models \Sigma\}$?

FIRST APPROXIMATION: What is the structure of the class $\{\text{card}(\mathcal{N}) : \mathcal{N} \models \Sigma\}$?

OBS: Henceforward assume AC.

Cardinality of the model $\mathcal{M}$ constructed in proof of compactness: Σ in $\mathcal{L}$, $|\mathcal{L}| = \lambda$; add to $\mathcal{L}$ a set C of constants, $|C| = \lambda$ (notice that $|C| \geq \lambda$ is absolutely necessary for proof one gave). $\mathcal{M} = \{[c] : c \in C\}$, so $|\mathcal{M}| \leq |C| = \lambda$.

6 (III-4) LEMMA: If Σ has a model, Σ has a model of cardinal less than, or equal to, $\text{card}(\mathcal{L})$.

NOTE: This is the best possible. Let κ be any infinite cardinal. Let $\mathcal{L}$ have constants d_μ, $\mu < \kappa$. Let Σ be the set of all $\neg(d_\mu = d_\nu)$, $\mu \neq \nu$. Σ obviously has a model, but no model can ~~be~~ have cardinal less than κ, because the d_μ's require distinct interpretations.

NOTE: One has not shown that in all cases $\text{card}(\mathcal{M}) = \text{card}(\mathcal{L})$ is possible. However if $\lambda = \aleph_0$ one does get that Σ has a finite or countable model.

7 (III-5) LEMMA: Let $\text{card}(\mathcal{L}) = \lambda$. Suppose that Σ has an infinite model. Let $\kappa \geq \lambda$ be a cardinal. Then Σ has a model $\mathcal{N}$, with $|\mathcal{N}| \geq \kappa$.

PROOF:

Let C be a new set of constants, with $|C| = \kappa$. Fix some $\mathcal{M}$, $\mathcal{M} \models \Sigma$, and $\mathcal{M}$ infinite.

Now consider $\Sigma' = \Sigma \cup \{\neg(c=d) : c, d$ distinct in $C\}$.

<u>CLAIM</u>: Σ' is finitely satisfiable.

Let $\Sigma'_0 \subseteq \Sigma'$ finite, $\Sigma'_0 = \Sigma_0 \cup \{\neg(c=d) : c, d \in C_0, c \neq d\}$, where $\Sigma_0 \subseteq \Sigma$, $C_0 \subseteq C$, C_0 finite ($|C_0| = n$).

Choose n distinct elements of $\mathcal{M}$. Match those elements to C_0 in some fixed way. Since $\mathcal{M} \models \Sigma_0$, get a model of Σ'_0 by interpreting $\forall c \in C_0$ by the matched elements. ////

By compactness, Σ' has a model $\mathcal{M}'$, $|\mathcal{M}'| \geq \kappa$, since the elements of C get different interpretations, and $\mathcal{M}' \models \Sigma$ ($\subseteq \Sigma'$). □

<u>DOWNWARD VERSION-PROBLEM</u>: Given $\mathcal{M}_1 \models \Sigma$, $\mathrm{card}(\mathcal{M}_1) = \kappa_1 \geq \mathrm{card}(\mathcal{L}) = \lambda$. Let $\lambda \leq \kappa \leq \kappa_1$. Construct $\mathcal{M} \models \Sigma$, $|\mathcal{M}| = \kappa$. Try for more, $\mathcal{M}$ is a substructure of $\mathcal{M}_1$, i.e.:

(i) the underlying set of $\mathcal{M}$ as a subset of the underlying subset of $\mathcal{M}_1$;

(ii) interpretation of constants, function and relation symbols are induced from that of $\mathcal{M}_1$.

(Alternatively, say that $\mathcal{M}_1$ is an extension of $\mathcal{M}$).

<u>SPECIAL CASE</u>: $\mathcal{L}$ has a single binary function symbol "$\cdot$", and Σ is the natural set of axioms for groups. Let G be a group of cardinal $\kappa_1 \geq \aleph_0$. Does G have a subgroup of cardinal κ, $\aleph_0 \leq \kappa \leq \kappa_1$? Yes.
Proof: Choose a subset $X \subseteq G$, $|X| = \kappa$. Let H be the subgroup generated by X. Then $\mathrm{card}(H) = \kappa$, since κ is infinite.

8 (III-6) <u>LEMMA (DOWNWARD LÖWENHEIM-SKOLEM)</u>: Let $\lambda = \mathrm{card}(\mathcal{L})$. Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure, $|\mathcal{M}| \geq \lambda$. Let X be a subset of $\mathcal{M}$ of cardinal κ. Then there is an elementary substructure $\mathcal{N}$ of $\mathcal{M}$ such that:
(i) X is contained in the underlying set of $\mathcal{N}$;
(ii) $\mathrm{card}(\mathcal{N}) = \max(\lambda, \kappa)$.

<u>NOTE</u>: $\mathcal{N} \prec \mathcal{M} \Rightarrow \mathrm{Th}(\mathcal{N}) = \mathrm{Th}(\mathcal{M})$.

PROOF:

Add to $\mathcal{L}$ constants $\overline{m}$ for $m \in \mathcal{M}$. This gives a language $\mathcal{L}(\mathcal{M})$ of cardinality equal to card$(\mathcal{M})$. Let $C = \{\overline{m} : m \in \mathcal{M}\}$. Interpret $\overline{m}$ by m, thereby enriching $\mathcal{M}$ to an $\mathcal{L}(\mathcal{M})$-structure. Let Σ be the set of all $\mathcal{L}(\mathcal{M})$-sentences true in $\mathcal{M}$. Then C is a set of witnesses for Σ in $\mathcal{L}(\mathcal{M})$.

GOAL: Construct a set $D \subseteq C$, of cardinal $\max(\lambda, |X|)$, with $\{\overline{x} : x \in X\} \subseteq D$, such that D is a set of witnesses for $\Sigma \cap$ (set of all $\mathcal{L}(D)$-sentences) $= \Sigma\restriction_D$.

Suppose this is done; note that for $c_1 \neq c_2$, c_i in C, Σ contains $\neg(c_1 = c_2)$. Also, Σ is maximal consistent.

As in compactness theorem, construct a model living on the set of equivalent classes $\{[c] : c \in C\}$; $[c]$ is a singleton in this case.

It is trivial that the resulting model is isomorphic to $\mathcal{M}$, via the map $m \mapsto [\overline{m}]$.

Now consider $\Sigma\restriction_D$ (as above). One has $\Sigma\restriction_D$ is maximal consistent and D is a witness set. Construct a model as in compactness theorem. Write $[c]_D$ for equivalence classes for relation $c_1 \equiv c_2$ iff $(c_1 = c_2) \in \Sigma\restriction_D$. Again, $[c] = \{c\}$. Moreover, the atomic structure on this new model is exactly the restriction of that given in the construction for Σ.

So, the map $[\overline{m}]_D \mapsto m$ defines an isomorphism of this new model with a substructure of $\mathcal{M}$, and that substructure contains X.

Let $\mathcal{N}_D$ be the structure on the $[c]_D$'s. Let $\mathcal{N}$ be the corresponding substructure of $\mathcal{M}$. Then $\mathcal{N} \cong \mathcal{N}_D$ as $\mathcal{L}(D)$-structures.

Let $m_1, \dots, m_k \in \mathcal{N}$, and $\Phi(v_1, \dots, v_k)$ be an $\mathcal{L}$-formula. Then

$$\begin{aligned}
\mathcal{N} \models \Phi(m_1, \dots, m_k) &\Leftrightarrow \mathcal{N}_D \models \Phi([\overline{m}_1]_D, \dots, [\overline{m}_k]) \Leftrightarrow \\
&\Leftrightarrow \Phi(\overline{m}_1, \dots, \overline{m}_k) \in \Sigma\restriction_D, \text{ by truth lemma} \Leftrightarrow \\
&\Leftrightarrow \Phi(\overline{m}_1, \dots, \overline{m}_k) \in \Sigma \Leftrightarrow \\
&\Leftrightarrow \mathcal{M} \models \Phi(m_1, \dots, m_k), \text{ by truth lemma.}
\end{aligned}$$

That is, $\mathcal{N} \prec \mathcal{M}$. And card$(\mathcal{N})$ = card$(\mathcal{N}_D) = |D|$.

To finish the proof, one needs to construct

D.

CONSTRUCTION OF D: (This is a slight generalization of group argument - see "special case" just before this lemma).

First, well-order the sentences of $\mathcal{L}(\mathcal{M})$ as $\{\sigma_\mu : \mu < |\mathcal{M}| = |\mathcal{L}(\mathcal{M})|\}$.

Define a series of sets Σ_μ, $\mu < \max(\lambda, |X|)$ of $\mathcal{L}(C)$-sentences, such that the series is increasing, one takes unions at limit ordinals, one adds at most two sentences at each sucessor stage, and $\Sigma\restriction_{\{\bar{z} : z \in X\}} \subseteq \Sigma_D$.

Optimal case: X is a set of witnesses for $\Sigma\restriction_X$, so one does not need to go on, and takes $X = D$.

Suppose that Σ_γ is constructed, $\Sigma_\gamma \subseteq \mathrm{Sent}(\mathcal{L}(\Gamma))$. If Σ_γ is maximal consistent in $\mathcal{L}(\Gamma)$ and has Γ as a set of witnesses, stop here and take $D = \Gamma$, if not, make (at most) two additions to Σ_γ, giving $\Sigma_{\gamma+1}$ as follows:

(i) Pick the least sentence in Σ_γ lacking a witness in Σ_γ. There is a witness implication in Σ. Add the least such to Σ_γ.

(ii) Pick least sentence in $\mathcal{L}(\Gamma)$ such that neither it nor its negation is in Σ_γ. Add to Σ_γ the one of those two wich lies in Σ.

EXERCISE: One stops after no more than $\max(\lambda, |X|)$.

$\square$

(III-7) THEOREM (MODIFICATION OF THEOREM (III-3)): Let $\mathcal{M}$ be an $\mathcal{L}$-structure of cardinality $\kappa \geq \lambda = |\mathcal{L}|$. Let κ_1, κ_2 be cardinals, $\kappa_1 \geq \kappa \geq \kappa_2 \geq \lambda$. Then there are models $\mathcal{M}_1, \mathcal{M}_2$, $|\mathcal{M}_1| = \kappa_1$, $|\mathcal{M}_2| = \kappa_2$, such that $\mathcal{M}_2 \prec \mathcal{M} \prec \mathcal{M}_1$.

PROOF:

To get $\mathcal{M}_1$, go to $\mathcal{L}(\mathcal{M})$ as usual, $(|\mathcal{M}| = |\mathcal{L}(\mathcal{M})|)$. Regard $\mathcal{M}$ as an $\mathcal{L}(\mathcal{M})$-structure. Get, by theorem (III-3), an $\mathcal{L}(\mathcal{M})$-structure $\mathcal{M}_1'$ of cardinality κ_1, with $\mathcal{M}_1' \models \mathrm{Th}_{\mathcal{L}(\mathcal{M})}(\mathcal{M})$. Define map $f : m \in \mathcal{M} \mapsto f(\bar{m}) \in \mathcal{M}_1'$, where $f(\bar{m}) =$ interpretation of $\bar{m}$ in $\mathcal{M}_1$; (this is an elementary map).

EXERCISE: Use naive set theory to get an actual elementary extension $\mathcal{M} \prec \mathcal{M}_1$.

Notice that one has already done the part $\mathcal{M}_2 \preceq \mathcal{M}$.

□

APPLICATIONS:

(1) Existence of a countable ($\in$)-model of set theory (Skolem).

"Paradox": How can one have a countable model and (by Cantor) uncountable sets?

Let Σ be the set of all sentences true in the universe V of sets. Then Σ has a countable model, but maybe with a weird interpretation of "$\in$". But apply Downward Löwenheim-Skolem to V to obtain a countable $\mathcal{M} \preceq V$ (and thus, $\in$ here is actually membership).

PROBLEM: V is not a set.

(2) Non-standard models of arithmetic and analysis:

(2.1) Arithmetic: $\mathcal{L}$ has $+, \cdot, 0, 1$. Let Σ be the set of sentences true in $\mathbb{N}$. Get $\mathbb{N} \prec \mathcal{M}$, with $\mathcal{M}$ of arbitrary cardinality (even $\mathbb{N} \not\cong \mathcal{M}$ by an auxiliary use of Downward Löwenheim-Skolem).

(2.2) Analysis: $\mathcal{L}$ has a primitive for each function from $\mathbb{R}$ to $\mathbb{R}$ and constants $\bar{r}$ for all $r \in \mathbb{R}$. Let $\Sigma = Th_{\mathcal{L}}(\mathbb{R})$. Get $\mathbb{R} \prec \mathcal{M}$, with $|\mathcal{M}| \geq 2^{\aleph_0}$.

29/10/86 OBSERVATION: Let $\mathcal{L}$ be countable, and $\mathcal{M}$ be a finite $\mathcal{L}$-structure, and $\Sigma = Th_{\mathcal{L}}(\mathcal{M})$. Then, part of Σ pins down the cardinality of any model of Σ in this case.

EXAMPLE: Suppose $card(\mathcal{M}) = n \in \mathbb{N}$; consider the sentence $\oplus = (\exists x_1 \ldots x_n)((x_1 \neq x_2) \wedge \ldots \wedge (x_1 \neq x_n) \wedge (x_2 \neq x_3) \wedge \ldots \wedge \wedge (x_{n-1} \neq x_n)) \wedge \forall x_{n+1}((x_{n+1} = x_1) \vee \ldots \vee (x_{n+1} = x_n))$.

9 (III-8) THEOREM (TARSKI'S LIMIT THEOREM): Suppose Λ is an ordered set; let M_λ ($\lambda \in \Lambda$) be an $\mathcal{L}$-structure, with $M_\lambda \prec M_\mu$, whenever $\lambda \leq \mu$. Let $M = \bigcup_{\lambda \in \Lambda} M_\lambda$, as an $\mathcal{L}$-structure. Then, for each $\lambda \in \Lambda$, $M_\lambda \prec M$.

WARNING: One cannot weaken theorem to
hypothesis: $M_\lambda \subseteq M_\mu$ and $Th(M_\lambda) = Th(M_\mu)$;
conclusion: $Th(M) = Th(M_\lambda)$, for all $\lambda \in \Lambda$.

COUNTEREXAMPLE: $\mathcal{L}$ has $=$ and $<$, $\Lambda = \mathbb{N}$. Let $M_n = [-(n+1), (n+1)] \cap \mathbb{Q}$ as an ordered set. Then $M_n \cong M_k$, $\forall n, k \in \mathbb{N}$, so $Th(M_n) = Th(M_k)$. However, $\bigcup_{n \geq 0} M_n = \mathbb{Q}$ as an ordered set, and $\mathbb{Q} \models (\forall x)(\exists y)(y < x)$, but no M_k satisfies it.

PROOF OF THEOREM (III-8):

Add to $\mathcal{L}$ constants $\overline{m}$ for $m \in M$. Let $\Sigma = Th_{\mathcal{L}(M)}(M)$ and $C = \{\overline{m} : m \in M\}$. Then Σ is maximal finitely satisfiable and C is its set of witnesses. As usual, the model M_C constructed from this is isomorphic to M. Let $C_\lambda = \{\overline{m} : m \in M_\lambda\}$, $\lambda \in \Lambda$.

CLAIM: $\Sigma \cap \mathcal{L}(C_\lambda)$ is maximal consistent, and has C_λ as a set of witnesses.

Notice that this statement is equivalent to the theorem.

Let $\Sigma_\lambda = Th(M_\lambda)$ in $\mathcal{L}(M_\lambda)$. Exactly because $M_\lambda \prec M_\mu$, $\Sigma_\lambda \subseteq \Sigma_\mu$.

Let $\Sigma_\Lambda = \bigcup_{\lambda\in\Lambda} \Sigma_\lambda$. Each Σ_λ is maximal finitely satisfiable (in $\mathcal{L}(M_\lambda)$), and has C_λ as a set of witnesses. So Σ_Λ is maximal finitely satisfiable (in $\mathcal{L}(M)$) and has C as a set of witnesses.

Construct a model $\mathfrak{N}$ from Σ_Λ as in compactness theorem. Compare $\mathfrak{N}$ to M_C. Notice that Σ and Σ_Λ agree on atomic formulas. $\mathfrak{N}$ is constructed on the set of equivalence classes of $m \equiv n \iff (\overline{m} = \overline{n}) \in \Sigma_\Lambda$, and M_C is constructed on the set of equivalence classes of $m \approx n \iff (\overline{m} = \overline{n}) \in \Sigma$. Clearly $\equiv$ and $\approx$ are the same.

Definition, e.g., of the interpretation of a binary relation symbol R:

(a) in $\mathfrak{N}$: $R([\overline{m}_1], [\overline{m}_2]) \iff R(\overline{m}_1, \overline{m}_2) \in \Sigma_\Lambda$;

(b) in M_C: $R([\overline{m}_1], [\overline{m}_2]) \iff R(\overline{m}_1, \overline{m}_2) \in \Sigma$.

It follows that $\mathfrak{N} \cong M_C$ via the map $[\overline{m}]_\equiv \mapsto [\overline{m}]_\approx$.

But $M \cong M_C$ by $m \mapsto [\overline{m}]$.

Now, by truth lemma, suppose $\Phi(v_1, \dots, v_k)$ is an $\mathcal{L}$-formula, and $m_1, \dots, m_k \in M_\lambda$. Then

$$M \models \Phi(m_1, \ldots, m_k) \Leftrightarrow \Phi(\overline{m}_1, \ldots, \overline{m}_k) \in \Sigma \; (= \Sigma_\lambda) \Leftrightarrow$$
$$\Leftrightarrow \Phi(\overline{m}_1, \ldots, \overline{m}_k) \in \Sigma_\lambda \Leftrightarrow$$
$$\Leftrightarrow M_\lambda \models \Phi(m_1, \ldots, m_k) .$$

□

(IV) OMITTING TYPES AND K-CATEGORICITY

One wants to determine the structure of Mod(Σ). Assume that Σ has a model of cardinality greater than, or equal to, $|\mathcal{L}|$. Then, by Löwenheim-Skolem, for all cardinals $\kappa \geq \lambda = |\mathcal{L}|$, $Mod_\kappa(\Sigma) = \{\mathfrak{M} : \mathfrak{M} \models \Sigma \text{ and } |\mathfrak{M}| = \kappa\}$ is not empty.

PROBLEM: How many isomorphism types in $Mod_\kappa(\Sigma)$?

There are cases where there is only one isomorphism type.

One wants methods to construct non-isomorphic models of power κ, when this is possible.

Typical limitation (R.L. Vaught, 1960): Suppose $\mathcal{L}$ countable. For all $n \in \mathbb{N}$, $n \geq 1$ and $n \neq 2$, there is a set Σ_n such that $Mod_{\aleph_0}(\Sigma_n)$ contains exactly n isomorphism types. (Denumerable models of complete theories - In Infinitistic Methods - Pergamon Press, 1961.)

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NOTE: Whereas compactness theorem is about compactness of certain spaces, the omitting types theorem is essentially equivalent to the Baire Cathegory theorem for a related space.

(IV-1) DEFINITION: Suppose $\mathfrak{M}$ is an $\mathcal{L}$-structure. Let $\langle a_1, \ldots, a_n \rangle$ be a sequence of elements of $\mathfrak{M}$. Then the type of this sequence in $\mathfrak{M}$ is $\{\Phi(v_1, \ldots, v_n) : \mathfrak{M} \models \Phi(a_1, \ldots, a_n)\} = Type_{\mathfrak{M}}(\langle a_1, \ldots, a_n \rangle)$.

NOTE: ~~Type~~ $Type_{\mathfrak{M}}(a_1, \ldots, a_n)$ is consistent with $Th(\mathfrak{M})$ and is maximally so.

(IV-2) LEMMA: A set Γ of $\mathcal{L}$-sentences is a complete theory iff is maximal finitely satisfiable.

One aims for a similar characterization of an n-type over T, where T is a complete $\mathcal{L}$-theory.

(IV-3) DEFINITION: Let T be a complete $\mathcal{L}$-theory and Σ be a set of $\mathcal{L}$-formulas in free variables $v_1, \dots, v_n$, such that $\Sigma \cup T$ is finitely satisfiable and maximally so. Then Σ is called an <u>n-type</u> over T.

11 (IV-4) LEMMA: $\Sigma(v_1, \dots, v_n)$ is an n-type (over T) iff there is some model of T and elements $a_1, \dots, a_n$ in $\mathcal{M}$ such that $\Sigma = \mathrm{Type}_{\mathcal{M}}(a_1, \dots, a_n)$.

PROOF:

If $\Sigma = \mathrm{Type}_{\mathcal{M}}(a_1, \dots, a_n)$, then $\Sigma \cup T$ is satisfiable (in $\mathcal{M}$, using a_i for v_i), and is clearly maximally so.

Conversely, suppose Σ is an n-type over T. Add constants $c_1, \dots, c_n$ to $\mathcal{L}$, and consider $\Sigma(c_1, \dots, c_n) \cup T$. This is finitely satisfiable, so satisfiable. Let $\mathcal{N}$ be a model, and $a_1, \dots, a_n$ be the interpretation of $c_1, \dots, c_n$ in $\mathcal{N}$. Then $\mathcal{N} \models T$ and $\Sigma = \mathrm{Type}_{\mathcal{N}}(a_1, \dots, a_n)$. $\square$

(IV-5) DEFINITION: Suppose $\mathcal{M} \models T$. Then one says that $\mathcal{M}$ <u>realizes the n-type Σ</u> iff there are $a_1, \dots, a_n \in \mathcal{M}$ such that $\Sigma = \mathrm{Type}_{\mathcal{M}}(a_1, \dots, a_n)$. One says that $\mathcal{M}$ <u>omits</u> Σ if $\mathcal{M}$ does not realize Σ.

Lemma (IV-4) shows that any n-type Σ is realized in some model.

PROBLEM: When can one omit Σ in some $\mathcal{M}$?

A case when Σ cannot be omitted in any $\mathcal{M}$: Σ may be <u>principal</u>, i.e. there may be a formula $\Phi(v_1, \dots, v_n)$ in Σ such that for all $\Psi(\vec{v}) \in \Sigma$, $T \vdash \forall \vec{v}\, (\Phi(\vec{v}) \rightarrow \Psi(\vec{v}))$.

12 (IV-6) LEMMA: If Σ is principal then Σ is realized in all models of T.

PROOF:

Let Σ be principal, Φ be as above. Then $T \cup \{\Phi(\vec{v})\}$ is consistent, so $T \cup \{\exists \vec{v}\, \Phi(\vec{v})\}$

is consistent.

Either $\exists\vec{v}\,\Phi(\vec{v})$ or $\neg(\exists\vec{v}\,\Phi(\vec{v}))$ is in T, by maximality. So $\exists\vec{v}\,\Phi(\vec{v}) \in T$, so every model of T satisfies $\exists\vec{v}\,\Phi(\vec{v})$.

Let $\mathcal{M} \models T$. Choose $a_1, \ldots, a_n$ such that $\mathcal{M} \models \Phi(\vec{a})$. So $a_1, \ldots, a_n$ realize Σ.

$\square$

13 (IV-7) <u>THEOREM</u> (<u>OMITTING TYPES</u>): Let $\mathcal{L}$ be <u>countable</u>, T be a complete theory, and $\Sigma(v_1, \ldots, v_n)$ be a non-principal n-type. Then there exists a countable model of T omitting Σ.

<u>PROOF</u>:

<u>Strategy</u>: One builds $\mathcal{M}$ as in proof of compactness by adding countably many constants $C = \{c_n\}$ to $\mathcal{L}$ and forming a set Γ of $\mathcal{L}(C)$-sentences, such that:

(i) Γ is maximal consistent, with C as a set of witnesses;

(ii) $T \subseteq \Gamma$.

$\mathcal{M}$ will be constructed from Γ as always, based on equivalence classes of constants, and $\mathcal{M} \models T$ by truth lemma.

<u>Major Requirement</u>: no tuple $\vec{a} \in M^n$ realizes Σ, where $\vec{a} = \langle a_1, \ldots, a_n \rangle = \langle [c_{i_1}], \ldots, [c_{i_n}] \rangle$. (There are only countably many such tuples.)

Imagine meeting <u>one</u> requirement for each tuple $\langle [c_{i_1}], \ldots, [c_{i_n}] \rangle$. One has to find a formula $\Phi(v_1, \ldots, v_n)$ such that $\mathcal{M} \models \neg\Phi([c_{i_1}], \ldots, [c_{i_n}])$. This is equivalent to $\neg\Phi(c_{i_1}, \ldots, c_{i_n}) \in \Gamma$.

<u>Now, the proof in detail</u>:

Enumerate the n-tuples $\langle c_{i_1}, \ldots, c_{i_n} \rangle$ as $\tau_0, \tau_1, \ldots$. Also enumerate the set of $\mathcal{L}(C)$-sentences as $\sigma_0, \sigma_1, \ldots$. Construct Γ as an increasing union $\bigcup_m \Gamma_m$, so that:

(i) $T \subseteq \Gamma_0$;

(ii) each Γ_m is got from T by adding finitely many $\mathcal{L}(C)$-sentences;

(iii) Γ_m is consistent;

(iv) <u>for m even</u>: Γ_m contains one of $\sigma_{m/2}$ or $\neg\sigma_{m/2}$, and a witness for $\sigma_{m/2}$ if $\sigma_{m/2}$ is existencial;

(v) <u>for m odd</u>: in this case, Γ_m

05/11/86 should ensure that $\tau_{(m-1)/2}$ should not realize Σ.

Let $m = 2k+1$: ensure that τ_k does not realize Σ:

One has Γ_{m-1} already, a finite extension of $\Gamma_0 \supseteq T$, that is, $\Gamma_{m-1} = T$ plus a finite number of $\mathcal{L}(C)$-sentences, and w.l.o.g. (= = without loss of generality), one can suppose $\Gamma_{m-1} = T$ plus a single sentence Ψ (i.e., the conjunction of them).

Let $\tau_k = \langle c_{i_1}, \dots, c_{i_n} \rangle$. Ψ involves finitely many constants. Write Ψ as $\Psi(c_{i_1}, \dots, c_{i_n}, c_{i_{n+1}}, \dots, c_{i_{n+r}})$. (One does not suggest that all, or any, of $c_{i_1}, \dots, c_{i_n}$ occur in Ψ; $c_{i_{n+1}}, \dots, c_{i_{n+r}}$ are the constants that occur in Ψ but not in τ_k.)

One wants to add to Γ_{m-1} an $\mathcal{L}(C)$-sentence θ giving Γ_m, so that Γ_m is consistent, and such that no matter how one proceeds later towards $\bigcup_\ell \Gamma_\ell$, in the final model, the n-tuple τ_k (or rather, its interpretation) will not realize Σ. This is obviously accomplished as soon as one has a formula $\Phi(v_1, \dots, v_n) \in \Sigma$ such that Γ_m plus $\neg\Phi(c_{i_1}, \dots, c_{i_n})$ is consistent.

Suppose this is not possible. Then for each $\Phi \in \Sigma$, $\Gamma_{m-1} \vdash \Phi(c_{i_1}, \dots, c_{i_n})$, i.e., $T \cup \{\Psi(c_{i_1}, \dots, c_{i_{n+r}})\} \vdash \Phi(c_{i_1}, \dots, c_{i_n})$, or $T \vdash \Psi(c_{i_1}, \dots, c_{i_{n+r}}) \rightarrow \Phi(c_{i_1}, \dots, c_{i_n})$, and since T is a set of $\mathcal{L}$-sentences, obviously $T \vdash \forall v_1, \dots, v_n [\exists v_{n+1} \dots v_{n+r}\, \Psi(v_1, \dots, v_{n+r}) \rightarrow \Phi(v_1, \dots, v_n)]$.

<u>NOTE</u>: T plus $\exists v_{n+1} \dots v_{n+r} \Psi(v_1 \dots v_{n+r})$ (call this formula $\Delta(v_1 \dots v_n)$) is consistent, since Γ_m is consistent, and since T is complete, then $T \vdash \exists v_1 \dots v_n\, \Delta(v_1, \dots, v_n)$, and for all $\Phi \in \Sigma$, $T \vdash \Delta(v_1 \dots v_n) \rightarrow \Phi(v_1, \dots, v_n)$. So Σ is principal, contrary to the assumption. $\square$

<u>NOTE</u>: Countability was crucial in this proof (used in the conjunction of formulas to obtain Δ).

<u>REMARK</u>: One can replace Σ by a countable set $\{\Sigma_0, \Sigma_1, \dots\}$ of non-principal types varying "arities". It is the same proof,

with minor changes in order to meet requirements.

REMARK: By the above, one gets countable $\mathcal{M}$ omitting Σ, but no simple Upward Löwenheim-Skolem arguments gives $\mathcal{M}$ of arbitrary large cardinal.

14 (IV-8) LEMMA: If $\mathcal{M}$ omits Σ and $\mathcal{N} \prec \mathcal{M}$, then $\mathcal{N}$ omits Σ.

15 (IV-9) LEMMA: Let $\mathcal{L}$ be countable and T be a complete $\mathcal{L}$-theory with a non-principal n-type P. Then T has at least two non-isomorphic countable models.

PROOF:

There is a countable $\mathcal{M}_0 \models T$ realizing P, by (IV-4), and there is a countable $\mathcal{M}_1 \models T$ omitting P, by (IV-7). So $\mathcal{M}_0 \not\cong \mathcal{M}_1$. $\square$

(IV-10) DEFINITION: T is said κ-categorical if any two models of T of cardinal κ are isomorphic.

(IV-11) COROLLARY: If T is $\aleph_0$-categorical then all types over T are principal.

Now one begins to prove the converse:

MAIN PROBLEM: What is the structure of countable models $\mathcal{M}$ of an arbitrary theory T such that $\mathcal{M}$ realizes only principal types?

Such a model is called atomic (because of correspondence between principal types and principal ultrafilters in $\mathbb{B}_n(T)$ — see problem sheet 3 — and these in turn correspond to atoms — see problem sheet 1).

16 (IV-12) THEOREM: Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be countable atomic models of a complete theory T in a countable $\mathcal{L}$. Then $\mathcal{M}_1 \cong \mathcal{M}_2$.

NOTE: This theorem is an ultimately generalization of Cantor's theorem of dense linear orders.

PROOF:

Enumerate $\mathfrak{M}_i$ as $m_{i,0}, m_{i,1}, \ldots$, for $i=1,2$.

Construct isomorphism $f: \mathfrak{M}_1 \cong \mathfrak{M}_2$ in ω stages (using a back-and-forth argument), such that at any finite stage, f is defined only on a finite set.

PROBLEM: How to define $f(m_{1,0})$?

STAGE 0: Consider $\mathrm{Type}_{\mathfrak{M}_1}(m_{1,0})$. This is principal, determined by $\Phi(v)$ say. Since $\mathfrak{M}_2 \models T$ and $T \vdash \exists v \Phi(v)$ (for it is true in $\mathfrak{M}_1$ and T is complete), $\mathfrak{M}_2 \models \exists v \Phi(v)$. Choose the first element $m_{2,k}$ satisfying Φ. It follows that $\mathrm{Type}_{\mathfrak{M}_1}(m_{1,0}) = \mathrm{Type}_{\mathfrak{M}_2}(m_{2,k})$. Put $f(m_{1,0}) = m_{2,k}$.

07/11/86: STAGE 1: Now one wants to ensure that $m_{2,0} \in \mathrm{Range}(f)$. (If $k=0$ in stage 0, this is done.) Suppose $k \neq 0$.

Consider $\mathrm{Type}_{\mathfrak{M}_2}(m_{2,k}, m_{2,0}) = \Sigma(v_1, v_2)$, and $\Gamma_1(v_1) = \mathrm{Type}_{\mathfrak{M}_2}(m_{2,k}) \subseteq \Sigma(v_1,v_2)$. Then $\Gamma_1(v_1) = \mathrm{Type}_{\mathfrak{M}_1}(m_{1,0}) \subseteq \mathrm{Type}_{\mathfrak{M}_1}(m_{1,0}, x) \stackrel{?}{=} \mathrm{Type}_{\mathfrak{M}_2}(m_{2,k}, m_{2,0})$ (that is, is there any $x \in \mathfrak{M}_1$ such that the last equality above holds?).

$\Sigma(v_1,v_2)$ is principal, determined by $\Psi(v_1,v_2)$ say. In particular, for every $\theta(v_1) \in \Gamma(v_1)$, $T \vdash \Psi(v_1,v_2) \to \theta(v_1)$. Thus $T \vdash (\forall v_1)[(\exists v_2)\Psi(v_1,v_2) \to \theta(v_1)]$; but $(\exists v_2)\Psi(v_1,v_2) \to \theta(v_1) \in \Gamma(v_1) = \mathrm{Type}_{\mathfrak{M}_2}(m_{2,k}) = \mathrm{Type}_{\mathfrak{M}_1}(m_{1,0})$. So $\mathfrak{M}_1 \models (\exists v_2)\Psi(m_{1,0}, v_2)$. Choose least q such that $\mathfrak{M}_1 \models \Psi(m_{1,0}, m_{1,q})$ (notice that $q \neq 0$, since $(v_1 \neq v_2) \in \Sigma(v_1,v_2)$). Then $\Psi(v_1,v_2) \in \mathrm{Type}_{\mathfrak{M}_1}(m_{1,0}, m_{1,q})$. As Ψ "generates" Σ, and Σ is maximal, $\Sigma(v_1,v_2) = \mathrm{Type}_{\mathfrak{M}_1}(m_{1,0}, m_{1,q})$. Put $f(m_{1,q}) = m_{2,0}$. Now $\mathrm{Type}_{\mathfrak{M}_1}(m_{1,0}, m_{1,q}) = \mathrm{Type}_{\mathfrak{M}_2}(f(m_{1,0}), f(m_{1,q}))$.

A similar argument, alternating back-and-forth, will ensure that $\mathrm{dom}(f) = \mathfrak{M}_1$, $\mathrm{Range}(f) = \mathfrak{M}_2$, and for all $t_1, \ldots, t_k$ in $\mathfrak{M}_1$, $\mathrm{Type}_{\mathfrak{M}_1}(t_1,\ldots,t_k) = \mathrm{Type}_{\mathfrak{M}_2}(f(t_1),\ldots,f(t_k))$. In particular, $\mathfrak{M}_1 \models \Phi(t_1,\ldots,t_k) \Longleftrightarrow \mathfrak{M}_2 \models \Phi(f(t_1),\ldots,f(t_k))$ for all $\Phi(v_1,\ldots,v_k)$.

$\square$