Dynamics of Smart Systems and Structures

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Editors

Dynamics of Smart Systems and Structures

Concepts and Applications

Springer

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Preface

Dynamics of Smart Systems and Structures presents a general overview of smart material systems and structures. This book represents an effort related to the *First School of Smart Structures in Engineering* that was held at UNESP/Ilha Solteira—SP, Brazil, November 9–13, 2014. The event was an initiative of the *Committee of Smart Materials and Structures* of the *Brazilian Society of Mechanical Sciences and Engineering* (ABCM).

The subject of Smart Materials and Structures in Brazil was related to several disconnected groups. However, in 2008, Brazilian government decided to sponsor thematic projects that would be organized to form *National Institutes of Science and Technology*. One of these projects is the *National Institute for Smart Structures in Engineering (INCT-EIE)* that represents a network that puts together a number of scientists, engineers, and students working collaboratively on a number of topics related to smart structures in cooperation with international groups. This initiative changed the scenario of Smart Structures in Brazil.

Several projects were developed since the beginning of the INCT-EIE activities. This book is one of them, being prepared thinking on the beginner students and engineers interested on Smart Material Systems and Structures. The authors hope that this introductory text may encourage, motivate, and help readers to explore this challenging interdisciplinary area.

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Composite Structures Design and Analysis

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Abstract Recent improvements in manufacturing processes and materials properties associated with excellent mechanical characteristics and low weight have become composite materials very attractive for application on different types of structures. However, even new designs are still very conservative, because the composite structure failure phenomena are very complex. This chapter shows the principal fundamentals to design and analyze composite structures. In the introduction, there is a definition and a classification of composite materials, as well as motivation, considering advantages and challenges to design by using this type of material. Thus, it is presented a methodology to design composite structures in order to overcome the main challenges related to this task. In this methodology, it is found three important analyses: micromechanical, macromechanical, and failure analyses. In order to perform micromechanical analysis, it is necessary to know more about matrix, reinforcements, and interfaces. For example, in this chapter, it is addressed only polymeric matrix and long fibers as reinforcements, which are combined to create an orthotropic ply. Then, different plies can stack with fibers oriented in different directions, creating an anisotropic or orthotropic laminate. The material properties of the ply can be obtained by Rule of Mixture or via mechanical testing. Hence, it is commented some difficulties to carry out experiments on composite materials and how is complicated to obtain allowable values for laminates. Based on the material properties, it is possible to calculate strain in the laminate, as well as strain and stress distribution in each ply. To perform the macromechanical analysis, it is possible to use Classical Laminate Theory (CLT). Thus, it is shown all hypothesis adopted for that theory and the implications generated by these ones. Finally, based on the actuating stress or strain values in each ply and allowable values of the used composite material, it is calculated the margin of safety for the plies by applying a failure criterion. In fact, for laminate structures, failure phenomena include intralaminar damages and interlaminar failures (delaminations), which are very complicated to be predicted via any failure

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theory. Therefore, even nowadays, many researchers have developed different failure theories to improve the design and analysis of composite structures.

Keywords Composite materials • Composite structures • Composite design • Composite analysis • Design methodology

1 Introduction

The usage of composite materials is a reality nowadays, mainly in the aeronautical and aerospace engineering. During several years, it has been observed different designs, which were developed considering high performance provided by this type of material, such as F-111, Vought A-7, F-18, F-22, Lockheed L-1011, Rutan Voyager, Boeing 777, Airbus 380, Boeing 787, and others. A composite can be defined as a multiphase material, which has properties better than if each phase were used alone (Callister 1985).

According to this synergistic effect in composite materials, the engineers have tried to design very carefully the combination of the phases in order to obtain materials with very high performance. The phases, which form the composite material, can be classified as matrix, reinforcements, and interface. The matrix has the function to maintain the reinforcements together, transmitting the loadings applied on the structure by the interface. Then the reinforcements have the function to support these loadings. Due to the different types of composite materials, Callister (1985) classified them as composite reinforced by particles; composite reinforced by fibers; and structural composites. In this chapter, it will be addressed the laminate composite materials, which has polymer matrix and long fibers as reinforcements stacked in plies. Each ply has fibers in one specific direction and the stacked plies generate a composite structure as shown by Fig. 1a.

The natural anisotropy related to the laminate composite materials provides a unique way to design the material properties with the geometric characteristics in order to reach the performance required by the project. The combination of high

Fig. 1 Composite material: (**a**) fuselage made of laminate composite; (**b**) damage and failure in laminate composite materials

strength and stiffness, as well as the low volumetric mass density, become the composite materials very strategic for structural applications, mainly in aeronautical and aerospace designs. Regarding the strength and the stiffness of the structure, it is possible to design both characteristics, considering the project requirements. In other words, the material can be developed in function of the loadings, which actuate in the structure. In fact, the stiffness and strength can be improved without increasing weight of the structure. Thus, for automobiles and airplanes, the performance of the product can be improved, reducing the fuel usage. In addition, the ratio between weight of green material and weight of the final product is very low for composites (10.1.2–10.1.3) compared to metals (15–25) (Jones 1999). This shows that manufacturing processes for composite structures are more efficient than manufacturing processes for metals.

However, the anisotropy and heterogeneity in the composite structures could be seen as a positive or a negative aspect. By one side, it is feasible not only to select the materials of the phases, but also to select the orientation of the fibers in each ply. By the other side, it is very complicated to predict the failure modes in the structure (Fig. 1b). This challenge is related directly to the reliability of the structure and this is more critical for products, which suffer fatigue or damage by impact loadings. Thus, it is necessary to apply high safety factors during the design process, which reduce the potentialities of composite materials and increase the cost of the final product (Tita 2003). Therefore, this scenario motivates to understand better how to design and to analyze with more accuracy composite structures.

1.1 Composite Materials: Definition and Classification

As commented earlier, a composite can be defined as a multiphase material, which has properties better than if each phase were used alone (Callister 1985). And, the phases, which form the composite material, can be classified as matrix, reinforcements, and interface. According to Vinson and Sierakowski (1987), the laminate AU1 composite can be addressed by two different analyses: micromechanics and macromechanics approaches (Fig. 2). AU2

In the micromechanics approach, it is considered each phase in the analysis. Although the phases are frequently heterogeneous and non-isotropic, it is normally assumed the hypotheses of isotropy and homogeneity. This approach can be used to determine the elastic properties of the ply or to estimate the local damage in each phase when the ply is loaded.

In the macromechanics approach, it is considered that each ply is homogenous, and the orientation of the fibers in the plies is very important in the analysis, as well. In addition, the plies are frequently non-isotropic, so they are assumed to be orthotropic. This approach can be used to predict the stiffness of the laminate, as well as its mechanical behavior when the laminate structure is loaded.

Nowadays, many researchers have combined both approaches in order to analyze the composite structures, and this new approach is called multi-scale analysis.

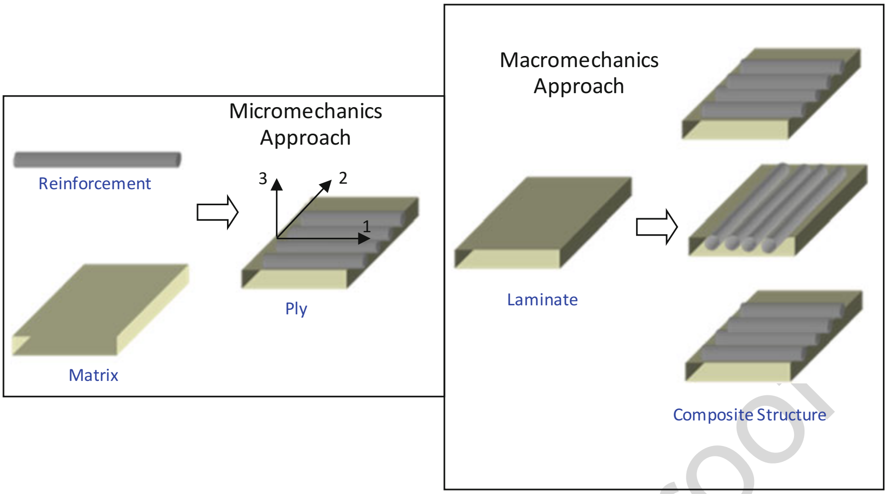

Fig. 2 Micro and macromechanics approaches (Vinson and Sierakowski 1986)

1.2 *Motivation: Advantages and Challenges*

For a long time, the man has combined different materials in order to obtain other materials. For example, in 4000 BC, Sumerians added straw in the mud in order to built better bricks. Although the benefits of composite materials are known for a long time, only recently, there was the development of manufacturing processes, which produce structures with high quality and high structural efficiency.

The structural efficiency is associated directly to the material used in the manufacturing process. This parameter is high when the relation between strength and stiffness per density is high and vice-versa. According to the literature, composite materials with 70 % of epoxy volume fraction and 30 % of carbon fiber volume fraction, or 40 % of epoxy volume fraction and 60 % of glass fiber volume fraction show stiffness close to aluminum, which is more density than both composite materials. In the same way, a composite with 40 % of epoxy volume fraction and 60 % of carbon fiber volume fraction shows stiffness close to steel (Engineered Materials Handbook 1987) (Fig. 3).

Beyond high specific strength (strength/density), composite materials show good performance under dynamic loadings. For example, in some products, it is necessary to avoid damage caused by vibrations. Thus, the plies can be stacked in order to obtain a laminate with natural frequencies different to the excitation frequencies (Tita et al. 2001). In the last years, the composite materials are not only used to guarantee high structural efficiency, but also the safety of passenger under impact loadings. Thus, the laminate is designed in order to absorb the maximum impact energy, controlling the collapse of the structure and reducing the accelerations after impact.

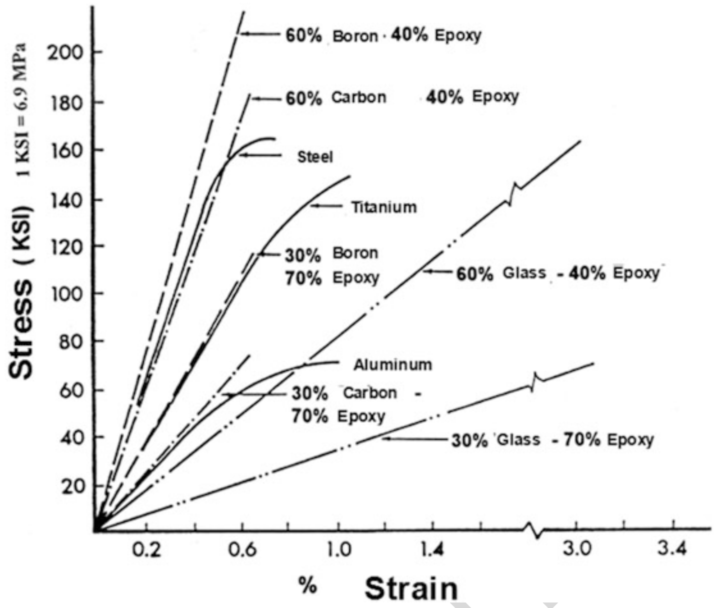

Fig. 3 Stress–strain curves: metals vs. composite materials (Adapted from Magagnin Filho 1996)

As commented earlier, the anisotropy related to the laminate composite materials provides a unique way to design the material properties with the geometric characteristics in order to reach the performance required. However, this inherent anisotropy and heterogeneity of the composite materials promote complex failure modes in the structures, which are very complicated to predict. Then, in the next section, it is shown a methodology to design composite structures in order to help engineers to overcome this challenge.

1.3 Methodology to Design Composite Structures

Figure 4 shows a procedure proposal to design laminate composite structures. It is verified that the procedure starts with the selection of the type of fibers and polymer matrix. Normally, the manufacturers of the fibers and the polymer provide the data sheet for each material. Then, by using the Rule of Mixture, which is based on *Micromechanics Analysis*, mechanical properties of each ply can be evaluated. However, it is recommended to perform experimental tests for determining not only the elastic properties of the plies, but also the allowable values (strength and strain limits) and the damage/failure modes of the composite material. In fact, the mechanical tests are very important, because the mechanical behavior of the real ply, which was manufactured by using specific values for process parameters (pressure, temperature, and time), can be investigated in details.

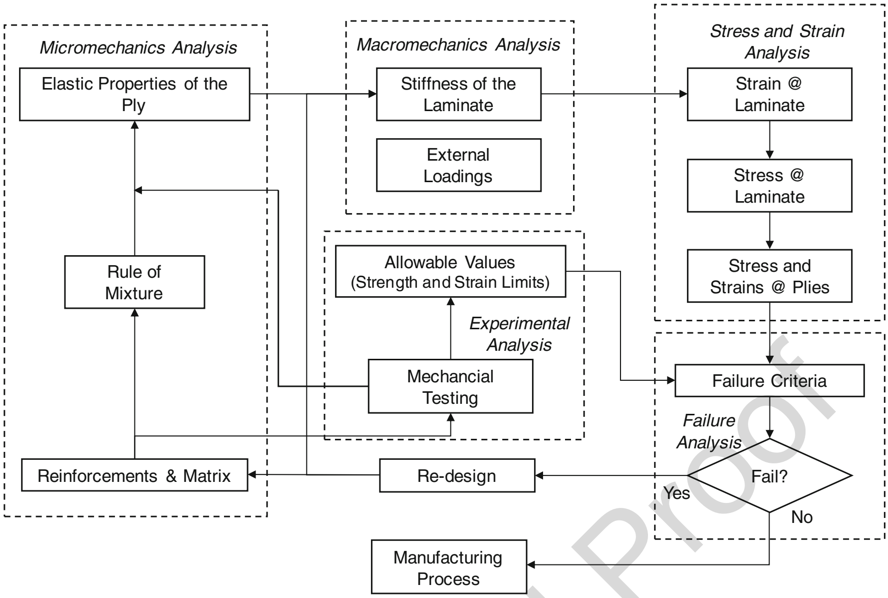

Fig. 4 Procedure proposal to design and analyze composite structures

Based on the elastic properties of each ply, it can be calculated the stiffness of the laminate via Classical Laminate Theory (CLT), for example. By using the stiffness and the external loadings applied in the laminate, it is calculated the strain components and curvatures for the Global Coordinate System for the laminate. This calculus can be named as *Macromechanics Analysis*, and based on the constitutive relations, the stress components for each ply for the Global Coordinate System can be determined. By using the transformation of coordinate systems, it is calculated not only the strain components, but also the stress components for the Local Coordinate System.

The next step in the procedure consists on carrying out *Failure Analysis*. Hence, the values of the strain or stress components for the Local Coordinate System and the allowable values determined via mechanical testing are used in the failure criterion, which is selected considering the mechanical behavior of the composite material shown during the tests. In case of failure, it is necessary to redesign the composite structure. Thus, there many options to do this, such as changing the stacking sequence of the plies; changing the fibers and/or the polymer matrix; and increasing the fiber volume fraction. Finally, if the composite structure does not fail, then it can be manufactured.

2 Micromechanical Analysis and Testing

Micromechanical analysis can be used for evaluating the mechanical properties for "one single ply" (stacked plies with the same fiber orientations), which is formed by the reinforcements (fibers), matrix (polymeric resin), and interface fiber-matrix.

2.1 Matrix, Reinforcements, and Interfaces

The matrix is the first phase in the composition of the composite materials. One of the most important functions of the matrix is to join the reinforcements. This guarantees the adequate position and orientation of the fibers such as the loads in the structure can be transferred to the reinforcements. Moreover, the matrix protects the fibers against environment effects and damages caused by hand contacts. In some cases, greater values of flexibility and damping can be obtained due to the polymeric resin. Then, this is good for attenuation of mechanical vibrations amplitudes.

The reinforcements are the second phase in the composition of the composite materials. They have an important mission, which consists on supporting the loads transferred by the matrix. In the case of long fibers, it is very important the orientation of the fibers in relation to applied design loadings. The final mechanical properties of the ply strongly depend on the fiber volume fraction and the polymer matrix processing, i.e., temperature, time, and pressures used during the manufacturing process of the composite material. Besides, it must consider the type of the fibers such as continuous (long) or discontinuous (short) and oriented or random.

The interface fiber-matrix is the third phase in the composition of the composite material. This phase is produced during the composite material processing and it is very important, because it quantifies the degree of interaction between reinforcements and matrix. Thus, in order to have a satisfactory performance by the composite material, it is necessary that there is a strong adhesion between fibers and matrix. According to Callister (1994), it is essential to have adhesive forces in the interface fiber-matrix, because the strength of the composite depends on these forces, as well.

2.1.1 Polymeric Matrix

Physics and chemical properties of the polymers influence a lot on the properties of the composite materials. For example, the maximum temperature in service of the composite material depends on the polymer used as matrix. Therefore, variations in the chemical formulations can affect the performance of the final composite

Table 1 Comparison between properties of thermoset and thermoplastic polymers

Property	Thermosetting polymer	Thermoplastic polymer
Young's modulus (GPa)	1.3–6.0	1.0–4.8
Tensile strength value (GPa)	0.02–0.18	0.04–0.19
Maximum temperature in service (°C)	50–450	25–230

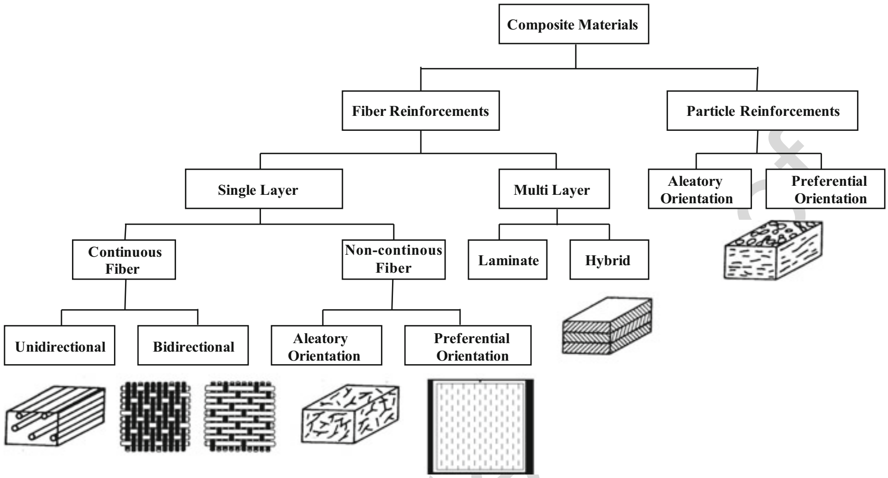

Fig. 5 Types of reinforcements

material. It is important to be careful to keep polymers, avoiding, for example, exposition to UV light.

In general, polymers can be classified as thermosetting or thermoplastic. In fact, one of the most important differences between both polymers consists on showing different behavior under heating. Thermoplastic polymers, such as PE, PP, and nylon, can suffer fusion (physic process) under heating, and the composite structure can be molded and solidified in a required geometry. Thermosetting polymers, such as epoxy and phenol resins, suffer cure (chemical process), creating cross-link between the polymer chains. Table 1 shows a comparison between properties of thermosetting and thermoplastic polymers.

Nowadays, thermosetting polymers are often applied on composite structures. However, due to reduced time to manufacture, the usage of thermoplastic polymers has been increased.

2.1.2 Reinforcements

Figure 5 shows different forms that can be used for reinforcements in the composite materials. In general, it is verified two relevant categories: fibers and particles. However, as commented earlier, this chapter is focused on the unidirectional (ply) and multidirectional (laminate) composite material.

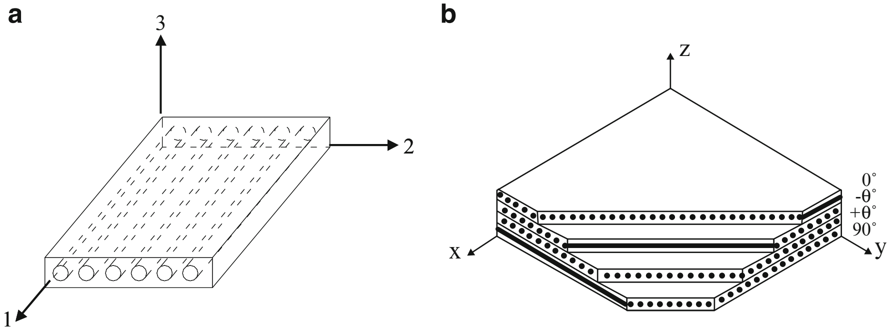

Fig. 6 (**a**) Unidirectional fibers: orthotropic material (ply); (**b**) multidirectional fibers: anisotropic material (laminate)

Table 2 Mechanical properties of fibers

Fiber	Density [10^6 g/m^3]	Young's modulus [GPa]	Tensile strength [MPa]
E-glass	2.54	70	2200
Kevlar 49	1.45	130	2900
SiC	2.60	250	2200
Alumina	3.90	380	1400
Boron	2.65	420	3500
Carbon	1.86	380	2700

In Fig. 6a, the unidirectional arrangement creates 3 (three) planes of symmetry, which are orthogonal each other (planes 1-2, 1-3, and 2-3). Hence, in this case (for the ply), it is assumed to have an orthotropic material. By other side, in Fig. 6b, it is observed multidirectional arrangement, which does not create any plane of symmetry. Thus, in this case (for the laminate), it is assumed to have an anisotropic material in the most of cases.

Table 2 shows some typical data about fibers, which can be found in the literature and data sheet of fiber manufacturers.

2.2 Rule of Mixture

The mechanical properties of the composite materials strongly depend on the properties and proportions of the 3 (three) phases (fiber, matrix, and interface) as well as the conditions of the manufacturing process (temperature, pressure, and time). The principal objective of the Rule of Mixture is the determination of the mechanical or thermal properties of the composite material by using micromechanical analysis. Indeed, this is the simplest analytical approach to homogenize a ply, which is formed by the 3 (three) phases as shown by Fig. 7a.

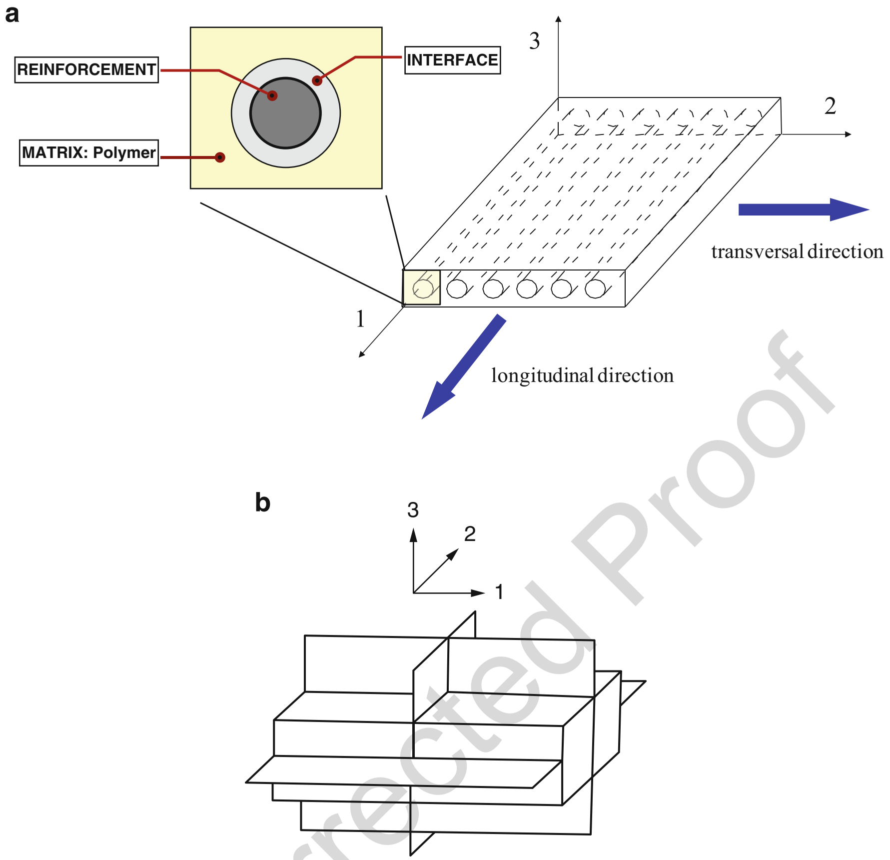

Fig. 7 (**a**) Ply: longitudinal and transversal directions; (**b**) orthotropy planes

And, this homogenized ply is assumed to be an orthotropic material with 3 (three) planes of symmetry as shown by Fig. 7b.

As the ply is assumed to be an orthotropic material, then it is necessary to determine 9 (nine) elastic constants:

- E_{11} = Young's modulus in the longitudinal direction
- E_{22} = Young's modulus in the transversal direction (in-plane of the ply)
- E_{33} = Young's modulus in the transversal direction (out-of-plane of the ply)
- G_{12} = shear modulus in plane 1-2
- G_{13} = shear modulus in plane 1-3
- G_{23} = shear modulus in plane 2-3
- ν_{12} = Poisson's ratio in plane 1-2
- ν_{13} = Poisson's ratio in plane 1-3
- ν_{23} = Poisson's ratio in plane 2-3

However, the orthotropic unidirectional ply is also transversely isotropic in the plane 2-3, so: $E_{22} = E_{33}$; $G_{12} = G_{13}$; and $\nu_{12} = \nu_{13}$. Thus, now, it is necessary to determine 6 (six) elastic constants.

The elastic properties obtained via Rule of Mixture are calculated in function of the fiber and matrix properties as well as their respective volume fractions and considering following hypotheses:

- The response of ply is linear elastic and there are not residual and thermal internal stresses.
- Fibers are uniform, homogenous, same diameter, continuous, parallels, and regularly spaced.
- The matrix is homogenous, isotropic, showing linear elastic response.
- There is a perfect interface fiber-matrix and there are not voids in the material.
- The interface is infinitely fine, being disregard in the calculus.

Considering the volume of the composite V_c and mass of the composite M_c with fiber volume V_f and fiber mass M_f, matrix volume V_m and matrix mass M_m, and voids volume V_v, it is written:

$$M_e = M_f + M_m \tag{1}$$

$$V_c = V_f + V_m + V_v \tag{2}$$

Dividing Eqs. (1) and (2) by M_c and V_c, respectively:

$$1 = \frac{M_f}{M_c} + \frac{M_m}{M_c} \tag{3}$$

$$1 = \frac{V_f}{V_c} + \frac{V_m}{V_c} + \frac{V_v}{V_c} \tag{4}$$

The mass and volume fraction can be defined as:

$$m_f = \frac{M_f}{M_c}; \quad m_m = \frac{M_m}{M_c} \tag{5}$$

$$v_f = \frac{V_f}{V_c}; \quad v_m = \frac{V_m}{V_c}; \quad v_v = \frac{V_v}{V_c} \tag{6}$$

Thus, rewriting (3) and (4):

$$\begin{aligned} m_f + m_m = 1 \quad \text{or} \quad \frac{\sum M_i}{M_c} = \sum m_i = 1 \\ v_f + v_m + v_v = 1 \quad \text{or} \quad \frac{\sum V_i}{V_c} = \sum v_i = 1 \end{aligned} \tag{7}$$

In order to calculate the mass and volume fractions, it is necessary to determine the composite density ρ_c. Based in the Eq. (1) or in the Eq. (2), it is written:

$$\begin{aligned}\rho_c &= \frac{M_c}{V_c} = \frac{1}{\frac{V_c}{M_c}} = \frac{1}{\frac{V_f}{M_c} + \frac{V_m}{M_c} + \frac{V_v}{M_c}} \\ \rho_c &= \frac{1}{\frac{M_f}{\rho_f M_c} + \frac{M_m}{\rho_m M_c} + \frac{v_v}{\rho_c V_c}} = \frac{1}{\frac{m_f}{\rho_f} + \frac{m_m}{\rho_m} + \frac{v_v}{\rho_c}}\end{aligned} \tag{8}$$

or:

$$\begin{aligned}\rho_c &= \frac{M_c}{V_c} = \frac{M_f + M_m}{V_c} = \frac{\rho_f V_f + \rho_m V_m}{V_c} \\ \rho_c &= \rho_f v_f + \rho_m v_m\end{aligned} \tag{9}$$

The voids volume fraction v_v is given by:

$$v_v = 1 - (v_f + v_m) \tag{10}$$

or, by using Eq. (8), it is obtained:

$$v_v = 1 - \left(\frac{m_f}{\rho_f} + \frac{m_m}{\rho_m}\right)\rho_{c\,(\text{experimental})} \tag{11}$$

Besides, the theoretical density is calculated via:

$$\rho_{c\,(\text{theoretical})} = \frac{1}{\frac{m_f}{\rho_f} + \frac{m_m}{\rho_m}} \tag{12}$$

Therefore, Eq. (12) can be written as:

$$v_v = 1 - \frac{\rho_{c\,(\text{experimental})}}{\rho_{c\,(\text{theoretical})}} \tag{13}$$

After determining the matrix and fiber volume fractions, it is necessary to have the matrix and fiber properties, such as Young's moduli of the matrix (E_m) and fiber (E_f), Poisson's ratios of the matrix (ν_m) and the fiber (ν_f). Frequently, these properties are provided by the manufacturers of the polymers and fibers. Otherwise, it should be carried out experimental tests in order to obtain these data.

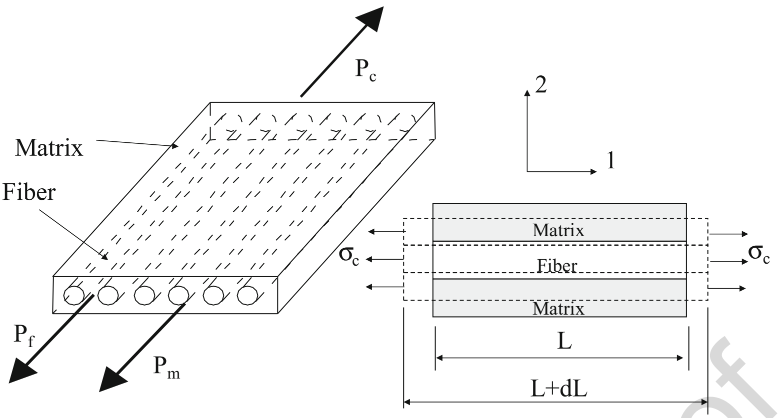

Fig. 8 Ply loaded in the longitudinal direction

2.2.1 Longitudinal Young's Modulus

Considering a loading P_c applied in the direction of the fiber, the strains in the fibers, matrix, and composite are assumed to be equals (Fig. 8):

$$\varepsilon_\mathrm{c} = \varepsilon_\mathrm{f} = \varepsilon_\mathrm{m} \tag{14}$$

Considering elastic response, stresses can be calculated by Hooke's Law:

$$\sigma_\mathrm{f} = E_\mathrm{f}\varepsilon_\mathrm{f} \quad \text{and} \quad \sigma_\mathrm{m} = E_\mathrm{m}\varepsilon_\mathrm{m} \tag{15}$$

Stresses σ_f and σ_m actuate on the A_f and A_m, respectively. Based on Fig. 8, the loading P_c can be calculated as follows:

$$P_\mathrm{c} = P_\mathrm{f} + P_\mathrm{m} \tag{16}$$

Moreover:

$$P_\mathrm{f} = \sigma_\mathrm{f}A_\mathrm{f} = E_\mathrm{f}\varepsilon_\mathrm{f}A_\mathrm{f} \quad \text{and} \quad P_\mathrm{m} = \sigma_\mathrm{m}A_\mathrm{m} = E_\mathrm{m}\varepsilon_\mathrm{m}A_\mathrm{m} \tag{17}$$

Applying (17) into (16):

$$P_\mathrm{c} = \sigma_\mathrm{c}A_\mathrm{c} = \sigma_\mathrm{f}A_\mathrm{f} + \sigma_\mathrm{m}A_\mathrm{m} \quad \text{or} \quad \sigma_\mathrm{c} = \sigma_\mathrm{f}\frac{A_\mathrm{f}}{A_\mathrm{c}} + \sigma_\mathrm{m}\frac{A_\mathrm{m}}{A_\mathrm{c}} \tag{18}$$

The volume of the fiber can be calculated as follows:

$$V_f = A_f L_f \tag{19}$$

By using the same way, it is calculated matrix and composite volume. Thus, based on Fig. 8:

$$L_f = L_m = L_c \tag{20}$$

Replacing (19) into (18) and considering (20):

$$\sigma_{\mathrm{c}} = \sigma_{\mathrm{f}} v_{\mathrm{f}} + \sigma_{\mathrm{m}} v_{\mathrm{m}} \tag{21}$$

Since the ply has an elastic behavior, then $\sigma_{\mathrm{c}} = E_{\mathrm{c}}\varepsilon_{\mathrm{c}}$ and $\varepsilon_{\mathrm{c}} = \varepsilon_{\mathrm{f}} = \varepsilon_{\mathrm{m}}$, so:

$$\begin{gathered} \sigma_{\mathrm{c}} = E_{\mathrm{c}}\varepsilon_{\mathrm{c}} = E_{\mathrm{f}}\varepsilon_{\mathrm{f}} v_{\mathrm{f}} + E_{\mathrm{m}}\varepsilon_{\mathrm{m}} v_{\mathrm{m}} \\ E_{\mathrm{c}} = E_{\mathrm{f}} v_{\mathrm{f}} + E_{\mathrm{m}} v_{\mathrm{m}} \quad \text{or} \quad E_{11} = E_{\mathrm{f}} v_{\mathrm{f}} + E_{\mathrm{m}} v_{\mathrm{m}} \end{gathered} \tag{22}$$

Finally, Eqs. (21) and (22) can be rewritten:

$$\sigma_{11} = \sum_{i=1}^{n} \sigma_i v_i \quad \text{and} \quad E_{11} = \sum_{i=1}^{n} E_i v_i \tag{23}$$

It is important to notice that the Rule of Mixture calculates de elastic properties of the ply by using the weighted average of the volume fractions for n constituents of the composite material.

2.2.2 Transversal Young Modulus

Considering the hypotheses used by Rule of Mixture, if a transversal loading P_{c} is applied in the transversal direction, then the actuating stresses in the fibers, matrix, and composite are assumed to be the same in this direction (Fig. 9):

$$\sigma_{\mathrm{c}} = \sigma_{\mathrm{f}} = \sigma_{\mathrm{m}} \tag{24}$$

Thus, the transversal elongation in the ply δ_{c} is given by the sum of elongations of the fibers δ_{f} and the matrix δ_{m}:

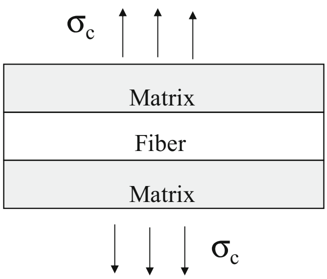

Fig. 9 Ply loaded in the transversal direction

$$\delta_c = \delta_f + \delta_m \tag{25}$$

As $\varepsilon = \delta/t$, where t is thickness of the phase or the composite, then:

$$\varepsilon_c t_c = \varepsilon_f t_f + \varepsilon_m t_m \tag{26}$$

Since the matrix and fibers volume fraction can be written as:

$$v_f = \frac{t_f}{t_c} \quad \text{and} \quad v_m = \frac{t_m}{t_c} \tag{27}$$

Replacing (27) into (26):

$$\varepsilon_c = \varepsilon_f v_f + \varepsilon_m v_m \tag{28}$$

As the actuating transversal stresses in the fibers are equal in the matrix, then:

$$\varepsilon_f = \frac{\sigma_c}{E_f} \quad \text{and} \quad \varepsilon_m = \frac{\sigma_c}{E_m} \tag{29}$$

Replacing (29) into (28):

$$\frac{1}{E_c} = \frac{1}{E_f} v_f + \frac{1}{E_m} v_m \tag{30}$$

Finally, Eqs. (28) and (30) can be rewritten:

$$\varepsilon_{22} = \sum_{i=1}^{n} \varepsilon_i v_i \quad \text{and} \quad E_{22} = \frac{1}{\sum_{i=1}^{n} \frac{1}{E_i} v_i} \tag{31}$$

Due to the transversal isotropy of the ply, the Transversal Young Modulus in the ply plane (E_{22}) is equal to the Transversal Young Modulus out of the ply plane (E_{33}).

2.2.3 Shear Modulus

For the determination of the shear modulus of the ply, it is assumed that the shear strains are linear and the actuating stresses are the same in the fibers and matrix (Fig. 10).

The total displacement of the ply u_c is calculated by the sum of the displacements of the fibers u_f and the matrix u_m, thus:

$$u_c = u_f + u_m \quad \text{or} \quad u_c = t_f \gamma_f + t_m \gamma_m \tag{32}$$

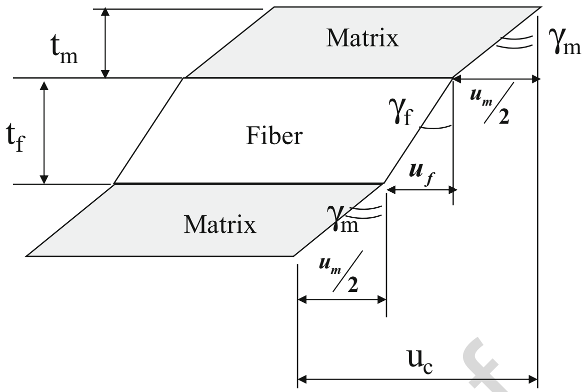

Fig. 10 Ply deformed due to shear loading

where γ_{f} is the angle for fibers and γ_{m} is the angle for the matrix. Applying (27) into (32):

$$u_{\mathrm{c}} = v_{\mathrm{f}} t_{\mathrm{c}} \gamma_{\mathrm{f}} + v_{\mathrm{m}} t_{\mathrm{c}} \gamma_{\mathrm{m}} \tag{33}$$

γ_{12} for the ply can be calculated as follows:

$$\gamma_{12} = \frac{u_{\mathrm{c}}}{t_{\mathrm{c}}} \tag{34}$$

Applying (34) into (33):

$$\gamma_{12} = v_{\mathrm{f}} \gamma_{\mathrm{f}} + v_{\mathrm{m}} \gamma_{\mathrm{m}} \tag{35}$$

Based on the linear hypotheses, then:

$$\gamma_{\mathrm{f}} = \frac{\tau_{\mathrm{f}}}{G_{\mathrm{f}}}, \quad \gamma_{\mathrm{m}} = \frac{\tau_{\mathrm{m}}}{G_{\mathrm{m}}} \quad \text{and} \quad \gamma_{12} = \frac{\tau_{12}}{G_{12}} \tag{36}$$

Considering that the actuating shear stresses in the fibers, matrix, and composite are equal and replacing Eq. (36) into Eq. (35), it is calculated the shear modulus of the ply in the plane 1-2:

$$\frac{1}{G_{12}} = v_{\mathrm{f}} \frac{1}{G_{\mathrm{f}}} + v_{\mathrm{m}} \frac{1}{G_{\mathrm{m}}} = \sum_{i=1}^{n} \frac{v_i}{G_i} \tag{37}$$

Due to the transversal isotropy of the ply, it is assumed that G_{12} is equal to G_{13} (shear modulus of the ply in the plane 1-3). However, G_{23} (shear modulus of the ply in the plane 2-3) is much more complicated to calculate, and, normally, it is required experimental tests.

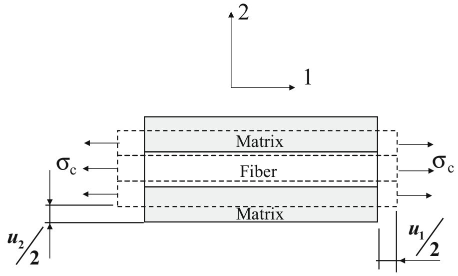

Fig. 11 Poisson's effect in the ply

2.2.4 Poisson's Coefficient

If a normal stress σ_c is applied in the longitudinal direction of the fibers, there will be a contraction of the ply in the transversal direction (Fig. 11), which is calculated by:

$$u_2^c = u_2^f + u_2^m \tag{38}$$

Contractions of the fibers and matrix can be calculated via Poisson's ratios:

$$\begin{aligned} \nu_m &= -\frac{\varepsilon_2^m}{\varepsilon_1^m} = -\frac{u_2^m/t_m}{\varepsilon_1^m} \quad \text{or} \quad u_2^m = -\nu_m \varepsilon_1^m t_m \\ \nu_f &= -\frac{\varepsilon_2^f}{\varepsilon_1^f} = -\frac{u_2^f/t_f}{\varepsilon_1^f} \quad \text{or} \quad u_2^f = -\nu_f \varepsilon_1^f t_f \end{aligned} \tag{39}$$

where ν_m and ν_f are Poisson's ratio for fibers and matrix, respectively. And, t_f and t_m are thickness of the fibers and matrix, respectively.

Replacing (39) into (38):

$$u_2^c = -\nu_m u_1^m - \nu_f u_1^f = -\left(\nu_m \varepsilon_1^m t_m + \nu_f \varepsilon_1^f t_f\right) \tag{40}$$

Considering that the strains in the fibers, matrix, and composite are equal, then:

$$\varepsilon_1^m = \varepsilon_1^f = \varepsilon_1^c = \varepsilon_{11} \tag{41}$$

Applying (41) into (40) and operating t_c (thickness of the ply) in the both sides of the equation:

$$t_c u_2^c = -(\nu_m t_m + \nu_f t_f) t_c \varepsilon_{11} \tag{42}$$

or:

$$u_2^c = -\left(\nu_m \frac{t_m}{t_c} + \nu_f \frac{t_f}{t_c}\right) t_c \varepsilon_{11} \tag{43}$$

Since the fiber and matrix volume fraction can be written as:

$$v_{\mathrm{f}} = \frac{t_{\mathrm{f}}}{t_{\mathrm{c}}} \quad \text{and} \quad v_{\mathrm{m}} = \frac{t_{\mathrm{m}}}{t_{\mathrm{c}}} \tag{44}$$

Thus, Eq. (43) can be rewritten:

$$\frac{u_2^{\mathrm{c}}}{t_{\mathrm{c}}} = -(\nu_{\mathrm{m}} v_{\mathrm{m}} + \nu_{\mathrm{f}} v_{\mathrm{f}})\varepsilon_{11} = \varepsilon_{22} \tag{45}$$

The Poisson's ratio ν_{12} calculated in the ply plane (plane 1-2) is given by:

$$\nu_{12} = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = \nu_{\mathrm{m}} v_{\mathrm{m}} + \nu_f v_f = \sum_{i=1}^{n} \nu_i v_i \tag{46}$$

Due to the transversal isotropy of the ply, it is assumed that ν_{12} is equal to ν_{13} (Poisson's ratio of the ply in the plane 1-3). However, ν_{23} (Poisson's ratio of the ply in the plane 2-3) is much more complicated to calculate, and, normally, it is required experimental tests.

2.3 *Mechanical Testing*

Regarding the hypothesis used in the Rule of Mixture, sometimes, the values of mechanical properties obtained by this approach are very different when compared to the experimental values. This occurs because different effects influence on the final properties of composite materials. For example, parameters of material processing (time, pressure and temperature) are very important, because, a composite plate made of a kind of fiber, matrix, and volume fractions can show totally different properties than other composite plate with the same fiber, matrix, and volume fractions of phases manufactured on different conditions. Therefore, it is almost impossible to avoid experimental tests for determination of elastic properties, strength and strain limit values of composite materials.

For an isotropic material, a tensile test in one direction can provide: Young Modulus, Poisson's ratio, strength values, and strain limits. However, for orthotropic materials, it is necessary 6 (six) experimental tests as shown by Table 3.

Moreover, the experimental tests provide the stress–strain curves, which helps to identify different mechanisms in the ply, such as micro-damages or macro-failures (delamination). This will be very important to select a failure criterion for designing a composite structure. However, to carry out experimental tests on composite materials is a hard task, because there are many particularities:

1. The experimental tests are based on the concepts of the basic mechanic theory, which are applied for isotropic, elastic, homogeneous materials. However, composite materials are anisotropic, heterogeneous, and inelastic. Thus, the application of these concepts is not direct.

Table 3 Experimental testing for orthotropic materials

Mechanical testing	Elastic properties	Strength value	Strain limit
(1) *Tension 0°*: tension in the longitudinal direction	E_{11}; ν_{12} $(=\nu_{13})$	X_T	X'_T
(2) *Tension 90°*: tension in the normal direction.	E_{22} $(=E_{33})$	Y_T	Y'_T
(3) *Compression 0°*: compression in the longitudinal direction	–	X_C	X'_C
(4) *Compression 90°*: compression in the normal direction	–	Y_C	Y'_C
(5) *Shear in plane 1-2*: shear loading in the plane 1-2	G_{12} $(=G_{13})$	S_{12}	S'_{12}
(6) *Shear in plane 2-3*: shear loading in the plane 2-3	G_{23}	–	–

2. During the tests, many difficulties can take place, such as:
   - Influence of end-effects, which produces regions with stress concentration close to the edges of the specimen
   - How to apply acceptable load levels without creating premature fails in the material
   - How to determine the correct dimensions of the specimen (mainly thickness), regarding the heterogeneity
3. Problems caused by the anisotropy:
   - Increase the problem related to the end-effects
   - Promote premature fails in regions close to the clamps
   - Promote premature delaminations close to the edge of the specimen
4. Experimental tests of composite materials are expensive and take long time, mainly the manufacturing of the specimens.
5. For some cases, traditional standards (ASTM, ISO, DIN, etc.) work, but for others, these standards are completely inappropriate.

In fact, in the literature, different standards to perform experimental tests in composite materials can be found. However, it is better to use these standards as a guide to carry out the tests, because, for some composite materials, it is necessary to change some parameters specified in the standard, such as the dimensions and/or test speed.

3 Macromechanical Analysis

In the macromechanical analysis, it is considered not only the ply properties, but also the stacking sequence of the laminate.

3.1 Classical Laminate Theory

First, it is important to assume a code to identify the stacking sequence used in the laminate. In this chapter, it is used the SLC (*Standard Laminate Code*), which requires:

- Orientation of each ply, considering the global coordinate system ($x-y-z$).
- Number of the plies for a given orientation.
- Stacking sequence of the plies to obtain the laminate.

For example, a laminate with orientation angles for fibers equal 0°, 90°, 90°, and 0° can be represented by different ways: [0/90/90/0]; $[0/90_2/0]$; $[0/90]_s$; $[0/90/90/0]_T$. The subscripts of the angles specify the number of the plies with fibers oriented in that direction. The subscript S indicates symmetry of the laminate, and T shows that the laminate has the total number of the plies used to manufacture the structure.

The composite structure [0/90/90/0] is a symmetric laminate, because the plane, which split the thickness in two parts is like a mirror, i.e., the laminate is symmetric in relation to its medium plane. Other example is the laminate in Fig. 12, which is represented by $[0_3/90_2/45/-45_3/-45_3/45/90_2/0_3]_T$ or $[0_3/90_2/45/-45_3]_s$.

Beyond symmetric laminates, there are the antisymmetric laminates and the asymmetric laminates. However, in the literature, it can be found a large number of classifications for laminates. Regarding antisymmetric laminates, the plies are stacked in order to create antisymmetry in relation of medium plane. By one side, a laminate with orientation angles of fibers in 0°, 90°, 0°, and 90° can be considered antisymmetric. By the other side, an asymmetric laminate has a random stacking sequence, and there is none rule of stacking related to the medium plane.

At this moment, there is a question: How to determine the laminate stiffness considering the plies stacked in different directions?

One approach to do this consists on using the CLT, which is based on Theory of Elasticity. Therefore, considering a solid (continuous media) loaded, this body produces internal stresses in order to equilibrate the applied loadings (Fig. 13).

A point in the body has the 3D stress state represented by the following stress tensor:

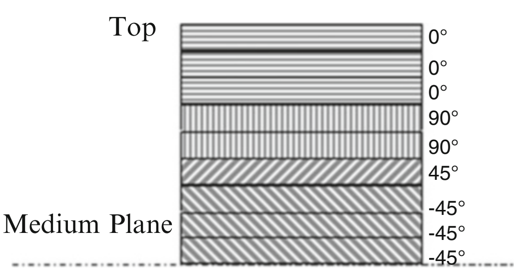

Fig. 12 Symmetric laminate

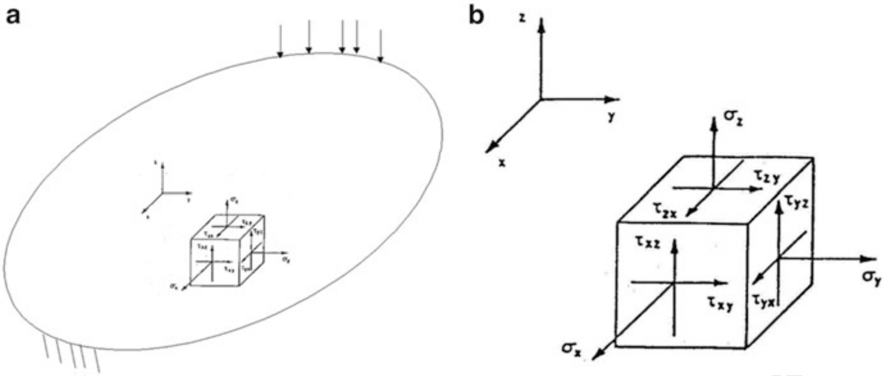

Fig. 13 (**a**) Solid loaded; (**b**) 3D stress state

$$\sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{47}$$

By using the equilibrium equations of momentum, it is obtained:

$$\tau_{xy} = \tau_{yx} \quad \text{and} \quad \tau_{xz} = \tau_{zx} \quad \text{and} \quad \tau_{yz} = \tau_{zy} \tag{48}$$

Thus, the stress tensor is symmetric and it can be represented mathematically by a vector with 6 (six) positions:

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} \quad \text{or} \quad \sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} \quad \text{or} \quad \sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{49}$$

An analog approach can be used for the strain tensor:

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \varepsilon_{yy} & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \varepsilon_{zz} \end{bmatrix} \tag{50}$$

Thus, the strain tensor is also symmetric and it can be represented mathematically by a vector with 6 (six) positions:

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz}/2 \\ \gamma_{zx}/2 \\ \gamma_{xy}/2 \end{bmatrix} \quad \text{or} \quad \varepsilon = \begin{bmatrix} \varepsilon_{x} \\ \varepsilon_{y} \\ \varepsilon_{z} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{bmatrix} \quad \text{or} \quad \varepsilon = \begin{bmatrix} \varepsilon_{1} \\ \varepsilon_{2} \\ \varepsilon_{3} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{bmatrix} \quad \text{or} \quad \varepsilon = \begin{bmatrix} \varepsilon_{1} \\ \varepsilon_{2} \\ \varepsilon_{3} \\ \varepsilon_{4} \\ \varepsilon_{5} \\ \varepsilon_{6} \end{bmatrix} \tag{51}$$

The constitutive equation relates the stress and strain vectors. For anisotropic materials, this relation is given by the Hooke's Law Generalized as follows (for index notation):

$$\sigma_i = D_{ij}\varepsilon_j \quad i,j = 1,2,\ldots,6$$

For matrix notation, it is observed the constitutive tensor D with 36 (thirty six) components:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{31} & D_{32} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{bmatrix} \tag{52}$$

However, it is shown that the constitutive tensor D is symmetric ($D_{ij} = D_{ji}$) and, in fact, the number of components is equal to 21 (twenty-one). Moreover, D represents the stiffness of the material and D^{-1} represents the compliance. Thus, D can be written in function of the material properties of composite phases (matrix, reinforcements, and interface).

As commented earlier, a ply of the laminate is assumed to be orthotropic material. Then, this ply has 3 (three) planes of symmetry. Also, an orthotropic material does not show coupling between normal stresses and shear strains (γ), as well as between shear stresses and normal strains (ε). Thus, the tensor D for this type of material has only 9 (nine) components:

$$D = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{12} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{13} & D_{23} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \tag{53}$$

By analogy, the tensor C has 9 (nine) components:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{54}$$

$$C_{11} = \frac{1}{E_{11}}; \quad C_{21} = \frac{-\nu_{12}}{E_{11}}; \quad C_{31} = \frac{-\nu_{13}}{E_{11}}$$

$$C_{12} = \frac{-\nu_{21}}{E_{22}}; \quad C_{22} = \frac{1}{E_{22}}; \quad C_{32} = \frac{-\nu_{23}}{E_{22}}$$

$$C_{13} = \frac{-\nu_{31}}{E_{33}}; \quad C_{23} = \frac{-\nu_{32}}{E_{33}}; \quad C_{33} = \frac{1}{E_{33}}$$

$$C_{44} = \frac{1}{G_{23}}; \quad C_{55} = \frac{1}{G_{31}}; \quad C_{66} = \frac{1}{G_{12}}$$

Considering the symmetry of the tensor C, then:

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \tag{55}$$

The matrix inverse of compliance is the stiffness, and for composites, this matrix will be named by Q:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & Q_{23} & 0 & 0 & 0 \\ Q_{31} & Q_{32} & Q_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{bmatrix} \tag{56}$$

where:

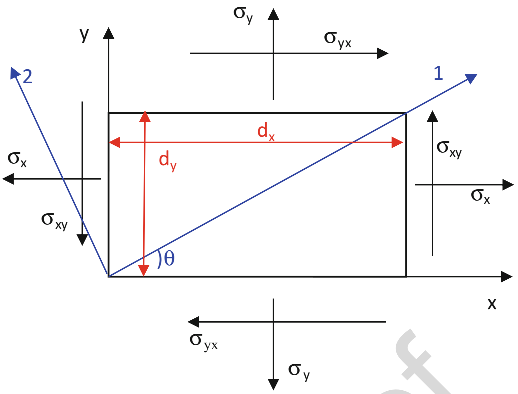

Fig. 14 Thin ply of composite material: plane stress state. Local coordinate system (1-2) and Global coordinate system ($x-y$)

$$Q_{11} = \frac{E_{11}(1-\nu_{23}\nu_{32})}{\Delta} \qquad Q_{12} = \frac{E_{11}(\nu_{21}+\nu_{31}\nu_{23})}{\Delta} = \frac{E_{22}(\nu_{12}+\nu_{32}\nu_{13})}{\Delta}$$

$$Q_{22} = \frac{E_{22}(1-\nu_{31}\nu_{13})}{\Delta} \qquad Q_{13} = \frac{E_{11}(\nu_{31}+\nu_{21}\nu_{32})}{\Delta} = \frac{E_{22}(\nu_{13}+\nu_{12}\nu_{23})}{\Delta}$$

$$Q_{33} = \frac{E_{33}(1-\nu_{12}\nu_{21})}{\Delta} \qquad Q_{23} = \frac{E_{22}(\nu_{32}+\nu_{12}\nu_{31})}{\Delta} = \frac{E_{33}(\nu_{23}+\nu_{21}\nu_{13})}{\Delta}$$

$$Q_{44} = G_{23} \qquad Q_{55} = G_{13} \qquad Q_{66} = G_{12}$$

$$\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13}$$

However, for a ply reinforced by fibers in one direction, it is considered a transversally isotropic material, so: $E_{22}=E_{33}$; $G_{13}=G_{23}$ and $\nu_{12}=\nu_{13}$. Besides, the thickness of the ply is very thin compared to the length and the width, then it is assumed plane stress state (Fig. 14).

Thus, the Hooke's Law can be written by using the Reduced Stiffness Stress:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_6 \end{bmatrix} \tag{57}$$

where:

$$Q_{11} = \frac{E_{11}}{1-\nu_{12}\nu_{21}} = \frac{E_{11}^2}{E_{11}-\nu_{12}^2 E_{22}} \qquad Q_{12} = Q_{21} = \frac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}} = \frac{\nu_{12}E_{11}E_{22}}{E_{11}-\nu_{12}^2 E_{22}}$$

$$Q_{22} = \frac{E_{22}}{1-\nu_{12}\nu_{21}} = \frac{E_{11}E_{22}}{E_{11}-\nu_{12}^2 E_{22}} \qquad \frac{\nu_{12}}{E_{11}} = \frac{\nu_{21}}{E_{22}}$$

$$Q_{66} = G_{12}$$

Considering the axes 1 and 2 and that 1 is aligned to the fibers and 2 is normal to the fibers, it can be used the transformation matrix of coordinates in order to write the stress components in Local or Global coordinate systems:

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_{12}\end{bmatrix}_{\text{Local}} = [T]\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{bmatrix}_{\text{Global}} \quad \text{or} \quad \begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{bmatrix}_{\text{Global}} = [T]^{-1}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_{12}\end{bmatrix}_{\text{Local}} \tag{58}$$

where: AU4

$$[T] = \begin{bmatrix} m^2 & n^2 & 2mn\\ n^2 & m^2 & -2mn\\ -mn & mn & (m^2-n^2)\end{bmatrix}; \quad m = \cos(\theta) \quad \text{and} \quad n = \text{sen}(\theta)$$

By analogy, the strain relations can be given by:

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_6/2\end{bmatrix}_{\text{Local}} = [T]\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}/2\end{bmatrix}_{\text{Global}} \quad \text{or} \quad \begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}/2\end{bmatrix}_{\text{Global}} = [T]^{-1}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_6/2\end{bmatrix}_{\text{Local}} \tag{59a}$$

Replacing (58) and (59a) into (57), it is obtained the constitutive equation for the Global coordinate system by using the Transformed Reduced Stiffness Matrix:

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{bmatrix}_{\text{Global}} = \begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16}\\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26}\\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}_{\text{Global}} \tag{59b}$$

or:

$$\begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16}\\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26}\\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66}\end{bmatrix} = [T]^{-1}\begin{bmatrix}Q_{11} & Q_{12} & Q_{16}\\ Q_{12} & Q_{22} & Q_{26}\\ Q_{16} & Q_{26} & Q_{66}\end{bmatrix}[T] \tag{59c}$$

Thus, the matrix components $\boxed{\overline{Q}}$ are given by:

$$\begin{aligned}
\overline{Q}_{11} &= Q_{11}m^4 + 2m^2n^2(Q_{12} + 2Q_{66}) + Q_{22}n^4\\
\overline{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})n^2m^2 + Q_{12}(n^4 + m^4)\\
\overline{Q}_{22} &= Q_{11}n^4 + 2(Q_{12} + 2Q_{66})n^2m^2 + Q_{22}m^4\\
\overline{Q}_{16} &= (Q_{11} - Q_{12})nm^3 + (Q_{12} - Q_{22})n^3m - 2mn(m^2 - n^2)Q_{66}\\
\overline{Q}_{26} &= (Q_{11} - Q_{12})n^3m + (Q_{12} - Q_{22})nm^3 + 2mn(m^2 - n^2)Q_{66}\\
\overline{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})n^2m^2 + Q_{66}(n^4 + m^4)
\end{aligned} \tag{60}$$

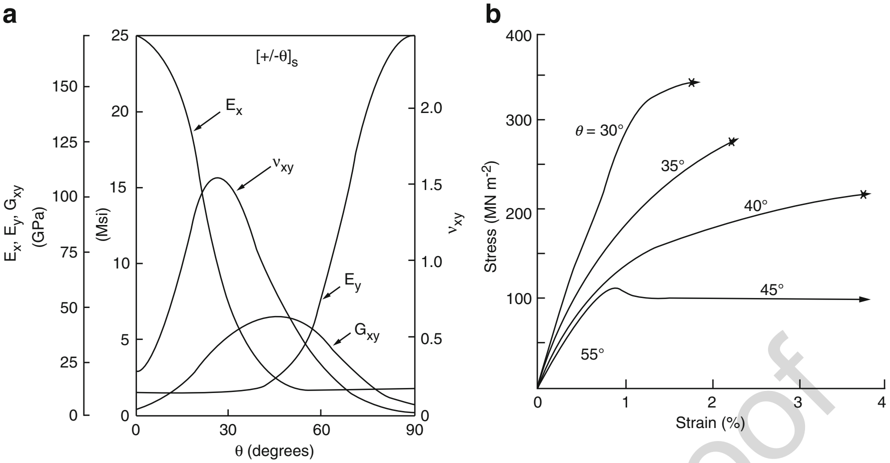

Fig. 15 Influence of the fiber orientation: (**a**) in the elastic properties (Jang 1994); (**b**) in the ply stiffness (Hull 1981)

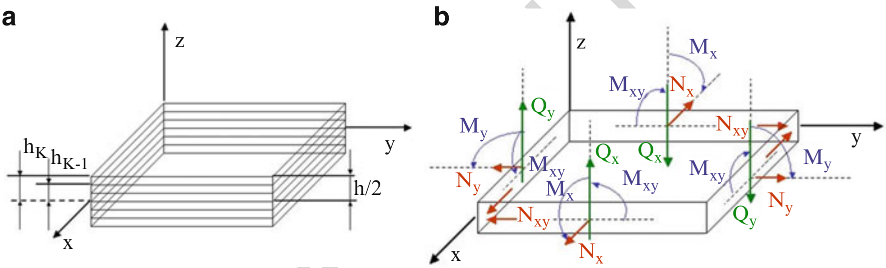

Fig. 16 (**a**) Laminate structure; (**b**) membrane loadings, shear forces, and bending moments

Therefore, $\overline{Q}$ has the influence of the orientation of the fiber in the ply (Fig. 15). It is verified that the orientation of the fiber influences in the mechanical properties and, consequently, in the ply stiffness, which will influence in the laminate stiffness.

Considering a laminate with h thickness and N plies, where the top of each k ply is distant h_k from the medium plane of the laminate as show by Fig. 16a, it will be calculated its stiffness by using CLT.

In this laminate, Membrane Loadings (N_x; N_y; and N_{xy}), Shear Forces (Q_x and Q_y), Bending Moments (M_x and M_y), and Torsion Moments (M_{xy}) can actuate as shown by Fig. 16b. These loadings can be calculated in function of the intern stresses of the laminate as follows:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ Q_x \\ Q_y \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix} \mathrm{d}z[\mathrm{N/m}] \tag{61}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} z\mathrm{d}z\,[\mathrm{Nm/m}] \tag{62}$$

Therefore, it is necessary to obtain the intern stresses of the laminate, which can be calculated by using CLT. And, this theory is based on the Kirchhoff's and other hypotheses, as well.

- The laminate is considered plane (as a plate) and the medium plane (medium surface), which split the laminate, is in the middle of the laminate and contains the plane $x-y$.
- The plies are perfectly linked and there is not relative displacement between plies, so the displacements are continuous.
- The matrix, which is between two plies, is very thin and it is not deformed by shear stress.
- The laminate is thin and Kirchhoff's kinematic hypotheses are applied. Therefore, these promotes $\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_z = 0$ and $\sigma_{xz}, \sigma_{yz}, \sigma_z << \sigma_{xy}, \sigma_y, \sigma_x$.

It is important to highlight that the Kirchhoff's kinematic hypotheses do not make account the transversal shear stress (Fig. 17). Hence, the transversal sections of the medium plane, which were plane and normal to the medium plane, remain plane and normal to the medium plane after the applied loading. Therefore:

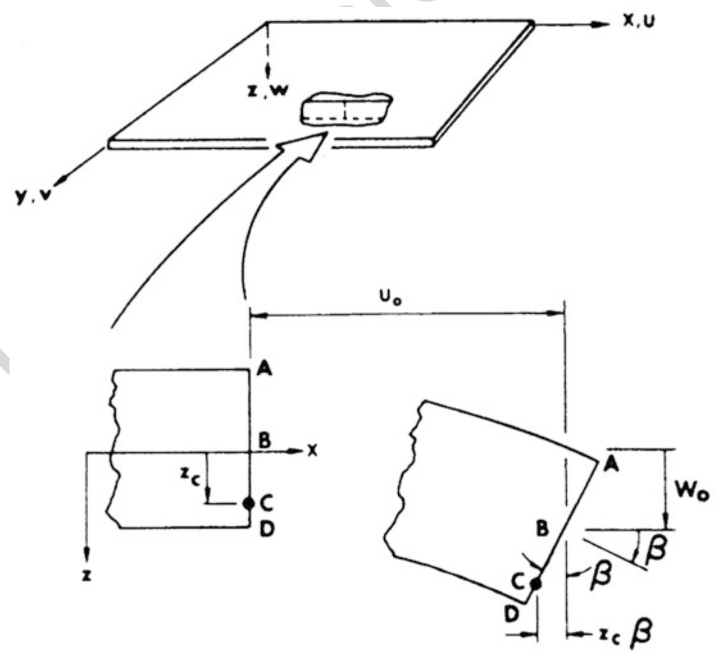

Fig. 17 Kirchhoff's kinematic hypotheses (Keunings 1992)

$\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_z = 0$. However, the stresses σ_{xz}, σ_{yz}, and σ_z are very important for delamination analyses. Moreover, if the structure is thick, the structural analyses should be affected in case of the transversal shears are not considered. Thus, for thick laminates or delamination analyses, it is necessary to use other kinematic hypotheses such as Mindlin-Reissner or Higher-order Shear deformation Theory—HST. However, in this chapter, it is considered mainly thin laminates, i.e., the relation length (or width) per thickness is minimum higher than 10.

Considering Fig. 17, for the point C with distance equal to z_c from the medium plane, the displacement u_c in the x direction is given by:

$$u_c = u_0 - z_c \beta \tag{63}$$

Thus:

$$\beta = \frac{\partial w_0}{\partial x} \tag{64}$$

Therefore, the displacements u and v in the directions x and y, respectively, are given by:

$$u(x, y, z) = u_0(x, y) - z \frac{\partial w_0(x, y)}{\partial x} \tag{65}$$

$$v(x, y, z) = v_0(x, y) - z \frac{\partial w_0(x, y)}{\partial y} \tag{66}$$

where:

u_o and v_o are displacements measured in the medium plane.
w is the displacement in z direction:

$$w(x, y, z) = w_0(x, y) \tag{67}$$

Thus, the strain for k ply can be calculated as follows:

$$\varepsilon_x(x, y, z) = \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2} = \varepsilon_{x0} + zK_x \tag{68}$$

$$\varepsilon_y(x, y, z) = \frac{\partial v_0}{\partial y} - z \frac{\partial^2 w_0}{\partial y^2} = \varepsilon_{y0} + zK_y \tag{69}$$

$$\begin{aligned} 2\varepsilon_{xy}(x, y, z) &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z \frac{\partial^2 w_0}{\partial x \partial y} = 2\varepsilon_{xy0} + zK_{xy} \\ \text{or} \quad \gamma_{xy} &= \gamma_{xy0} + zK_{xy} \end{aligned} \tag{70}$$

where:

ε_{xo}, ε_{yo}, and ε_{xyo} are strains related to extensional or distortional deformation in plane $x-y$.

It is observed that Kirchhoff's kinematic hypotheses results on a linear variation of the displacements and strains along the thickness. Hence, for a laminate, the strain vector can be written for the Global Coordinate System ($x-y$) as follows:

$$[\varepsilon]_{\text{Global}} = [\varepsilon_0]_{\text{Global}} + z[K]_{\text{Global}} \tag{71}$$

Therefore, the stress distribution varies from one ply to another along the thickness. Replacing (71) into (59b), it is calculated the stress vector for each k ply for the Global Coordinate System:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}^k_{\text{Global}} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}^k \begin{bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{bmatrix}_{\text{Global}} + z \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}^k \begin{bmatrix} K_x \\ K_y \\ K_{xy} \end{bmatrix}_{\text{Global}} \tag{72}$$

Considering the compact form:

$$[\sigma]^k_{\text{Global}} = \left[\overline{Q}\right]^k_{\text{Global}} \left[[\varepsilon_0]_{\text{Global}} + z[K]_{\text{Global}}\right] \tag{73}$$

where:

$[\varepsilon_o] =$ strains
$[K] =$ curvatures
$k =$ ply in the k position.

Replacing (73) into (61) and into (62):

$$\begin{bmatrix} N_X \\ N_Y \\ N_{XY} \end{bmatrix} = \sum_{K=1}^{n} \left\{ \int_{h_{K-1}}^{h_K} \left[\overline{Q}\right]_K \begin{bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{bmatrix} \mathrm{d}z + \int_{h_{K-1}}^{h_K} \left[\overline{Q}\right]_K \begin{bmatrix} K_x \\ K_y \\ K_{xy} \end{bmatrix} z\mathrm{d}z \right\} \tag{74}$$

$$\begin{bmatrix} M_X \\ M_Y \\ M_{XY} \end{bmatrix} = \sum_{K=1}^{n} \left\{ \int_{h_{K-1}}^{h_K} \left[\overline{Q}\right]_K \begin{bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{bmatrix} z\mathrm{d}z + \int_{h_{K-1}}^{h_K} \left[\overline{Q}\right]_K \begin{bmatrix} K_x \\ K_y \\ K_{xy} \end{bmatrix} z^2\mathrm{d}z \right\} \tag{75}$$

The matrix $\left[\overline{Q}\right]$ remains constant for each ply, because it is only function of the elastic properties of plies and fiber orientation in each ply. The strain components $[\varepsilon_o]$ and the curvature $[K]$ of the laminate remains constant for each ply, also. Therefore, Eqs. (74) and (75) can be written as follows:

$$[N] = [A][\varepsilon_0] + [B][K] \tag{76a}$$

$$[M] = [B][\varepsilon_0] + [D][K] \tag{76b}$$

where:

$[A] = \sum_{k=1}^{n} \int_{h_{K-1}}^{h_K} \left[\overline{Q}\right]_K \mathrm{d}z =$ membrane stiffness matrix.

$[B] = \sum_{k=1}^{n} \int_{h_{K-1}}^{h_K} \left[\overline{Q}\right]_K z\mathrm{d}z =$ coupling stiffness matrix.

$[D] = \sum_{k=1}^{n} \int_{h_{K-1}}^{h_K} \left[\overline{Q}\right]_K z^2\mathrm{d}z =$ bending/torsion stiffness matrix.

or:

$$\begin{bmatrix} [N] \\ [M] \end{bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{bmatrix} [\varepsilon_o] \\ [K] \end{bmatrix} \tag{77}$$

If the coupling matrix $[B]$ is not null, then membrane loadings can cause not only normal and shear strains, but also curvatures K_x, K_y, and K_{xy}. By analogy, moments loadings can cause not only curvatures K_x, K_y, and K_{xy}, but also normal and shear strains. By the other side, if the coupling matrix $[B]$ is null, these effects cannot occur. In fact, matrix $[B]$ is null for symmetric laminates, and this is easily proved by verifying that stiffness part related to z positive values are canceled by stiffness part related to z negative values.

In case of thick laminate analysis, it is necessary to consider the shear forces (Q_x and Q_y). Thus, one simple approach consists on assuming parabolic distribution along of the laminate thickness:

$$f(z) = \frac{5}{4}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \tag{78}$$

Integrating this equation, it is obtained:

$$Q_x = \left(A_{55}\gamma_{xz} + A_{45}\gamma_{yz}\right) \tag{79}$$

$$Q_y = \left(A_{45}\gamma_{xz} + A_{44}\gamma_{yz}\right) \tag{80}$$

where:

$$A_{ij} = \frac{5}{4}\sum_{k=1}^{n}\left(\overline{Q}_{ij}\right)_k\left[h_k - h_{k-1} - \frac{4}{3}\left(h_k^3 - h_{k-1}^3\right)\frac{1}{h^2}\right]$$

Therefore:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{x_0} \\ \varepsilon_{y_0} \\ \gamma_{xy_0} \\ K_x \\ K_y \\ K_{xy} \end{bmatrix} \tag{81}$$

$$\begin{bmatrix} Q_y \\ Q_x \end{bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \tag{82}$$

It is concluded that for thin laminates, it should be used only Eq. (81), and, for thick laminates, it is necessary to use at least Eq. (82), as well. AU5

In terms of design, the equations above should be written in inverse format, because, normally, the loadings are provided and it is required to calculate the strains and curvatures. However, these values are obtained for each ply, considering the Global Coordinate System, and, now, it is necessary to calculate these values for Local Coordinate System.

3.2 *Strain and Stress Analyses in the Ply*

The determination of stress and strain components for each ply for the Local Coordinate System is very important to evaluate the failure or not of a laminate, considering a load case.

The failure mechanisms and failure criteria will be addressed in the next section, but the criteria are normally verified in each ply of the laminate considering the stress and strain components for the Local Coordinate System (1-2). Thus, in order to obtain these values, it is initially written Eq. (76a) in the following format: AU6

$$[\varepsilon_0] = [A]^{-1}[N] - [A]^{-1}[B][K] \tag{83}$$

Replacing (83) into (76b), there is:

$$[M] = [B][A]^{-1}[N] - \left\{[B][A]^{-1}[B] - [D]\right\}[K] \tag{84}$$

Equations (83) and (84) can be combined:

$$\begin{bmatrix} [\varepsilon_o] \\ [M] \end{bmatrix} = \begin{bmatrix} [A^*] & [B^*] \\ [C^*] & [D^*] \end{bmatrix} \begin{bmatrix} [N] \\ [K] \end{bmatrix} \tag{85}$$

where:

$$\begin{aligned}[A^*] &= [A^{-1}] \\ [B^*] &= -[A^{-1}][B] \\ [C^*] &= [B][A^{-1}] = -[B^*]^T \\ [D^*] &= [D] - [B][A^{-1}][B]\end{aligned}$$

Thus, Eqs. (83) and (84) can be written as follows:

$$[\varepsilon_0] = [A^*][N] + [B^*][K] \tag{86}$$

$$[M] = [C^*][N] + [D^*][K] \tag{87}$$

Solving the system above for the curvatures K:

$$[K] = [D^*]^{-1}[M] - [D^*]^{-1}[C^*][N] \tag{88}$$

Replacing Eq. (88) into (86):

$$[\varepsilon_0] = \left\{[A^*] - [B^*][D^*]^{-1}[C^*]\right\}[N] + [B^*][D^*]^{-1}[M] \tag{89}$$

Combining Eqs. (88) and (89), it is obtained the system of equation completely inverted:

$$\begin{bmatrix}[\varepsilon_o] \\ [K]\end{bmatrix} = \begin{bmatrix}[A'] & [B'] \\ [C'] & [D']\end{bmatrix}\begin{bmatrix}[N] \\ [M]\end{bmatrix} \tag{90}$$

where:

$$\begin{aligned}[A'] &= [A^*] - [B^*][D^*]^{-1}[C^*] = [A^*] + [B^*][D^*]^{-1}[B^*]^T \\ [B'] &= [B^*][D^*]^{-1} \\ [C'] &= -[D^*]^{-1}[C^*] = [B']^T = [B'] \\ [D'] &= [D^*]^{-1}\end{aligned}$$

Hence, it is calculated the strain components $[\varepsilon_o]$ and the curvatures $[K]$ of the laminate for the Global Coordinate System, considering a loading state. Based on these values, it is calculated the stress components for each k ply for the Global Coordinate System (Fig. 18):

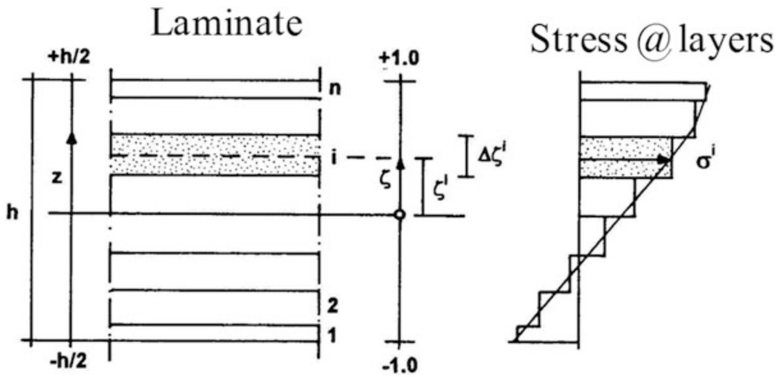

Fig. 18 Distribution of stress along of thickness

$$[\sigma]^k_{\text{Global}} = [\overline{Q}]^k_{\text{Global}}\left[[\varepsilon_0]_{\text{Global}} + z[K]_{\text{Global}}\right] \tag{91}$$

By using the equations for coordinate transformation, it is determined the stress and strain components for the Local Coordinate System:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{bmatrix}^K_{\text{Local}} = [T] \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}^k_{\text{Global}} \quad \text{and} \quad \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_6/2 \end{bmatrix}^k_{\text{Local}} = [T] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{bmatrix}^k_{\text{Global}} \tag{92}$$

where:

$$[T] = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & (m^2 - n^2) \end{bmatrix}; \quad m = \cos(\theta) \quad \text{and} \quad n = \text{sen}(\theta)$$

Hence, the calculation of the stress and strain components for the Local Coordinate System can be summarized in 7 (seven) steps:

Step 1: Determine the elastic properties of each ply (E_{11}; E_{22}; G_{12}; and ν_{12}).

Step 2: Calculate the Reduced Stiffness Matrix for each ply in relation of Local Coordinate System.

$$[Q]_{\text{Local}} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}$$

where:

$$
\begin{aligned}
Q_{11} &= \frac{E_{11}}{1-\nu_{12}\nu_{21}} = \frac{E_{11}^2}{E_{11}-\nu_{12}^2E_{22}} \\
Q_{22} &= \frac{E_{22}}{1-\nu_{12}\nu_{21}} = \frac{E_{11}E_{22}}{E_{11}-\nu_{12}^2E_{22}} \\
Q_{66} &= G_{12} \\
Q_{12} &= Q_{21} = \frac{\nu_{12}E_{22}}{1-\nu_{12}\nu_{21}} = \frac{\nu_{12}E_{11}E_{22}}{E_{11}-\nu_{12}^2E_{22}} \\
\frac{\nu_{12}}{E_{11}} &= \frac{\nu_{21}}{E_{22}}
\end{aligned}
$$

Step 3: Calculate the Transformed Reduced Stiffness Matrix for each ply in relation of Local Coordinate System.

$$
\left[\overline{Q}\right]_k = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}
$$

where:

$$
\begin{aligned}
\overline{Q}_{11} &= Q_{11}m^4 + 2m^2n^2(Q_{12}+2Q_{66}) + Q_{22}n^4 \\
\overline{Q}_{12} &= (Q_{11}+Q_{22}-4Q_{66})n^2m^2 + Q_{12}(n^4+m^4) \\
\overline{Q}_{22} &= Q_{11}n^4 + 2(Q_{12}+2Q_{66})n^2m^2 + Q_{22}m^4 \\
\overline{Q}_{16} &= (Q_{11}-Q_{12})nm^3 + (Q_{12}-Q_{22})n^3m - 2mn(m^2-n^2)Q_{66} \\
\overline{Q}_{26} &= (Q_{11}-Q_{12})n^3m + (Q_{12}-Q_{22})nm^3 + 2mn(m^2-n^2)Q_{66} \\
\overline{Q}_{66} &= (Q_{11}+Q_{22}-2Q_{12}-2Q_{66})n^2m^2 + Q_{66}(n^4+m^4) \\
m &= \cos(\theta) \quad \textit{and} \quad n = \mathrm{sen}(\theta)
\end{aligned}
$$

Step 4: Calculate matrixes A, B, and D in relation of Global Coordinate System.

$$
[A] = \sum_{k=1}^{n} \left[\overline{Q}\right]_k (h_k - h_{k-1})
$$

$$
[B] = \frac{1}{2}\sum_{k=1}^{n} \left[\overline{Q}\right]_k \left(h_k^2 - h_{k-1}^2\right)
$$

$$
[D] = \frac{1}{3}\sum_{k=1}^{n} \left[\overline{Q}\right]_k \left(h_k^3 - h_{k-1}^3\right)
$$

Step 5: Calculate the strain components $[\varepsilon_o]$ and the curvatures $[K]$ of the laminate for the Global Coordinate System.

$$\begin{bmatrix} [\varepsilon_o] \\ [K] \end{bmatrix}_{\text{Global}} = \begin{bmatrix} [A'] & [B'] \\ [C'] & [D'] \end{bmatrix} \begin{bmatrix} [N] \\ [M] \end{bmatrix}$$

where:

$$[A'] = \left[A^*\right] - \left[B^*\right]\left[D^*\right]^{-1}\left[C^*\right] = \left[A^*\right] + \left[B^*\right]\left[D^*\right]^{-1}\left[B^*\right]^T$$
$$[B'] = \left[B^*\right]\left[D^*\right]^{-1}$$
$$[C'] = -\left[D^*\right]^{-1}\left[C^*\right] = [B']^T = [B']$$
$$\left[D^*\right] = \left[D^*\right]^{-1}$$

Step 6: Calculate the stress components for each k ply for the Global Coordinate System.

$$[\sigma]^k_{\text{Global}} = \left[\overline{Q}\right]^k_{\text{Global}}\left[[\varepsilon_0]_{\text{Global}} + z[K]_{\text{Global}}\right]$$

Step 7: Calculate the stress components for each k ply for the Local Coordinate System.

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{bmatrix}_{\text{Local}} = [T]\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}_{\text{Global}}$$

where:

$$[T] = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & (m^2 - n^2) \end{bmatrix}$$

These stress or strain components will be used in the Failure Criteria, and the engineer will be able to evaluate if the composite structure will fail or not under a specific load case.

4 Failure Analysis

Based on the stress or strain components values for each ply for the Local Coordinate System, it is carried out the failure analysis of the laminate. However, it is necessary to know previously the different failure modes, which can be found in the composite structures. Thus, based on the failure modes, which can occur, the failure criterion should be selected.

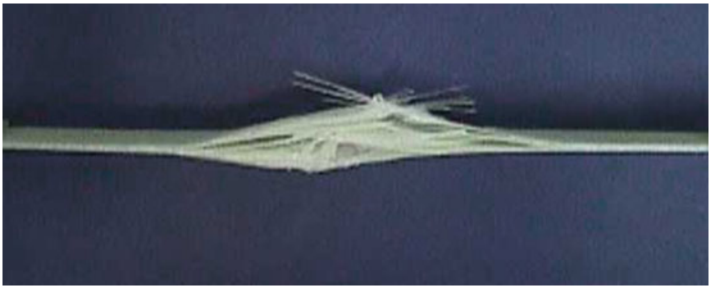

Fig. 19 Damage and failure mechanisms

4.1 Laminate Failure Modes

In this chapter, the failure/damage mechanisms are classified in two types:

- *Intralaminar damage*: occur inside the ply;
- *Interlaminar failure*: occur between plies.

The intralaminar damages correspond to the damage in the matrix, fibers, or interface fiber-matrix. The interlaminar failures correspond to the delaminations between plies, which consists on the separation of plies (Fig. 19).

4.1.1 Intralaminar Damage

The intralaminar damages can be divided in three different mechanisms:

- Mechanism of fiber damage.
- Mechanism of damage damage.
- Mechanism of interface matrix-fiber damage.

The mechanism of fiber damage depends on different aspects, such as diameter and length of fibers, volume fraction of fibers, and orientation of fibers. However, the damage modes are also related to the applied loadings. For examples, compression loading can produce fails in the fibers through micro-buckling or shearing (Fig. 20).

Tension loading can promote the rupture of the fibers and depends on the level of the adhesion between fibers and polymer matrix. In other words, if the loading, which acts in the matrix, is transferred to the fibers in an efficient way, then the fibers can fracture, depending on the level of load.

The matrix damage modes depend on the physic-chemical properties of the polymer, which can be fragile or ductile and have linear elastic or viscoelastic response. Moreover, this behavior depends on the environment temperature. However, in general way, the rupture of the matrix occurs close to a fractured fiber or close to a void created during the material processing. These regions show stress

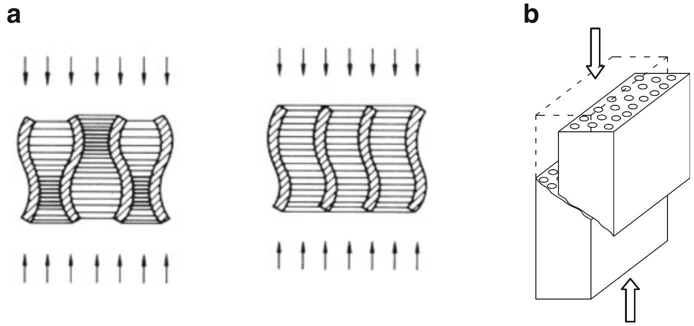

Fig. 20 Fiber damage mode under compression: (**a**) micro-buckling (Agarwal and Broutman 1990); (**b**) shearing (Adapted from Agarwal and Broutman 1990)

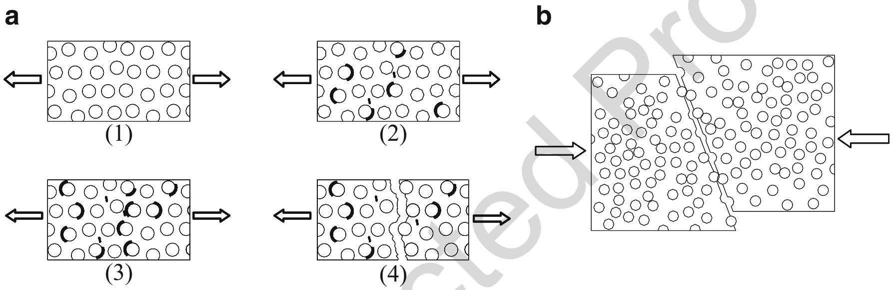

Fig. 21 Damage process in the matrix: (**a**) under tension; (**b**) under compression (adapted from Agarwal and Broutman 1990)

concentration, which causes failure of the matrix. Therefore, under tension loading, the damage process in the matrix, as shown by Fig. 21a, starts close to micro-failures (1), then propagates (2) and, finally, coalesces (3) until creating a catastrophic macro-failure (4). By the other side, under compression loading, the matrix can fail by shearing (Fig. 21b).

For the ply under shear loading, the damage mode will occur as shown by Fig. 22a. As it is observed, this damage mode depends mainly on the polymer matrix behavior, which can be non-linear due to inelastic strains.

The damage process of the ply is strongly influenced by the orientation of the fibers. For example, the ply can show a linear response when the loading is applied in the direction 1 (0°) or in the direction 2 (90°) due to the relevance of normal stresses. However, for the loadings applied close to the angle 15°, it is observed a non-linear response, because there is an important contribution of the shear stresses as shown by Fig. 22b.

Regarding the damage modes of the interface, it is confirmed that these modes depend on physic-chemical interaction between fiber and matrix. In fact, the quality of the interface is a parameter that it is used to evaluate the toughness of the

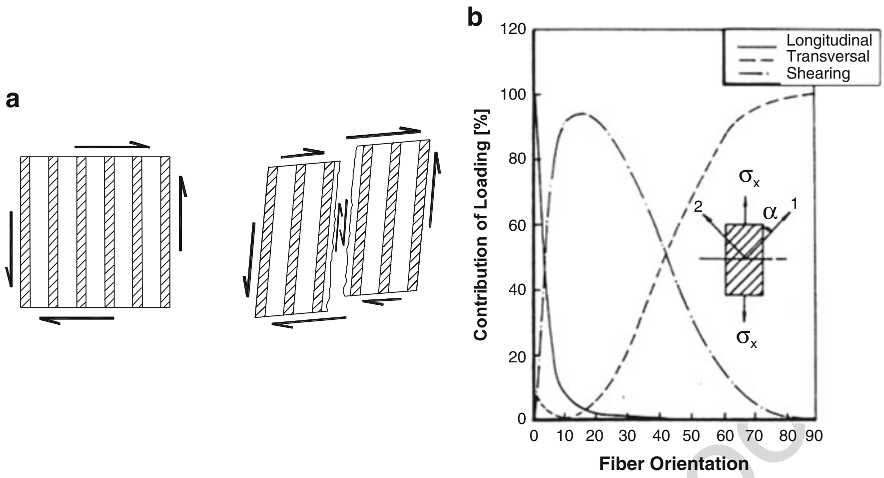

Fig. 22 (**a**) Damage of matrix under shear loadings (adapted from Agarwal and Broutman 1990); (**b**) influence of the fiber orientation in the damage process (Hahn and Tsai 1973)

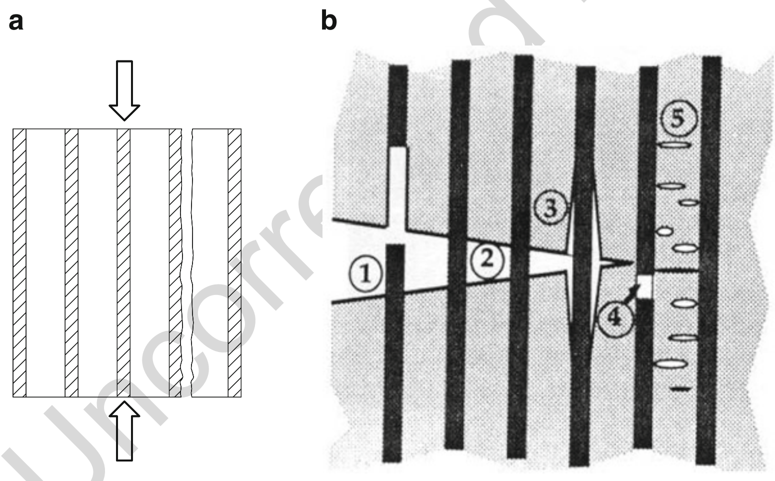

Fig. 23 (**a**) Debonding due to weak interface; (**b**) damage mechanisms in the ply (Anderson 1995)

composite material. Thus, if there is a weak interaction between fiber and matrix, then it occurs "*debonding*" as shown by Fig. 23a.

Figure 23b shows different damage mechanisms in the ply. If there is a weak interface, after the fiber failure, "*Pull-Out*" (mechanism 1) can take place. Before this mechanism, it is possible to occur "*Fiber Bridging*" (mechanism 2), since the composite has fragile fibers, ductile matrix, and strong interface. Thus, the crack propagation creates like bridges by using the fibers. As commented earlier, if the interface is weak, then "*debonding*" (mechanism 3) can occur. By the other side, if the interface is strong, then facture of the fiber (mechanism 4) and the damage process of the matrix (mechanism 5) are verified. However, these all damage mechanisms are random and depend on several aspects:

- Physic-chemical properties of the fibers and polymer matrix.
- Alignment and strength of the fibers.
- Orientation and volume fraction of the fibers.
- Type of loading: tension, compression, shear, or combined.
- Environment effects: temperature, humidity, corrosion, etc.

4.1.2 Interlaminar Failure (Delaminations)

In composite materials, the failure starts with micromechanisms (intralaminar damages) and, after that, it is observed the macromechanisms like delaminations. In general, the damage evolution starts in the plies with fiber orientation close to 90° in relation to the loading. After the first damage, stresses are redistributed in the laminated and new failure mechanisms can occur in the same ply or in other plies. This failure process evolutes until the damage to reach the interface between two plies, creating a discrete crack. In fact, the frontiers of the cracks, which were created in one ply, propagate until to find adjacent ply with fiber oriented in other direction (Fig. 24a). At this moment, the interlaminar shear stresses increase abruptly and the laminate suffers the delamination as shown by Fig. 24b. Considering the increment of the loadings, the delaminations increase (initiation) and evolve (propagation).

Researchers proved that the interlaminar failure is promoted by the interlaminar shear stress and normal stress in direction z as shown by Fig. 25a.

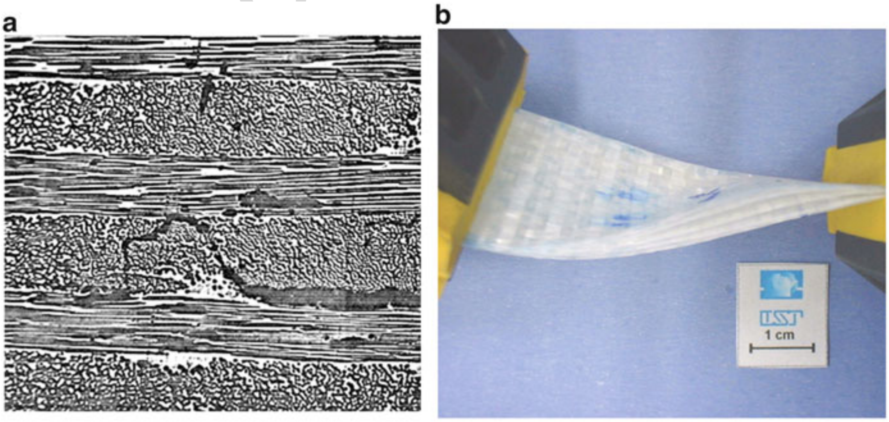

Fig. 24 Mechanisms of damage and failure in the plies: (**a**) evolution of the failure process (Hull 1981); (**b**) laminate with delaminations

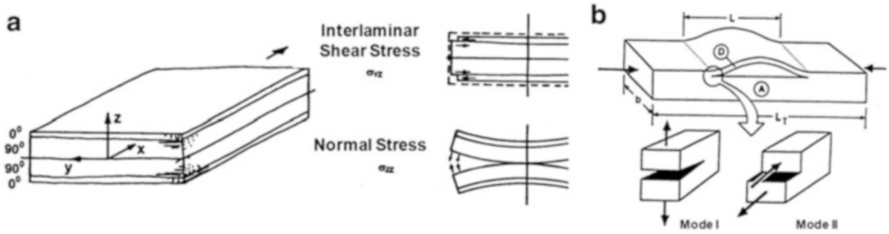

Fig. 25 (**a**) Delamination: interlaminar shear stress and normal stress; (**b**) modes of delamination (adapted from Magagnin Filho 1996)

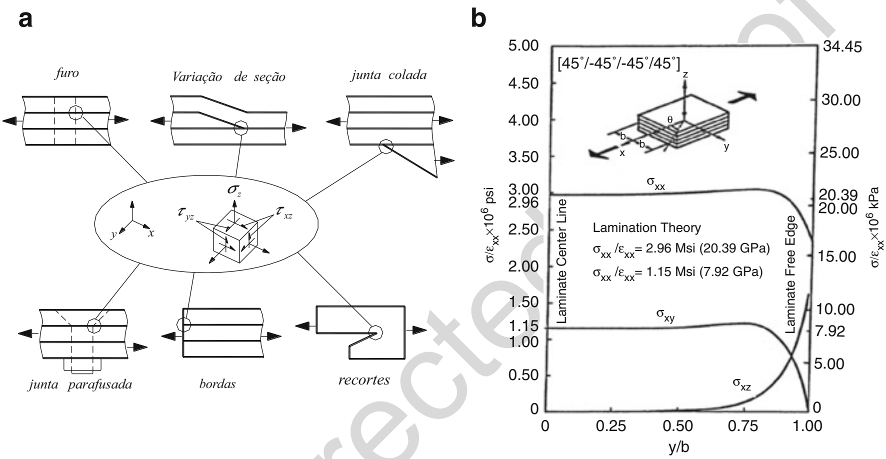

Fig. 26 (**a**) candidate regions of delamination (adapted from Jang 1994). (**b**) stress distribution along the ply length—edge effects (Keunings 1992)

According to the Fracture Mechanics, laminate material composites, normally, show two classic modes of delamination: Mode I and Mode II (Fig. 25b). The Mode I is created by tension loadings and Mode II is created by shear loadings. Thus, during the delamination process is common to observe the Mixed Mode, i.e., Mode I and Mode II are coupled.

In practical terms, the engineer should be pay attention, mainly in the geometrical discontinuities in the composite structures, such as holes and ply drop. In these regions, there is a 3D stress state, which promotes delamination (Fig. 26a). Another important region consists of the edge of the laminate. In fact, in this portion of the laminate, edge-effects can increase the transversal shear stress close to the edges (Fig. 26b).

4.2 Procedure to Analyze Failure in Laminates

It is considered that a structure fails when this one cannot satisfy the design criteria. Thus, failure criterion goals to provide an interpretation of the damages promoted by the loadings, showing if there is a local or a global failure in the structure. However, for laminate composite structures, there is a large number of damage and failure mechanisms, which occur in a random way. Thus, different approaches can be applied to design composite structures. One approach consists on carrying out micromechanics analyses in order to identify the local failure of fibers, matrix, or interface. By the other side, there is the macromechanics analysis, which consists on using a failure criterion in order to identify the failure of the ply.

The failure criterion can be written by using mathematical expressions (the criterion function), considering the stress or strain components for the Local Coordinate System (1-2) and allowable values for the ply:

$$\begin{aligned}&\text{If} \quad f(\sigma_1, \sigma_2, \sigma_3) \geq 0 \quad \text{then the ply fails.}\\&\text{If} \quad f(\sigma_1, \sigma_2, \sigma_3) < 0 \quad \text{then the ply does not fail}\end{aligned} \tag{93}$$

Associated to the failure criterion, there are two methods of approaching the problem:

- *FPF Method (First Ply Failure)*: the laminate fails when the first ply fails.
- *LPF Method (Last Ply Failure)*: the laminate fails when the last ply fails.

LPF Method can be summarized in 9 (nine) steps (Fig. 27):

1. *Stress analyses*: calculate the stress components in each ply.
2. *Failure criterion selection*: select the most adequate criterion, considering the failure modes observed during the experimental tests for determination of allowable values and elastic properties.
3. *Calculate the criterion function*: use the stress components and allowable values to calculate the value for the criterion function.
4. *Verify the failure plies*: identify the plies, which fail.
5. *If there is not failure—increase the loading*: increase the loading in order to re-calculate the stress components in each ply.
6. *If there is failure—reduce the mechanical properties*: before increasing the loading, the mechanical properties of the plies, which failed, should be reduced.
7. *Total failure?*: check if all plies fail.
8. *If there is not total failure—re-calculate the stress distribution*: re-calculate the stress components in each ply, considering the reduction of laminate stiffness.
9. *If there is total failure—THE END*: finalize the analyses.

The FPF Method is strongly safety, because the failure of only single ply implies in to have the failure of the entire laminate. By the other side, the LPF Method overestimates the strength of the laminate. Therefore, the engineer must be careful when choosing the method, mainly the failure criterion. However, due to the

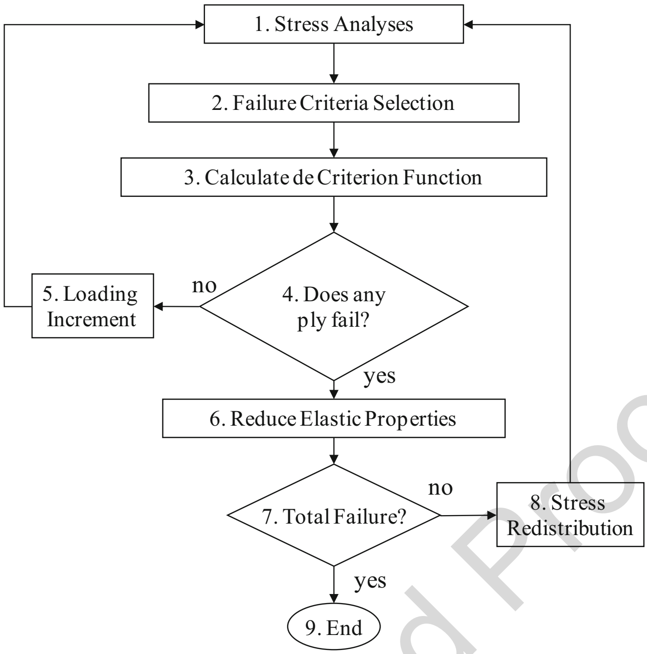

Fig. 27 Procedure to perform failure analyses by using Last Ply Failure Method

complexity to predict the failure mechanisms on composite structures, there is a large number of failure criteria to address this problem. In the next sub-items, it will be shown 3 (three) different failure criteria.

4.2.1 Maximum Stress Criterion

This failure criterion consists of 5 (five) sub-criteria and each one corresponds to the 5 (five) fundamental damage mode of the ply. If, at least, one allowable stress limit is exceeded, then the ply fails:

$$\sigma_1 \geq X_{\mathrm{T}} \quad \text{or} \quad \sigma_1 \leq -X_{\mathrm{C}} \quad \text{or} \quad \sigma_2 \geq Y_{\mathrm{T}} \quad \text{or} \quad \sigma_2 \leq -Y_{\mathrm{C}} \quad \text{or} \quad |\sigma_{12}| \geq S_{12} \tag{94}$$

where:

σ_1: normal stress component in direction 1.
σ_2: normal stress component in direction 2.
σ_{12}: shear stress component in the plane 1-2.
$X_{\mathrm{T,C}}$: strength value for tension or compression in direction 1.
$Y_{\mathrm{T,C}}$: strength value for tension or compression in direction 2.
S_{12}: strength value for shear in plane 1-2.

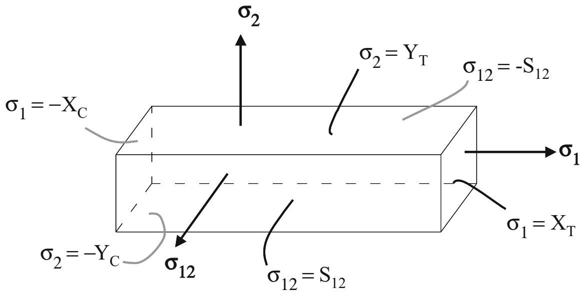

Fig. 28 Failure surface of maximum stress criterion

The failure surface for this criterion is a parallelepiped in the space of stresses (Fig. 28). Due to the difference between strength values for tension and compression, the geometric center of the parallelepiped does not coincide to the origin of space of stresses.

4.2.2 Maximum Strain Criterion

This failure criterion also consists of 5 (five) sub-criteria and each one corresponds to the 5 (five) fundamental damage mode of the ply. However, in this case, the criterion is written in terms of strains. Thus, if, at least, one allowable strain limit is exceeded, then the ply fails:

$$\varepsilon_1 \geq X_T^{'} \quad \text{or} \quad \varepsilon_1 \leq -X_C^{'} \quad \text{or} \quad \varepsilon_2 \geq Y_T^{'} \quad \text{or} \quad \varepsilon_2 \leq -Y_C^{'} \quad \text{or} \quad |\gamma_{12}| \geq S_{12}^{'} \tag{95}$$

where:

ε_1 = normal strain component in direction 1.
ε_2 = normal strain component in direction 2.
ε_{12} = shear strain component in plane 1-2.
$X_{T,C}^{'}$ = strain limit value for tension or compression in direction 1.
$Y_{T,C}^{'}$ = strain limit value for tension or compression in direction 2.
$S_{12}^{'}$ = strain limit value for shear in plane 1-2.

In general, the Maximum Stress Criterion and Maximum Strain Criterion provide similar predictions, but when the composite material shows non-linear behavior, it is better to use the second one. Also, these criteria are not interactive, i.e., the stress component in one direction does not influence the failure mode caused by a stress component in other direction and vice-versa, but the mode failure of the ply can be identified.

4.2.3 TSAI-HILL Criterion

Based on HILL criterion, Tsai proposed a failure criterion for composite materials, especially for laminates with orthotropic plies. Thus, TSAI-HILL criterion for plane stress state can be written as follows:

$$f(\sigma) = \left(\frac{\sigma_1}{X}\right)^2 + \left(\frac{\sigma_2}{Y}\right)^2 - \left(\frac{\sigma_1\sigma_2}{X^2}\right) + \left(\frac{\sigma_{12}}{S_{12}}\right)^2 = 1 \tag{96}$$

where σ_1 and σ_2 are the normal stress components in the ply. Besides, in this criterion, it is necessary to use different values for compression and tension, not only for actuating stresses, but also for allowable values. Thus, re-organizing the equation above, it is obtained 4 (four) different equations in the space of stresses $(\sigma_1 - \sigma_2)$:

1. For the First Quadrant $(\sigma_1, \sigma_2 > 0)$:

$$\frac{\sigma_1^2}{X_T^2} + \frac{\sigma_2^2}{Y_T^2} - \frac{\sigma_1\sigma_2}{X_T^2} = 1 - \frac{\sigma_{12}^2}{S_{12}^2} \tag{96a}$$

2. For the Second Quadrant $(\sigma_1 < 0, \sigma_2 > 0)$:

$$\frac{\sigma_1^2}{X_C^2} + \frac{\sigma_2^2}{Y_T^2} + \frac{\sigma_1\sigma_2}{X_C^2} = 1 - \frac{\sigma_{12}^2}{S_{12}^2} \tag{96b}$$

3. For the Third Quadrant $(\sigma_1, \sigma_2 < 0)$:

$$\frac{\sigma_1^2}{X_C^2} + \frac{\sigma_2^2}{Y_C^2} - \frac{\sigma_1\sigma_2}{X_C^2} = 1 - \frac{\sigma_{12}^2}{S_{12}^2} \tag{96c}$$

4. For the Fourth Quadrant $(\sigma_1 > 0, \sigma_2 < 0)$:

$$\frac{\sigma_1^2}{X_T^2} + \frac{\sigma_2^2}{Y_C^2} + \frac{\sigma_1\sigma_2}{X_T^2} = 1 - \frac{\sigma_{12}^2}{S_{12}^2} \tag{96d}$$

Based on the equations above, it is obtained the failure surface for TSAI-HILL criterion as shown by Fig. 29. It is verified that the increase of shear stress causes the contraction of the failure surface, becoming the failure process easier to occur for lower values of normal stresses.

In practical, it is used the definitions of Factor of Safety (FS) and Margin of Safety (MS) to determine if a ply fails or not by using TSAI-HILL criterion:

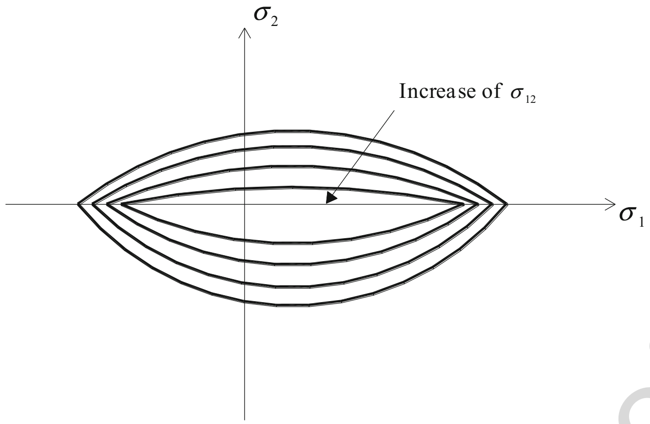

Fig. 29 Failure surface of TSAI-HILL criterion

$$\mathrm{FS} = \sqrt{f(\sigma)} = \sqrt{\left(\frac{\sigma_1}{X}\right)^2 + \left(\frac{\sigma_2}{Y}\right)^2 - \left(\frac{\sigma_1 \sigma_2}{X^2}\right) + \left(\frac{\sigma_{12}}{S_{12}}\right)^2} \tag{97}$$

$$\mathrm{MS} = \frac{1}{\mathrm{FS}} - 1 \tag{98}$$

If MS is lower than zero then the ply fails. By the other side, if the MS is much greater than zero, then it is concluded that the laminate should be optimized. This criterion is used a lot by the engineers, but it is important to highlight that it is not recommended for laminates with non-linear behavior. However, it is an interactive criterion; so a stress component in one direction can influence the failure mode caused by a stress component in other direction and vice-versa, but it is not possible to identify the failure mode for the ply.

In fact, advances in procedure to analyze failure in laminates have been performed by different research groups in the World for a long time. The research group coordinate by Professor Volnei Tita at University of São Paulo has worked in this way, as well. Therefore, some scientific contributions can be found in the literature, such as Tita et al. (2008, 2012), Angelo et al. (2012, 2015), Sartorato et al. (2012), and Ribeiro et al. (2012a, b, 2013a, b, 2015).

Finally, if a failure occurs, then the engineer can redesign the laminate composite structure as shown by Fig. 4. Thus, the stacking sequence of the laminate should be modified in order to change the stiffness, or it is necessary to change the type of polymer matrix or the fibers, or to increase the volume fraction of the fibers.

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