Example 1.1. *The Levi–Civita symbol and the volume tensor ϵ on $\mathbb{R}^3$*

You may have encountered the Levi–Civita symbol in coursework in classical or quantum mechanics. We will denote it here[1] as $\bar{\epsilon}_{ijk}$, where the indices i, j, and k range from 1 to 3. The symbol takes on different numerical values depending on the values of the indices, as follows:

$$\bar{\epsilon}_{ijk} \equiv \begin{cases} 0 & \text{unless } i \neq j \neq k \\ +1 & \text{if } \{i,j,k\} = \{1,2,3\},\ \{2,3,1\},\ \text{or } \{3,1,2\} \\ -1 & \text{if } \{i,j,k\} = \{3,2,1\},\ \{1,3,2\},\ \text{or } \{2,1,3\}. \end{cases} \tag{1.1}$$

Sometimes one sees this defined in words, as follows: $\bar{\epsilon}_{ijk} = 1$ if $\{i,j,k\}$ is a "cyclic permutation" of $\{1,2,3\}$, -1 if $\{i,j,k\}$ is an "anti-cyclic permutation" of $\{1,2,3\}$, and 0 otherwise.

The Levi–Civita symbol is usually introduced to physicists as a convenient shorthand that simplifies expressions and calculations; for instance, it allows one to write a simple expression for the components of the cross product of two vectors v and w:

$$(v \times w)^i = \sum_{j,k=1}^{3} \bar{\epsilon}_{ijk} v^j w^k .$$

It also allows for a compact expression of the quantum-mechanical angular momentum commutation relations:

$$[L_i, L_j] = \sum_{k=1}^{3} i\,\bar{\epsilon}_{ijk} L_k .$$

Despite its utility, however, the Levi–Civita symbol is rarely given any mathematical or physical interpretation, and (like tensors more generally) ends up being something that students know how to *use* but don't have a feel for. In this example we'll show how our new point of view on tensors sheds considerable light on both the mathematical nature and the geometric interpretation of the Levi–Civita symbol.

To begin, let's define a rank-three tensor, denoted ϵ, where ϵ eats three vectors u, v, and w and produces a number $\epsilon(u,v,w)$. We'd like to interpret $\epsilon(u,v,w)$ as the (oriented) volume of the parallelepiped spanned by u, v, and w; see Fig. 1.1. From vector calculus we know that we can accomplish this by defining

$$\epsilon(u,v,w) \equiv (u \times v) \cdot w. \tag{1.2}$$

[1] Usually the "–" across the top is omitted, but we will need it to conceptually distinguish the Levi–Civita symbol from the epsilon tensor defined below.

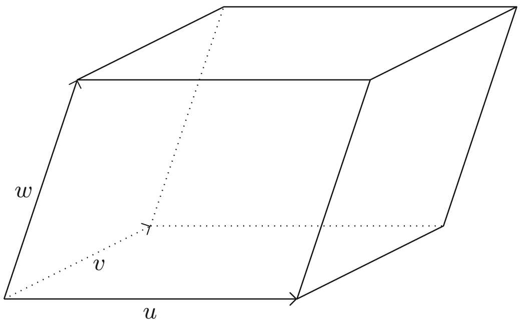

Fig. 1.1 The parallelepiped spanned by u, v, and w

For ϵ to really be a tensor, however, it must be ***multilinear***, i.e. linear in each argument. This means

$$\epsilon(u_1 + cu_2, v, w) = \epsilon(u_1, v, w) + c\epsilon(u_2, v, w) \tag{1.3a}$$

$$\epsilon(u, v_1 + cv_2, w) = \epsilon(u, v_1, w) + c\epsilon(u, v_2, w) \tag{1.3b}$$

$$\epsilon(u, v, w_1 + cw_2) = \epsilon(u, v, w_1) + c\epsilon(u, v, w_2) \tag{1.3c}$$

for all numbers c and vectors u, v, w, v_1, etc. Let's check that (1.3a) holds:

$$\begin{aligned}\epsilon(u_1 + cu_2, v, w) &= ((u_1 + cu_2) \times v) \cdot w \\ &= (u_1 \times v + cu_2 \times v) \cdot w \\ &= (u_1 \times v) \cdot w + c(u_2 \times v) \cdot w \\ &= \epsilon(u_1, v, w) + c\epsilon(u_2, v, w).\end{aligned}$$

Thus ϵ really is linear in the first argument. The check for (1.3b) and (1.3c) proceeds similarly and is left as an exercise. Thus, ϵ satisfies our definition of a tensor as a multilinear function. But, how does this square with our usual notion of a tensor as a set of numbers with some specified transformation properties? We claimed above that these numbers, known as the *components* of a tensor, are nothing but the tensor evaluated on sets of basis vectors. So, let's evaluate ϵ on three arbitrary basis vectors. Ordinarily, a basis vector is one of either $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, or $\hat{\mathbf{z}}$, but for the purposes of this example it will be easier to call these e_1, e_2, and e_3, respectively. We'll arbitrarily choose three of them (since ϵ is rank three) and call these choices e_i, e_j, and e_k, where it's possible that i, j, and k are not all distinct. Then we leave it as an exercise for you to check, using (1.2), that

$$\epsilon(e_i, e_j, e_k) = \begin{cases} 0 & \text{unless } i \neq j \neq k \\ +1 & \text{if } \{i, j, k\} = \{1, 2, 3\},\ \{2, 3, 1\},\ \text{or } \{3, 1, 2\} \\ -1 & \text{if } \{i, j, k\} = \{3, 2, 1\},\ \{1, 3, 2\},\ \text{or } \{2, 1, 3\}. \end{cases} \tag{1.4}$$

If, as mentioned above, we then define the ***components*** of the ϵ tensor to be the numbers

$$\epsilon_{ijk} \equiv \epsilon(e_i, e_j, e_k), \tag{1.5}$$

then (1.4) tells us that the components of the ϵ tensor are nothing but the Levi–Civita symbol! This is a major shift in perspective, and tells us several things. First:

1. The Levi–Civita symbol is not merely a mathematical convenience or shorthand; it actually represents the components of a tensor, the volume tensor (or ***Levi–Civita tensor***).

Furthermore, Eq. (1.5) tells us that:

2. **The components of a tensor are just the values of the tensor evaluated on a corresponding set of basis vectors**.

Combining 1 and 2 above then gives the following geometric interpretation of the Levi–Civita symbol:

3. ϵ_{ijk} **is the volume of the oriented parallelepiped spanned by** e_i, e_j, **and** e_k.

Another property worth noting is that the definition (1.2) of ϵ does not require us to choose a basis for our vector space. This makes sense, because ϵ computes volumes of parallelepipeds, which are geometrical quantities which exist independently of any basis. We can thus add a fourth observation to our list:

4. **The** ϵ **tensor exists independently of any basis**. This is in contrast to its components, which by (1.5) are manifestly basis-dependent.

While all this may illuminate the nature of the Levi–Civita symbol, and tensors more generally, we still don't know that the ϵ_{ijk} as defined here "transform" in the manner specified by the usual definition of a tensor. We'll see how this works in our next example, that of a generic rank-two tensor.

Exercise 1.1. Complete the proof of multilinearity by verifying (1.3b) and (1.3c), using the definition (1.2). Also use (1.2) to verify (1.4).

Example 1.2. *A generic rank-two tensor*

In this example we'll analyze a generic rank-two tensor, using our modern definition of a tensor as a multilinear function. This new viewpoint will clear up some of the pervasive and perennial confusion related to tensors, as we'll see.

Consider a rank-two tensor T, whose job it is to eat two vectors v and w and produce a number $T(v, w)$. In analogy to (1.3), multilinearity for this tensor means

$$\begin{aligned} T(v_1 + cv_2, w) &= T(v_1, w) + cT(v_2, w) \\ T(v, w_1 + cw_2) &= T(v, w_1) + cT(v, w_2) \end{aligned} \tag{1.6}$$

for any number c and all vectors v and w. An important consequence of multilinearity is that if we have a coordinate basis for our vector space, say $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, then