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## Chapter 3

# Conceptual Models of Legged Locomotion

Justin Seipel, Matthew Kvalheim, Shai Revzen, Maziar A. Sharbafi, and André Seyfarth

This chapter provides an overview of simple conceptual models of locomotion at the scale of whole body movements. First, conceptual models of locomotion are introduced along with a few key empirical observations that support the construction of simple conceptual models. Next, a theoretical perspective is offered based on “templates and anchors” theory, where templates are related to simple conceptual models. Commonly used models of legged locomotion are then presented: The Spring-Loaded Inverted Pendulum (SLIP) model of running and the Inverted Pendulum (IP) model of walking. Legged locomotion is next presented in terms of oscillatory behavior and oscillatory-based analysis. Finally, readers are taken on a tour of a “model zoo” featuring many extensions of the SLIP and IP models to more complex and realistic models.

## A Role for Simple Conceptual Models

Justin Seipel

*Purdue University, West Lafayette, IN, United States*

Legged locomotion of humans and other animals relies on a currently incomprehensible complex of underlying physiological systems. Though we have learned a lot about what is happening inside the body when humans or other animals move, we remain far from a coherent and complete understanding of how all the underlying processes integrate and contribute to whole-body motion.

Despite the overwhelming complexity of the internal processes of legged locomotion, the overall behavior on the level of whole body motion has remarkable coherence and regularity that can be understood using measures and models at the whole-body scale. As a complimentary approach to the direct study of the

full complexity of locomotion, it can be helpful to develop relatively simple conceptual models that capture the overall, whole-body characteristics of locomotion. These models may also be more likely to be tractable mentally and mathematically. Further, many simple conceptual models can also be related to physical experiments and corresponding mathematical governing equations that provide powerful capabilities of prediction and scientific analysis. Such models tend to be simple in the sense that they often have a small number of elements and degrees of freedom. They nonetheless often exhibit nonlinear dynamic behavior that requires sophisticated investigation and analysis. Also, the way these models relate to and are applied to biological and robotic systems often requires sophistication.

In this chapter we present conceptual models of whole-body locomotion. Further, these models are shown to be related to physical observations and experiments, as well as mathematical governing equations based on physical laws of motion. Subchapter 3.1 provides an introduction to conceptual models of locomotion and key empirical observations that support simple conceptual model building on a scientific basis. Subchapter 3.2 provides a perspective on “templates and anchors” theory and how it relates to simple conceptual models. Subchapters 3.3 and 3.4 present the Spring-Loaded Inverted Pendulum (SLIP) model of running and the Inverted Pendulum (IP) model of walking. Subchapter 3.5 introduces locomotion in terms of oscillatory behavior and related approaches to analysis. Finally, Subchapter 3.6 presents a “model zoo” featuring many extensions of the SLIP and IP models.

## Chapter 3.1

# Conceptual Models Based on Empirical Observations

Justin Seipel

*Purdue University, West Lafayette, IN, United States*

## 3.1.1 OBSERVING, IMAGINING, AND GAINING INSIGHTS INTO LOCOMOTION

Legged locomotion is in many ways familiar to us. Consider human running, as shown in [Fig. 3.1.1](#). We can recognize the scene of a human in motion, even if it is only a snapshot in time. We can likely recognize the basic anatomical segments of the trunk and limbs as well as the basic patterns of movement that are exhibited. We are likely able to form a kind of mental model of locomotion. Perhaps we can even conjure mental images or a movie related to the overall motions we observe when others run, or the experiences we have when we run.

Despite the familiarity of locomotion, more aspects may remain fuzzy or even foreign to us and become apparent only when explored further with trained observations and/or special tools and techniques. Questions may also help guide observations further. We might ask ourselves: Do we know what is happening when we move? Can we provide an explanation for it? Can we build a system that moves like we do? Do we know how major parts and processes of the body integrate into a coherent movement pattern?

Here we seek to develop a conceptual understanding of locomotion that includes and goes beyond our everyday observations. We also seek to provide a modeling framework that has predictive and design-aiding capabilities. Towards achieving these goals, it can be helpful to develop models of locomotion that provide both simple conceptual understanding as well as clear relationships to physical systems and physical laws.

## 3.1.2 LOCOMOTION AS A COMPLEX SYSTEM BEHAVIOR

There are significant challenges to developing scientific theories and models of legged locomotion. Movement in humans and other animals relies on a complex integration of skeletal, muscular, neurological, and other physiological systems. The skeletal system is organized with many complex joints, and multiple muscles are organized into groups and can span different numbers of joints and wrap in complex geometries. Also, the connectivity between neurons is beyond anything we understand. As complex as this anatomical perspective already is,



**FIGURE 3.1.1** Human locomotion. Modified photograph by William Warby, Heat 1 of the Womens 100 m Semi-Final. Cropped and converted to grayscale. CC BY 2.0.

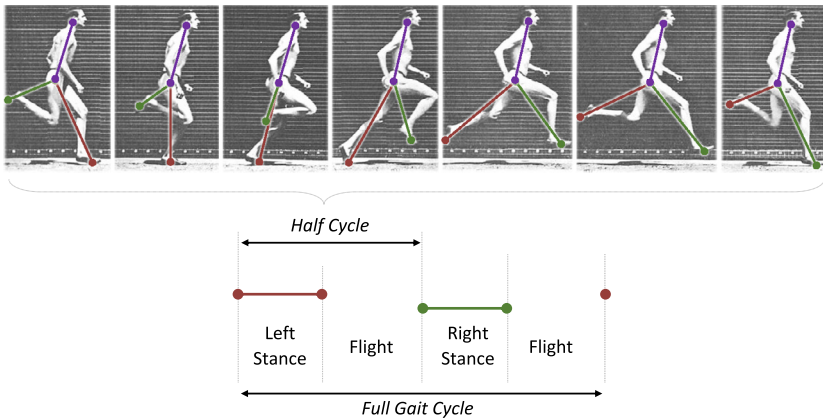
there remains additional complexity as seen from other perspectives, such as when considering information processes or feedback dynamics. Overall, it is a major challenge to derive simple models directly from the composition of a large number of biophysical parts and integrated processes.

An alternative and complimentary approach to simple model development is to use direct empirical study of overall, whole-body motion, as well as inspiration from mental models and intuition we may have. Other alternatives are possible too, such as attempting to model at an intermediate scale somewhere between the smallest underlying physiological processes and the whole body. All of these approaches can ultimately contribute to the development of a more comprehensive and integrative understanding of biological movement. For now, the focus of this chapter is on simple models that primarily capture overall whole-body movements of legged locomotion and that are related to physical experiments and physical laws.

### 3.1.3 SOME CHARACTERISTICS OF WHOLE-BODY LOCOMOTION

Models of legged locomotion can be conceived based on direct observations. Here, we focus on empirical observations of both the movement of the main body (trunk) and the corresponding movement of the legs, to reveal overall kinematic patterns that are characteristic of locomotion. Other measurements such as ground reaction forces and energetic consumption can be correlated with body movements to provide insights into kinetic processes influencing motion.

Overall movements of the body can be tracked from one or more points on the trunk, such as a marker on or near the hip (which is in the vicinity of the mass center in humans). The overall motion of the legs may be tracked relative to the



**FIGURE 3.1.2** Running sequence. Photographs by E. Muybridge. Markers added by eye.

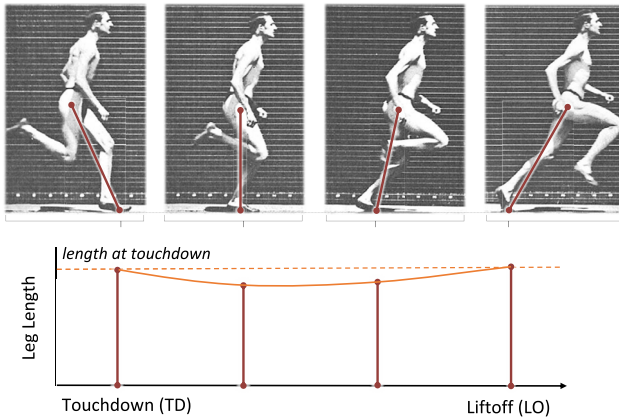
main body, such as by tracking points on the feet that help indicate whether a leg is in stance or swing, and where it is relative to the body.

Many observations of whole-body movement can be made with the unaided eye, but motion capture and tracking techniques have clarified what is otherwise fuzzy or too fast to see and has enabled significantly greater accuracy and quantification of movement (e.g., photographic techniques developed by E.J. Marey and E. Muybridge enabled and inspired new scientific and artistic works, [Marey, 1894](#); [Muybridge, 1979](#); [Silverman, 1996](#)).

An example of tracking and characterizing overall locomotion is provided in [Fig. 3.1.2](#). This illustration and analysis of human running is based on a sequence of Muybridge photographs. The original photographs have been modified with markers (dots) added manually, at the feet, hip, and top of the spine. These markers, connected by lines, indicate major segments of the body: the two legs and trunk. Legs are either functioning in stance (with foot on the ground) or in swing (with foot off the ground), where flight phases of motion occur when both legs are off the ground. The three segments or parts identified here—the trunk, and two legs in stance or swing—relate to differentiated “subfunctions” that integrate together into whole movement. This concept was introduced in [Chapter 2](#) from a motion control perspective, and is discussed again here from the complimentary perspective of three anatomical parts: trunk, stance leg, and swing leg.

### 3.1.3.1 The Trunk: Bouncing Along

During running, the trunk of the body appears to bounce along (here marked by the hip and top of spine): The trunk bounces or oscillates vertically between the



**FIGURE 3.1.3** Stance leg length shortening and lengthening.

action of gravity and stance leg forces, all the while making forward progress. Further, the trunk tends to be angled forward of vertical (averaging about 15 degrees in Fig. 3.1.2), and appears to be regulated such that it oscillates slightly about this average. In many cases, locomotion model development is focused on the translational movement of the body and in those cases rotations of the trunk are not included as one may focus on developing a “point-mass” model of the body. An example of a point-mass model of the body is provided in Subchapter 3.3.

### 3.1.3.2 The Stance Leg: Acting Like a Spring

During the stance phase of running, as shown in Fig. 3.1.3, the stance leg length shortens (compresses) and then lengthens (decompresses) while it pivots about the foot. Observations of ground reaction forces show that the direction of force is significantly aligned with the leg (though not entirely) and that the force magnitude  $F$  changes with leg length, emulating Hooke’s Law (Blickhan, 1989; Blickhan and Full, 1993):

$$F = k(l_0 - l).$$

Here,  $k$  is an effective leg spring stiffness,  $l_0$  is its resting length, and  $l$  is the leg length. This emulation of an effective leg spring has been observed across many species, including poly-pedal locomotion where multiple legs can act together as a single virtual leg-spring (Blickhan and Full, 1993). Further, effective leg stiffness behavior has been demonstrated during walking (Geyer et al., 2006). Though these observations provide us with a helpful model of the overall functional behavior of stance legs, and elastic tissues do exist in legs, actual animal

legs do not store and return energy the same way as an idealized spring. Animal legs generally require significant energy to operate.

### 3.1.3.3 The Swing Leg: Recirculating for Touchdown

During locomotion, the swing leg recirculates in order to be placed down at the next touchdown. For this, the prevailing movement of the swing leg is a swinging motion to a position forward of the body. This motion resembles the swinging of a pendulum. In addition to the forward swinging movement, the swing leg also goes through a significant retraction, both at the beginning and at the end of the swing phase: See [Fig. 3.1.2](#). While swing legs can move like a passive pendulum under their own weight, swing legs are also likely to be controlled. Simple models of swing legs could include some combination of passive pendulum-like dynamics and active leg placement control. One highly simplified swing leg model that has been used often results from assuming the swing leg mass is negligible compared with the main body mass, and that the swing leg angle is controlled to follow a prescribed trajectory (as simple as a constant angle) until touchdown. An example of this is provided in [Subchapter 3.3](#).

### 3.1.4 WHOLE-BODY CONCEPTUAL MODELS AS AN INTEGRATION OF PARTS OR SUBFUNCTIONS

An overall simple conceptual model of locomotion can be arrived at through integrating simple models/functions of the trunk (or main body), stance leg, and swing leg into a whole system. Further, by deriving governing equations of the whole system based on physical laws, we can predict its locomotion behaviors. Later, we present two well-established models of locomotion: the Spring-Loaded Inverted Pendulum (SLIP) model of running in [Subchapter 3.3](#), and the closely associated Inverted Pendulum (IP) model of walking in [Subchapter 3.4](#).

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## Chapter 3.2

# Templates and Anchors

Matthew Kvalheim and Shai Revzen

*University of Michigan, Ann Arbor, MI, United States*

## 3.2.1 A MATHEMATICAL FRAMEWORK FOR LEGGED LOCOMOTION

In this section we present a mathematical framework for analysis and modeling of legged locomotion. This framework is, for most applications, far too general. However, it will serve to provide a precise mathematical foundation, inside which other more practical models and approaches appear as special cases.

The study of legged locomotion is the study of how bodies move through space by deforming appendages we refer to as “legs” and using them to produce reaction forces from the environment that propel the body. Thus, the configuration of the system we seek to study comprises two parts—the location of the body in space, and the “shape” of that body with respect to a frame of reference that travels with the body. In mathematical terms, this means the overall configuration space  $\mathcal{Q}$  is:

$$\mathcal{Q} = \text{SE}(3) \times \mathcal{B}, \quad \mathcal{B} \subseteq \mathbb{R}^m \quad (3.2.1)$$

where  $\text{SE}(3)$  is the “special Euclidean group of dimension 3”, also known as “the space of rigid motions”, and  $\mathcal{B}$  is taken to be some bounded, continuous, closed, piecewise smooth surface in the space  $\mathbb{R}^m$ . Let us temporarily use  $q = (g, b) \in \mathcal{Q}$  to denote the instantaneous configuration.

In this book we are primarily concerned with legged locomotion that is generated by repeating patterns of motion called “gaits.” When an animal or robot executes a gait, it traces out a cycle with  $b$  in the shape space  $\mathcal{B}$ , while at the same time translating and/or rotating the body frame  $g \in \text{SE}(3)$  through the world. This form of a mathematical structure, in which a space is given by the Cartesian product of a “base space” (in our case, the shape space of the body) and a group,<sup>1</sup> here  $\text{SE}(3)$ , is called a (trivial) “principal fiber bundle”,<sup>2</sup> or simply a “principal bundle.” Subsets of  $\mathcal{Q}$  of the form  $\text{SE}(3) \times \{b\}$  for a fixed  $b \in \mathcal{B}$  are

1. More technically, the group is required to be a “Lie group”, and each fiber is a “principal homogeneous space” for this Lie group. Readers interested in these technicalities may consult, for example, [Steenrod \(1951\)](#), [Husemoller \(1994\)](#).

2. This bundle is called “trivial” because it is *equal* to a product  $\text{SE}(3) \times \mathcal{B}$ , whereas in general fiber bundles are spaces which are *locally trivial*, i.e., locally a product in some sense. See, for example, [Bloch et al. \(2003\)](#), [Husemoller \(1994\)](#), [Steenrod \(1951\)](#).

called “fibers.” A very readable introduction to the theory of fiber bundles may be found in Chapter 2 of Bloch et al. (2003).

In physics, principal bundles have been used to describe diverse phenomena in which cycles in the base space can be associated with a shift along a fiber. Names for some phenomena in the literature associated with this concept include “Berry phase”, “geometric phase”, “dynamical phase”, “Pancharatnam phase”, and “holonomy.”

In the study of locomotion, these ideas have been used to describe the maneuvers cats (Marsden et al., 1991) and geckos (Jusufi et al., 2008) use to land on their feet, and the choice of undulatory motions made by snakes and eels (Ostrowski and Burdick, 1998; Hatton and Choset, 2011). When the relationship between shape change and body frame velocity is linear, it is given by a “connection”:

$$g^{-1}\dot{g} = A(g, b)\dot{b} \quad (3.2.2)$$

While technical issues and high dimensions of the models create significant difficulties in applying “geometric mechanics” approaches in practice, this theoretical framework can in principal describe legged systems.

### 3.2.2 TEMPLATES AND ANCHORS: HIERARCHIES OF MODELS

One of the most influential insights allowing legged locomotion systems to be analyzed in practice was articulated in Full and Koditschek (1999), which proposed the use of “templates” for generating refutable, testable hypotheses for legged locomotion. While a “template” is defined as “*the simplest model (least number of variables and parameters) that exhibits a targeted behavior*”, the discussion and more recent treatments of the templates-and-anchors approach follow more closely the concept outlined in Full and Koditschek (1999) on page 3329: “*We will say that a more complex dynamic system is an ‘anchor’ for a simpler dynamic system if (1) motions in its high-dimensional space ‘collapse’ down to a copy of the lower-dimensional space of motions exhibited by the simpler system and (2) the behavior of the complex system mimics or duplicates that of the simpler system when operating in the relevant (reduced-dimensional copy of) motion space.*” In other words, animals have many degrees of freedom, but move “as if” they have only a few, and limit pose to a behaviorally relevant family of postures. One way to encapsulate this insight mathematically is to presume that animals occupy only a low-dimensional “behaviorally relevant” submanifold of  $\mathcal{B}$ , the space of possible “poses.” As an illustrative example of what this means in practice, consider a photo of a galloping horse. We know that

the horse is galloping, because the pose (“shape”) of the body that we see in that still image is one which is only used for galloping. In fact, there is a cycle of poses that is associated with that horse galloping, and if environmental circumstances contrive to perturb the horse’s body away from appropriate shapes for galloping, it quickly returns to some appropriate galloping pose.

However, the insight extends further: trotting quadrupeds such as horses and dogs, running bipeds such as humans and ostriches, insects like cockroaches employing alternating tripod gaits, and even running decapods like ghost crabs all employ similar center of mass dynamics—the “Spring Loaded Inverted Pendulum (SLIP)” (Dickinson et al., 2000; Blickhan, 1989). All these organisms exhibit similar center of mass dynamics: in each step, they bounce like a pogo stick. The center of mass slows down while descending closer to the ground, reaching its minimum speed at its lowest altitude, while ground reaction force in the normal direction is maximal. The center of mass continues, speeding up as it rises until the body entirely detaches from the ground into an aerial phase of ballistic motion leading to the next step.

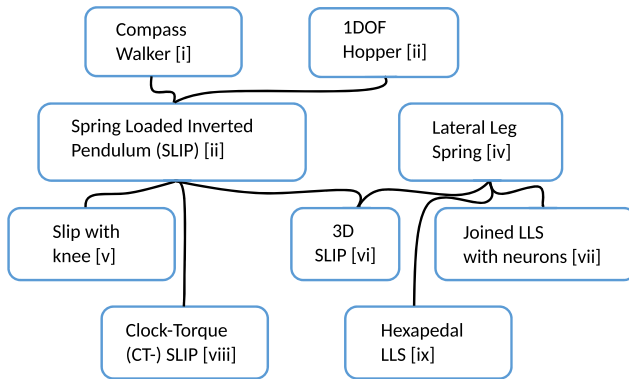
In this sense, the SLIP template represents a common governing feature appearing in many organisms when they run quickly. The template is not only a description of a typical subset of poses, but also a low dimensional dynamical model that captures features of the aggregate behavior of the body.

It would be tempting to assume that for every behavior or animal examined, there exists a specific “simplest” template model that governs that behavior. However, Full and Koditschek (1999) had already pointed out that the notion of “simplest” model is problematic, and that both the Lateral Leg Spring (LLS) and the Spring Loaded Inverted Pendulum (SLIP) are templates for running ( $H_3$ ,  $H_4$  in Full and Koditschek, 1999, Table 1). The specific formal definition of a “template” was left vague.<sup>3</sup>

As an illustrative example, both the “Clock-Torque (CT)-SLIP” (Seipel and Holmes, 2007) and “SLIP with knee” (Seyfarth et al., 2000; Rummel and Seyfarth, 2008) models may be considered to be anchors for the classical sagittal plane spring loaded inverted pendulum (SLIP) model; important features of this model can be further distilled (following Blickhan, 1989) into a vertical hopping model, or alternatively into a compass walker (Usherwood et al., 2008). The three-dimensional pogo stick-like SLIP template (Seipel and Holmes, 2005) may also be viewed as an anchor for the sagittal plane SLIP, but one may be equally justified in reducing this three-dimensional pogo stick to a horizontal plane Lateral Leg Spring (LLS) model (Schmitt and Holmes, 2000a, 2000b)

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3. This was intentional, based on personal communication with each of the authors.



**FIGURE 3.2.1** A collection of locomotion models with their template–anchor relationships indicated, showing a partial order structure. [i] Usherwood et al. (2008), [ii] & [iii] Blickhan (1989), [iv] Schmitt and Holmes (2000a, 2000b), [v] Rummel and Seyfarth (2008), Seyfarth et al. (2000), [vi] Seipel and Holmes (2005), [vii] Seipel et al. (2004), [viii] Seipel and Holmes (2007), [ix] Kukillaya and Holmes (2007).

which captures aspects of the horizontal motion such as steering, but ignores the importance of vertical bouncing. Additionally, both the “hexapedal lateral leg spring” (Kukillaya and Holmes, 2007) and “jointed lateral leg spring with neurons” (Seipel et al., 2004) models are extensions of the classical LLS which may be viewed as a template for these models. This hierarchy is depicted in Fig. 3.2.1.

We are led to the conclusion that rather than a template being a unique, ultimate object, “template and anchor” is a relationship between models. A given model Y can be a template for a more anchored model X, while Y itself may be an anchor for a further template Z. We will use the term “template” to imply that this model is “simpler” than its “anchor.” Usually, one aspect of this simplicity is a reduction of dimension, and quantities in a template often represent aggregates of quantities from the underlying anchor. For example, both SLIP and LLS reduce the mass distribution of the body to a concentrated mass with or without rotational inertia; both discard modeling the kinetic energy and momentum associated with the legs themselves. An insightful discussion on the design and control of legged robots using template–anchor notions is given in Blickhan et al. (2007).

In the remainder of this chapter we will discuss several of the ways in which a template-and-anchor hierarchy can be constructed to facilitate the understanding of legged locomotion.

As a cautionary note, it should be pointed out that the term “template” has sometimes been used to mean “a spring mass model of center of mass dynamics.” In this book, we will use it in the much broader meaning described above.

### 3.2.3 TEMPLATES IN DYNAMICS, CONTROL, AND MODELING

There are several ways to approach template–anchor relationships which have been used successfully. Mathematicians and physicists studying dynamical systems theory have constructed a variety of notions of dimensionality reduction. From this perspective, the primary object of study is an elaborate mathematical “anchor” model comprising a set of equations, the solutions of which are shown to be approximately or exactly modeled by a simpler “template” model comprising fewer equations with fewer parameters. Several examples of this approach can be found in [Holmes et al. \(2006\)](#). In particular, [Kukillaya and Holmes \(2007\)](#) have shown an example of a hexapedal cockroach model with jointed legs and neuronal control, which can be formally reduced and shown to behave similarly to the far simpler LLS model.

Engineers building robots have looked to templates as “targets of control”, i.e., as descriptions of desirable behaviors to be emulated ([Westervelt et al., 2003](#); [Revzen et al., 2012](#); [Ames, 2014](#)), or as simplifications to be used for quickly estimating an appropriate control policy ([Raibert et al., 1984](#)). Here, the primary object of study is not the template itself, as much as it is the means by which template dynamics are elicited from an anchor.

A more explicit focus on templates is found in work by engineers employing “template-based” strategies for the bio-inspired design and control of robots. Here, the goal is to embed well-known templates in more complex, anchored locomotion systems. Controllers have been designed to embed the dynamics encoded in SLIP and its three-dimensional analog in bipedal robots ([Wensing and Orin, 2014](#); [Dadashzadeh et al., 2014](#)). In [Poulakakis and Grizzle \(2009\)](#), the SLIP model is explicitly mathematically embedded as the “hybrid zero dynamics” of an asymmetric version of the SLIP model. [Ankarali and Saranli \(2011\)](#) have used an extended SLIP model involving torque actuation at the hip for designing a controller to achieve underactuated planar pronking in the robot RHex ([Saranli et al., 2001](#)). Other researchers have considered the combination of several different templates in the same robot in order to render it capable of achieving multiple goals, such as running/climbing ([Miller and Clark, 2015](#)) and running/reorientation/vertical hopping ([De and Koditschek, 2015](#)). [Lee et al. \(2008\)](#) have even worked to embed a cockroach-inspired antenna-based wall-following template in a robot with a bio-inspired antenna.

Biomechanists have looked to templates from a data-driven, experiment-centric perspective. Here, the primary objects of study are the locomotion data themselves. The goal is to find low-dimensional models which represent observational data, accounting for both trends and variability with a few meaningful parameters. Data-driven templates have been used successfully to predict how

animals recover from perturbations (Revzen et al., 2013) and how humans control and stabilize their running gait (Maus et al., 2014). These ideas are elaborated upon in Section 3.2.4.3.

### 3.2.4 SOURCES OF TEMPLATES; NOTIONS OF TEMPLATES

Given the three approaches to templates described in the previous section, it is hardly surprising that there are many mathematical notions of being a template-and-anchor pair. In this section we point to some of the literature in the field. The subtle differences and technical caveats associated with applying these notions are outside the scope of our exposition.

#### 3.2.4.1 Dimensionality Reduction in Dynamical Systems

As a simple example, templates exist in the dynamics of linear systems (see, e.g., the textbook of Hirsch and Smale, 1974, for an introduction to linear systems). When a stable Linear Time Invariant (LTI) system of differential equations  $\dot{x} = Ax$  has a large “spectral gap”—some modes (projections of solutions onto the generalized eigenspaces of  $A$ ) collapse much faster than others—the slower modes can justifiably be viewed as a template for the complete higher-dimensional system. This expresses itself as a large difference in the real part of the eigenvalues of the matrix  $A$ , with the eigenvalues corresponding to slow template modes having a real part close to zero.

Dynamicists have extended this idea to nonlinear systems in multiple ways using the notion of “invariant manifolds”, of which the generalized eigenspaces in the previous example are a special case. A positively (negatively) invariant manifold is a smooth submanifold of the state space of a dynamical system for which any initial condition belonging to this submanifold remains in the submanifold as it evolves forward (backward) in time. An invariant manifold is a smooth submanifold of the state space of a dynamical system which is both negatively and positively invariant; in other words, an invariant manifold is a union of trajectories. (Positively) invariant manifolds are often useful notions of templates—here, the template appears in a form which guarantees that the anchor dynamics restricted to template states are invariant, meaning that if the anchor begins in a state belonging to the template it can no longer escape back to exhibiting more complex behaviors. An excellent survey of the many ways invariant manifold methods have been useful in science and engineering is given in Chapter 1 of Wiggins (1994).

One well-known class of invariant manifolds which can be used to form useful templates are the asymptotically stable normally hyperbolic invariant manifolds (NHIMs) (Hirsch et al., 1970; Eldering, 2013; Wiggins, 1994); by

“asymptotically stable”, we mean that they attract all nearby trajectories asymptotically. Special cases of NHIMs include hyperbolic fixed points and hyperbolic periodic orbits (Hirsch and Smale, 1974). A particularly nice property of NHIMs is that they persist under small smooth perturbations of the equations defining the dynamical system (Hirsch et al., 1970), and the compact invariant manifolds which persist under smooth perturbations are normally hyperbolic (Mané, 1978). This makes NHIMs useful from a modeling perspective. Since physical measurements cannot determine parameters of a mathematical model with perfect accuracy, any physically meaningful feature of a mathematical model must persist under small perturbations.

Viewed as infinite-dimensional dynamical systems, even certain partial differential equations admit a template-like structure both through the theory of normal hyperbolicity (Bates et al., 1998, 2000) and the related theory of “inertial manifolds”, the second class of (positively) invariant manifolds we mention here (Constantin et al., 2012; Foias et al., 1988b). Inertial manifolds, when they exist, are finite-dimensional positively invariant manifolds containing the global attractor of a (possibly infinite-dimensional) dynamical system and attracting all solutions at an exponential rate (Foias et al., 1988b). If an inertial manifold exists for a given partial differential equation, it governs the long-term dynamics. Examples of systems having an inertial manifold include dissipative systems such as those that appear in elasticity and fluid dynamics (Constantin et al., 2012). Techniques for computationally producing approximate inertial manifolds have been studied (Foias et al., 1988a).

“Center manifolds” are the third class of invariant manifolds we mention here. We briefly describe the most basic notion of center manifold at the level of generality relevant for our discussion; see, for example, the discussion in Section 3.2 of Guckenheimer (1983) for more details. Given a system of differential equations  $\dot{x} = f(x)$  and a stable equilibrium point  $x_0$  with  $f(x_0) = 0$ , the eigenvalues of the linearization  $Df(x_0)$  split into collections of eigenvalues having negative and zero real part. These collections of eigenvalues respectively determine stable and center subspaces. The center manifold theorem states that there exist “stable” and “center” invariant manifolds respectively tangent to these subspaces. Trajectories in the stable manifold approach  $x_0$  exponentially in positive time. While the stable manifold is always unique, in general the center manifold need not be. Center manifolds may also be defined for periodic orbits (see Theorem 4 of Section 3.5 in Perko, 2001) and more general attractors (Chow et al., 2000). Center manifolds and NHIMs have somewhat similar spectral properties, but they differ in that NHIMs have an intrinsic global definition whereas center manifolds are only defined locally. This local definition manifests itself in the fact that center manifolds are in general nonunique (see Section 1.1.2 of Eldering, 2013 for more discussion).

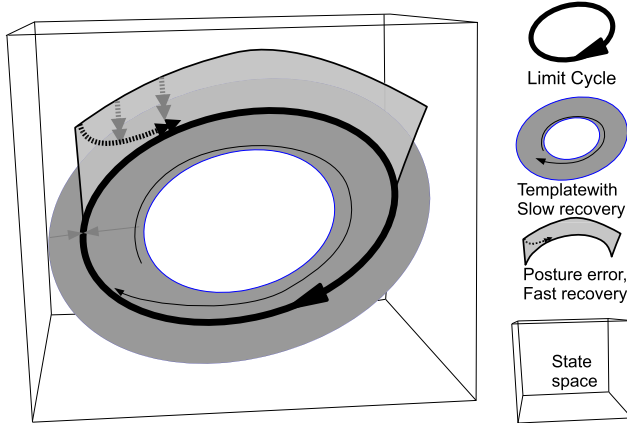
All of the classes of (positively) invariant manifolds we have mentioned have the property that they attract all nearby states. The template is stable in the sense that anchor states which are near template states will asymptotically approach the template. However, one important reason these notions of templates are so useful is more subtle than this; not only do nearby anchor states approach these invariant manifold templates, they approach *specific trajectories* in the template. This provides justification for the approximation of anchor dynamics by template dynamics. For inertial manifolds, this property is known in the literature as “asymptotic completeness” (Robinson, 1996), and the fact that center manifolds have this property is shown, for example, in Carr (1982). For NHIMs, this property is often noted by referring to the existence of an “invariant foliation” or “invariant fibration” of the basin of attraction of the invariant manifold (Hirsch et al., 1970), and is also sometimes referred to as “asymptotic phase” in the literature (Bronstein and Kopanskii, 1994) (we also refer to this as “dynamical phase” in Subchapter 3.5). Guckenheimer (1975) contains a simpler discussion of the properties of asymptotic phase for the special case of exponentially stable limit cycles.

As an illustrative example of the utility of invariant manifold notions of templates in the analysis of legged locomotion, consider an oscillator. As explained in Subchapter 3.5, an oscillator, by definition, consists of the dynamics in the basin of attraction of an exponentially stable periodic orbit (also known as a limit cycle).<sup>4</sup> The image of the periodic orbit, or set of points traced out by the limit cycle, is itself a normally hyperbolic invariant manifold. Defining the anchor to be the dynamics on the entire basin of attraction, a template may be taken to consist of the dynamics restricted to the image of the periodic orbit. Explicitly, the existence of asymptotic phase on the basin of attraction implies that each anchor state will asymptotically coalesce with a (in this case, unique) template state which may be represented by assigning to each anchor state a number  $\theta \in [0, 2\pi)$ . This is the “phase oscillator” template explained in Subchapter 3.5. However, for many practical applications, this particular template approximation of the anchor dynamics may be too coarse. As explained in Subchapter 3.5, the theory of normal forms (Bronstein and Kopanskii, 1994) shows that large “spectral gaps” in the “Floquet multipliers” of an oscillator yield additional invariant “slow manifolds” corresponding to slow “Floquet modes.” Anchor states will again asymptotically approach particular template trajectories in such a way that the dynamics restricted to such an invariant manifold constitutes a good template approximation of the anchor dynamics. Physically, the limit cycle may

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4. As in Subchapter 3.5, an “oscillator” is a *deterministic* system as defined here, while in Subchapter 3.5 we use the term “rhythmic system” to refer to a *nondeterministic* system resulting from perturbations of a (deterministic) oscillator by (“relatively small”) noise.

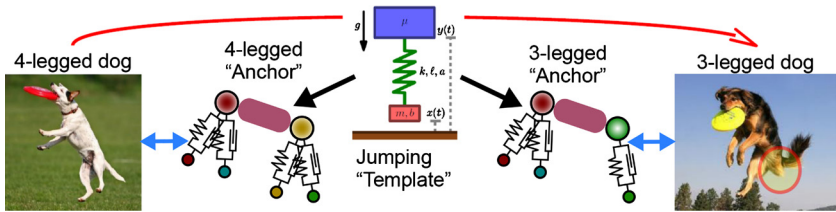




**FIGURE 3.2.2** An example of an invariant manifold template–anchor relationship in the context of modeling legged locomotion shape-space dynamics by an oscillator. The collection of states corresponding to “Floquet multipliers” with relatively large magnitude form an invariant “slow manifold.” The states belonging to this invariant manifold may be thought of as the states which return slowly to an unperturbed gait, modeled by the limit cycle. Taking the anchor to be the dynamics on the entire state space, the dynamics restricted to this invariant slow manifold may serve as a template. Alternatively, the dynamics restricted to the states traced out by the limit cycle itself may serve as a “phase oscillator” template which is a coarser approximation of the anchor dynamics.

be viewed as representing a perfectly periodic gait subject to no environmental or neuromuscular perturbations. The invariant slow manifold template may then be viewed as the collection of anchor states having “slow recovery” when perturbed from this steady gait. Any anchor states not belonging to this template will quickly return to the template and may be viewed as “posture errors.” This is illustrated in Fig. 3.2.2.

Yet another source of templates comes from mechanical models possessing symmetries. Roughly speaking, a differential equation is said to possess a “symmetry” if it is invariant under the action of a “Lie group” (Lee, 2012) on its state space. For the case of mechanical systems, reduction tools such as Noether’s Theorem from geometric mechanics (Abraham and Marsden, 1978; Bloch et al., 1996) yield conserved quantities (e.g., energy, momentum, angular momentum) which constrain trajectories of the dynamical system to lower-dimensional submanifolds. Dynamics restricted to these lower-dimensional submanifolds form templates for the original anchored mechanical system, and one can understand the behavior of the template in terms of the anchor and vice-versa. We note that other reduction methods in the spirit of Galois’ work on algebraic equations (Dummit and Foote, 2004) also exist for the analysis of ordinary differential equations possessing symmetries but not necessarily arising from mechanical systems (Olver, 2000).



**FIGURE 3.2.3** An uninjured dog and a 3-legged dog can both jump to catch a Frisbee. The ability to do so well can be expressed (red arrow) by taking the dynamics anchored in the 4-legged morphology, abstracting them as a jumping template and embodying this template in a 3-legged anchor reflecting the new morphology. The quality of this abstraction and embodiment can be quantified in a formal way within the framework of approximate bisimulation.

An even greater focus on templates as approximations can be found in the theory of “bisimulation” appearing in its original form in the study of discrete state transition systems in computer science (Park, 1981). Intuitively, two systems are bisimilar if they cannot be distinguished by an “external observer.” Bisimulation has been generalized to apply to continuous-time and hybrid dynamical systems. In fact, the template notions previously mentioned in this section are bisimulations of their anchor dynamics, which follows from Proposition 11 of Haghverdi et al. (2005). Bisimulation provides a formalism for discussing templates and anchors for situations more general than the case in which the template is an invariant submanifold of the anchor.

Despite the level of generality afforded by the framework of bisimulation, requiring bisimilarity between models as a criterion for template–anchor relationships can sometimes arguably be too restrictive for modeling physical systems. Bisimilarity relations are not necessarily robust to noise, measurement error, or other perturbations to physical models. Recent work has extended the notion of bisimilarity by providing a definition of “approximate bisimulation” (Girard and Pappas, 2007). The utility of approximate bisimulation lies in its ability to quantify the quality of approximation by one mathematical model of another. In particular, the language of approximate bisimulation can be used to quantify the degree to which some mathematical model is a template for another anchored model. As a simple example, a double pendulum with one small mass  $m$  and one large mass  $M$  can be approximated by a single pendulum of mass  $M + m$ . For a more interesting example, consider the following. Animals, such as dogs, are able to instantiate the same template despite seemingly catastrophic injury such as limb loss. Fig. 3.2.3 illustrates the approximate template–anchor relationships between relevant models for this case. These examples are hardly surprising from the perspective of mechanical intuition, but the theory of approximate bisimulation renders these observations formal, testable, and quantifiable in a computational framework.

### 3.2.4.2 Templates Based on Mechanical Intuition

By far, the most prolific source of models intended as templates has been the insight of researchers. As described in Section 3.2.2, the insight of Blickhan led to the introduction of the SLIP template in his seminal work (Blickhan, 1989). Despite being an energetically conservative model without control inputs, the SLIP has enjoyed enormous success in making tractable the tasks of animal locomotion analysis (Section 3.2.2) and robot design and control (Section 3.2.3). The success of the sagittal plane spring-loaded inverted pendulum as a mathematical model inspired various extensions of SLIP, such as CT-SLIP (Seipel and Holmes, 2007), as well as three-dimensional (Seipel and Holmes, 2005), bipedal (Geyer et al., 2006), and segmented versions of SLIP (Seyfarth et al., 2000; Rummel and Seyfarth, 2008). Other templates such as LLS (Schmitt and Holmes, 2000a, 2000b) and its extensions (described in Section 3.2.2) such as a model with additional joints and neuronal interactions (Seipel et al., 2004) and a hexapedal version of LLS (Kukillaya and Holmes, 2007) were developed to specifically capture the horizontal component of locomotion. Thoughtful consideration of modeling has produced a plethora of additional templates of varying levels of complexity appropriate for other situations. Inspired by the climbing aptitude of insects and geckos, Goldman et al. (2006) proposed a template for describing rapid vertical climbing. Observations of cockroaches using their antennae to follow walls motivated the introduction of an antenna-based wall following template (Cowan et al., 2006). Human walking inspired a template based on the notion of “virtual pivot points” (Maus et al., 2010). Examples of other templates proposed for specific classes of models include a quadrupedal running template for robotic systems with articulated torsos (Cao and Poulakakis, 2013) and a kinematic template proposed for an eight-legged miniature octopedal robot assumed to be in quasi-static motion (Karydis et al., 2015).

Typically, templates have been proposed without explicitly formulating the anchor model to which they relate, although there are exceptions. In other work, there is an emphasis placed on exploring relationships between various templates and their anchors. To name but a few examples, see Seipel and Holmes (2005, 2006), as well as Chapter 5 of Holmes et al. (2006) and the references therein.

### 3.2.4.3 Data-Driven Model Reduction

Data-driven dimensionality and model reduction has emerged as an industrious and interdisciplinary field of research, having broad applications to the science and engineering fields and drawing upon techniques from optimization, statistics, dynamical systems theory, and machine learning. Classical approaches

to dimensionality reduction include linear subspace projection methods such as “principal component analysis” and “factor analysis” (Jolliffe, 2002); one active area of current research concerns nonlinear dimensionality reduction approaches such as “manifold learning” (Lee and Verleysen, 2007), which generalize linear projection methods by replacing linear subspaces with submanifolds. Projection methods such as these identify a small (relative to the dimensionality of the raw data) collection of parameters which may accurately represent the raw data, and this collection of parameters is optimal in some sense depending on the projection method used. Such a small parameter set may accurately capture the spatial information present in time series data and motivate the construction of reduced-order spatio-temporal mathematical models. Givon et al. (2004) contains a review of several other algorithmic approaches to dimensionality reduction, focusing on methods specifically aimed at model reduction of general dynamical systems.

In the context of legged locomotion, there has been work on the construction of templates directly motivated from data. Operating under the assumption that the underlying mathematical model is an oscillator (see Subchapter 3.5), several researchers have performed nonparametric system identification of biomechanical systems (Ankarali, 2015; Wang, 2013; Revzen, 2009; Hurmuzlu and Basdogan, 1994; Hurmuzlu et al., 1996). Researchers have additionally attempted to find nonlinear coordinate systems directly from data in which oscillator dynamics are linear (see Revzen and Kvalheim, 2015, and references therein for more mathematical detail), and have coined the term “Data-Driven Floquet Analysis” (DDFA) to collectively refer to the computation of this linearizing coordinate system and to other oscillator system identification methods (Revzen, 2009). The linearizing coordinate change of DDFA can be viewed as a special case of finding linearizing “observables”, which are themselves eigenfunctions of the “Koopman operator” (Rowley et al., 2009; Koopman, 1931), and may in some cases be computed using “Dynamic Mode Decomposition” (Schmid, 2010) and its extensions. Using the techniques of DDFA, Revzen and Guckenheimer (2011) present a method for identifying appropriate dimensions of reduced-order models of legged locomotion and other rhythmic systems directly from noisy data and without explicit knowledge of governing equations. By exploiting the structure of the stability basin of an oscillator, they determine a candidate dimension for the slow manifold by examining the magnitudes of eigenvalues of Poincaré return maps. This candidate dimension serves as an upper bound for the dimension of a statistically significant template.

In the specific context of human walking, Wang used analysis of Poincaré maps to show a relationship between upper body/trunk motion and foot placements, providing a rigorous data-driven derivation of human walking features

previously conjectured (Wang, 2013). Maus et al. (2014) performed DDFA on human running data and showed that while the SLIP template predicts within-step kinematics of the center of mass, it fails to predict stability and behavior beyond one step. Furthermore, insights derived from DDFA enable Maus et al. (2014) to identify that swing-leg ankle states are important predictors of human locomotion beyond those present in the SLIP template. Augmenting the SLIP model with these predictors, the authors construct a model shown to have predictive power superior to SLIP for the available subject population.

### 3.2.5 CONCLUSION

The answer to the question of “which notion of template–anchor relationship should be used?” depends on one’s goals and on practical limitations of the application in mind.

As an example of one end of the spectrum, mathematicians wanting to explore mathematical relationships need to write down or use existing equations of motion, which may make many assumptions about the underlying physics and/or biology of a locomoting system. In this case, various templates may be amenable to discovery by theoretical consideration. For example, invariant manifolds may be found “by hand” or numerical methods. Alternatively, reduction tools from geometric mechanics and the theory of Lie groups may be used to produce templates if symmetries are present in the equations of motion. Notions such as bisimulation and approximate bisimulation from computer science are used to formalize template–anchor notions in some areas of the literature.

On the opposite end of the spectrum, experimental biologists deal with actual data and do not have access to explicit mathematical models *a priori*. For this reason, researchers have worked on data-driven methods of system identification and model reduction. As outlined in Section 3.2.4.3, many algorithms have been used in attempts to tackle this problem for real-world systems in general, and several researchers have worked on methods aimed specifically toward legged locomotion. In particular, there has been some success in using Data Driven Floquet Analysis both using data to directly explain previously conjectured features of human locomotion and in motivating new templates of human running which may outperform SLIP as predictive models.

In between these two extremes, engineers and control theorists need methods to obtain practical models amenable to computation for which they can produce their own template “targets of control” to achieve desirable behaviors in robotic systems. Some engineers have used “template-based” methods in the bio-inspired design and control of robots, attempting directly to embed the low-dimensional dynamics of classical templates such as SLIP in high-dimensional anchored robots in order to achieve useful behaviors.

There are a myriad of notions and examples of “templates and anchors” outlined in this chapter, and many of these notions appear, at least at first glance, to be quite distinct. Many engineers, scientists, and mathematicians may benefit from exposure to these ideas.

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## Chapter 3.3

# A Simple Model of Running

Justin Seipel

*Purdue University, West Lafayette, IN, United States*

## 3.3.1 RUNNING LIKE A SPRING-LOADED INVERTED PENDULUM (SLIP)

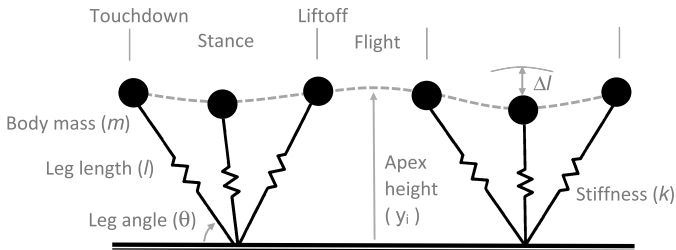
Humans and other animals run in a way that loosely resembles a pogo-stick bouncing along; see [Fig. 3.3.1](#) for an illustration of human running. This behavior is approximately captured in the spring–mass, or Spring-Loaded Inverted Pendulum (SLIP) model of running; see [Fig. 3.3.2](#). Further, passive dynamic running mechanisms can also embody SLIP-like running; see [Fig. 3.3.3](#).

There are several features of running behavior that are in common for animal, robot, and SLIP model running: During the *stance phase* of running, the body first moves downwards, reaching a minimum height at or near mid-stance, then moves upwards, all the while pivoting about the foot of the stance leg. After the stance leg lifts off, the trunk continues to rise during a *flight phase* of motion, until reaching a maximum height apex, then falls until the swing leg touches down to start the next stance. During stance, the length from trunk to the foot of the stance leg compresses (shortens) and then decompresses (lengthens), roughly in proportion to the ground reaction force acting on the leg, effectively like a spring. During flight, when all legs are off the ground, the leading swing leg (with foot off the ground) is moved into position for the next foot touchdown.

The overall behavior of running, as summarized here, can be captured in simple conceptual models of locomotion such as the SLIP model (e.g., spring–mass “SLIP” models by [Blickhan, 1989](#) and [McMahon and Cheng, 1990](#) and other SLIP models introduced in Subchapters [3.2](#) and [3.6](#)). SLIP models are often low-dimensional models, commonly using a single point-mass representing the body and a single massless leg that can represent key stance and swing leg functions during both stance and flight phases, respectively. Please see Subchapters [3.2](#) and [3.6](#) for an overview of SLIP-based models with varying degrees of complexity and realism. More realistic SLIP models might explicitly include more aspects of locomotion such as the movements of both legs during all phases of movement, as well as rotations and translations of the body/trunk. However, with increased realism there is often a trade-off in model complexity. Here a simple point-mass implementation of the SLIP model of running is presented.



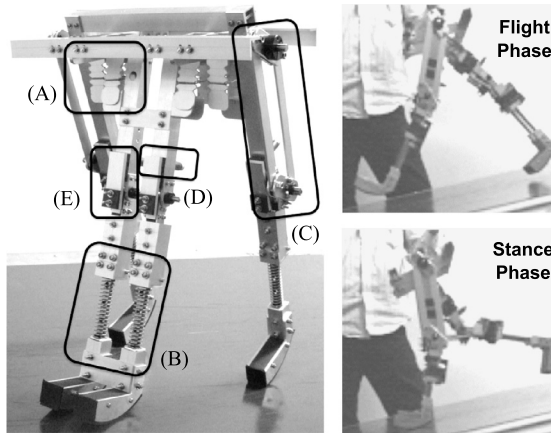
**FIGURE 3.3.1** Illustration of running (modified chronophotograph by Étienne-Jules Marey).



**FIGURE 3.3.2** An illustration of the Spring-Loaded Inverted Pendulum (SLIP) running. Here a running sequence is shown with descriptive labels of key events and phases of motion: touchdown, stance phase, liftoff, and flight phase prior to the next touchdown. The position of the body mass ( $m$ ) at any instant during stance is indicated by the leg length ( $l$ ) and leg angle ( $\theta$ ). During flight, the leg angle is controlled to be equal to the value  $\beta$  upon the next touchdown, where the leg length at touchdown is  $l_0$ . The leg stiffness ( $k$ ) and the leg compression ( $\Delta l$ ) are also indicated. The maximum height at apex is indicated by  $y_i$  (note that the subscript  $i$  indicates the  $i$ th apex event, to be followed by the  $(i + 1)$ th event). The timing of touchdown and liftoff events are close to those in Fig. 3.3.1 but do not exactly correspond.

### 3.3.1.1 Physical Mechanisms and Robots Related to the SLIP Model

The concept of pogo-stick locomotion or SLIP locomotion has also been influenced by the work of mechanicians and roboticists who were inspired by human and other animal motion to produce running machines and robots. For example, dynamic legged robots as described in Raibert (1986), and the Robotic Hexapod RHex as recorded in Saranli et al. (2001). More recently, passive running mechanisms have been demonstrated, with elements that directly relate to the spring-loaded inverted pendulum model. For example, the passive locomotion mechanism of Owaki et al. (2010) exhibits running-like behavior; see Fig. 3.3.3. Such running mechanisms and robots are essentially like pogo sticks bouncing along, and are similar to the Spring-Loaded Inverted Pendulum model of running.



**FIGURE 3.3.3** A passive dynamic legged mechanism. Images here are reproduced from [Owaki et al. \(2010\)](#) and displayed in a new arrangement. (Left panel) Image of a passive dynamic running mechanism, shown in a static position, with key system elements labeled: (A) hip springs to facilitate leg rotational oscillations, (B) leg springs to facilitate leg compression oscillations, (C) parallel link mechanism to synchronize the two outer legs, (D) shock absorber to dampen impact, and (E) a knee hyperextension mechanism to enable a form of mechanical support at the knee during stance, hypothesized to be needed in the absence of muscles or other actuators acting to transfer load across the knee joint ([Owaki et al., 2010](#)). (Right-top panel) A photograph captured during the flight phase, demonstrating one of the characteristic features of dynamic legged locomotion with a level of energy surpassing normal walking behavior. (Right-bottom panel) A photograph captured during the first half of stance, where the stance leg spring is clearly compressed. Note that in order for this system to maintain a steady stable gait, it runs on an inclined plane (here, running on an inclined treadmill). This mechanism exhibits flight phases and some characteristics of running, though currently does not produce maximum leg compression near mid-stance as is characteristic of SLIP running. Despite these differences with the classical assumptions and behaviors of the SLIP model, here we can see basic SLIP principles embodied in a physical system.

### 3.3.2 MATHEMATICAL AND PHYSICS-BASED SLIP MODEL

The SLIP model of running can be described in a more precise mathematical form based on physical laws of motion. The SLIP model, as presented in this chapter ([Fig. 3.3.2](#)), is composed of a point mass  $m$  representing the body, an effective leg stiffness  $k$  representing a massless stance leg, and a massless leg during flight representing a swing leg. Here, we derive the mathematical equations governing the motion of the SLIP model, based on a similar but more detailed presentation in [Shen and Seipel \(2016\)](#).

During the stance phase of motion the body mass  $m$  moves forward pivoting about the foot of the stance leg, and can be described by the leg length  $l$  and angle  $\theta$ . The Lagrangian  $L$  of the system, a description of the system's kinetic

energy  $T$  and potential energy  $V$ , is

$$L = T - V = \frac{1}{2}m \left( \dot{l}^2 + (l\dot{\theta})^2 \right) - \frac{1}{2}k(l - l_0)^2 - mgl \sin \theta.$$

Application of the Euler–Lagrange Equation to  $L$  yields the following equations governing stance:

$$\begin{aligned} m\ddot{l} &= m\dot{\theta}^2 - k(l - l_0) - mg \sin \theta, \\ ml^2\ddot{\theta} &= -mgl \cos \theta - 2ml\dot{\theta}. \end{aligned}$$

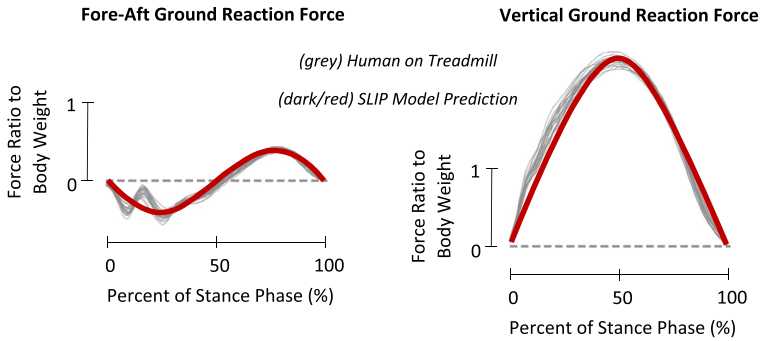
The stance phase of motion ends when the stance leg reaches its uncompressed length  $l = l_0$ . This event is called liftoff. After liftoff, the flight phase of motion follows.

During the flight phase of motion, the mass center is only affected by gravity, and so the motion is most simply described in terms of the height:  $\ddot{y} = -g$ . The horizontal component of velocity is constant during flight. During flight, the angle for the next touchdown leg is set to a specified value  $\beta$  and held there in preparation for the touchdown event, when the foot reaches the ground ( $y = l_0 \sin \beta$ ) and the flight phase ends. After touchdown, a new stance phase follows and the gait pattern repeats.

The governing equations of stance and flight, together with the event equations defining liftoff and touchdown, can be solved to determine the overall locomotion solutions of the SLIP model. In general, a numerical approach to solving the governing equations is used, though analytical solutions are possible for approximations of the SLIP model (e.g., Ghigliazza et al., 2005; Saranlı et al., 2010; Schwind and Koditschek, 2000; Geyer et al., 2005; Robilliard and Wilson, 2005; Altendorfer et al., 2004; Shen and Seipel, 2016).

### 3.3.2.1 Ground Reaction Forces During Stance

In addition to computing the solutions of the governing equations to yield position and velocity, other quantities such as ground reaction forces can be computed and predicted. Ground reaction forces are often measured in locomotion experiments and can provide insights into the kinetics of locomotion. A comparison of experimentally measured ground reaction forces of human running and predictions made by the SLIP model are presented in Fig. 3.3.4 (a modified reproduction of plots from Geyer et al., 2006). This demonstrates that multiple key features of ground reaction force in both the fore-aft (horizontal) and the vertical directions can be accurately predicted by the SLIP model.



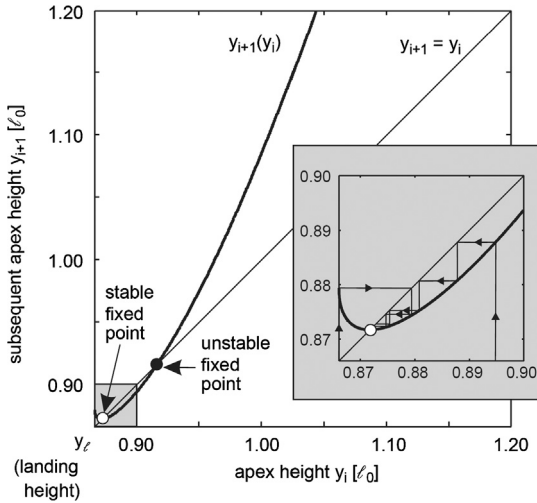
**FIGURE 3.3.4** The ground reaction forces of human running and SLIP model predictions. Human experimental traces and SLIP model traces are reproduced from [Geyer et al. \(2006\)](#).

### 3.3.2.2 Stride Maps: Behavior Investigated Step-by-Step

The dynamic nature of locomotion is often studied using a stride map: a function that governs how the system states, like position and velocity, change from one step to the next. In general, this is constructed using a Poincaré Return Map. In less precise terms, this is like taking a snapshot of the system at either a set interval of time, or alternatively, every time a well-defined event occurs (e.g., every time a foot touches down, or every time the trunk mass reaches a maximum height apex). The mapping that results is often referred to as a *stride map*.

### 3.3.2.3 Stability of Locomotion

The stability of running solutions can be determined using the stride map, which is a common approach for SLIP models. For a more general discussion of stability and analysis methods, please see [Full et al. \(2002\)](#), [Strogatz \(1994\)](#), or [Guckenheimer and Holmes \(1983\)](#). A common technique is to find periodic solutions and then determine whether small deviations to the periodic motion will lead to the system diverging away from the periodic motion or returning to it. This can be approximated by linearizing the stride map and evaluating it with respect to the periodic solution being investigated. The eigenvalues of the resulting linear system will indicate the kind of local stability that occurs in the neighborhood of the periodic locomotion (where if the magnitude of all eigenvalues is less than one there exists asymptotic stability; if greater than one, unstable; if equal to one, further analysis is needed). For example, for the SLIP model presented above, asymptotically stable periodic running exists for a wide range of system parameters, as described in [Geyer et al. \(2005\)](#) and reproduced here in [Fig. 3.3.5](#). In this figure, reproduced from [Geyer et al. \(2005\)](#), an apex-to-apex stride map is used. Here, two fixed points are shown, each representing differ-



**FIGURE 3.3.5** Here, a mapping from one apex to the next is displayed, reproduced from [Geyer et al. \(2005\)](#). There are two fixed points where the same apex height repeats each step, indicating a periodic locomotion solution. However, the stability of these two solutions differs. The stable fixed point is demonstrated by the inset figure, where an example sequence of steps is shown converging upon the stable fixed point value. Note that the analysis in [Geyer et al. \(2005\)](#) makes use of dimensionless parameters and some naming conventions that are different than those used here.

ent periodic locomotion solutions. One represents a stable limit cycle, or stable periodic locomotion. The other fixed point is an unstable periodic locomotion solution of the SLIP model. The SLIP model also exhibits other behaviors, such as higher period locomotion ([Ghigliazza et al., 2005](#)).

### 3.3.3 SOME INSIGHTS INTO RUNNING AIDED BY SLIP-BASED MODELS

#### 3.3.3.1 Adaptive, Resilient Locomotion Based on Open-Loop Stability

An aspect of locomotion theory influenced by SLIP or pogo-stick models is our understanding of how locomotion is regulated or controlled in animals, and how it could be regulated in robots or assistive devices. In particular, SLIP models have demonstrated that largely uncontrolled dynamics of running can be self-stabilizing, requiring minimal control sensing or actuation. Understanding how open-loop stability properties of running integrate with more active feedback and actuation layers of locomotion is still far from being understood (perhaps partly due to the complexity of neuromechanical systems). Nonetheless, many simple SLIP model analyses have demonstrated both basic stability properties (e.g., [Ghigliazza et al., 2005](#); [Geyer et al., 2005](#)) but also improved stability

properties by including features we know represent realistic biological strategies, such as swing leg placement control (e.g., [Knuesel et al., 2005](#) and other studies introduced in Subchapter 3.6), and inclusion of forcing and damping (e.g., [Shen and Seipel, 2012](#)). More examples of controlled and actuated SLIP models are presented in Subchapters 3.2 and 3.6.

### 3.3.3.2 Reducing Energetic Costs through Compliant Interaction

SLIP models of running have demonstrated clearly the theoretical possibility of locomotion with relatively small energetic cost (due to efficient energy storage in compliant legs and low-mass, low-impact legs that are idealized in many SLIP models). Robots, and prosthetic devices in particular, can be designed to efficiently store and return energy using elegant elastic structures inspired by SLIP models. Though the SLIP model is a highly idealized conception of running, and we know that animal and robot running generally involves many forms of energetic loss and actuation, the SLIP model can nonetheless provide insights into theoretical limiting cases that can influence and challenge our thinking about locomotion.

### 3.3.3.3 Momentum Trading to Benefit Stability

Another perspective on the mechanics of running is based upon momentum of the body (its mass times velocity) and angular momentum about the stance foot. During locomotion, there are transitions between flight phases where forward linear momentum is conserved, and stance phases where angular momentum is nearly conserved (or approximately conserved in the case of negligible gravity). At events like liftoff and touchdown, we can think of the system transitioning between these two modes. Whatever momentum was being conserved in one phase now gets “traded” or otherwise exchanged such that part of it contributes to a new conserved form of momentum. This has been referred to as “momentum trading” (e.g., [Holmes et al., 2006](#)). Without this aspect of the switching (or hybrid) dynamics of locomotion, the stability properties of an energy conserving SLIP system would not be possible ([Holmes et al., 2006](#)). The regulation of locomotion might be thought about in part as the regulation of traded momentum, from one step to the next.

### 3.3.3.4 Useful Inefficiency: Inefficiency can Benefit Robustness

An inefficient use of energy might sometimes be beneficial for creating more robust stability of legged locomotion. A common aspect of physical running, though less commonly represented in the simplest of SLIP models, is a significant energetic cost. While it is physically possible to demonstrate entirely



passive SLIP-based running mechanisms, even in these cases there are energy losses that are overcome by using an inclined plane (e.g., [Owaki et al., 2010](#)). In other words, some non-negligible amount of positive work on the system and negative work on the system appears to be a common feature of periodic running locomotion. Further, recent studies have suggested that this could play a substantial role in the stability of locomotion, helping to generate significantly greater robustness (e.g., it can contribute to significantly larger basins of attraction, [Shen and Seipel, 2012](#)). In addition to regulating momentum, locomotion might also be thought about as regulating the flow of energy from step to step.

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## Chapter 3.4

## Simple Models of Walking

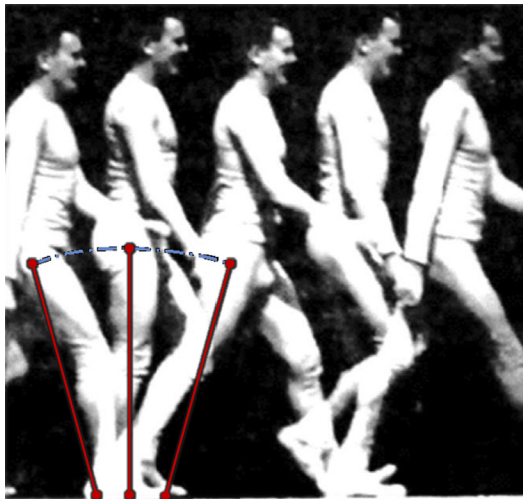
Justin Seipel

*Purdue University, West Lafayette, IN, United States*

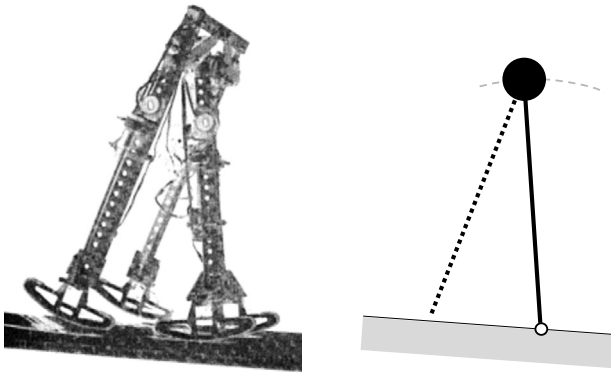
## 3.4.1 WALKING LIKE AN INVERTED PENDULUM

The movements of human walking, and animal walking more generally, have been likened to the motion of an inverted pendulum (e.g., Alexander, 1976; Mochon and McMahon, 1980); see Figs. 3.4.1 and 3.4.2 for illustrations of human and inverted pendulum walking, respectively. Further, some walking mechanisms and robots have exhibited similar walking behavior and are sometimes constructed in ways that are mechanically analogous to an inverted pendulum; see Fig. 3.4.2 for one example by McGeer (1990).

The overall motions of the trunk, stance leg, and swing leg of walking humans, walking mechanisms and robots, and the inverted pendulum model share many similarities: During the stance phase of human walking, when a single leg is on the ground, the body tends to rise and then fall as it pivots about the foot. This is similar to the way an inverted pendulum moves about its pivot. Walking is also described as a pattern or *gait* with alternating left and right legs (or sets



**FIGURE 3.4.1** Illustration of human walking based on a modified chrono-photograph taken by Étienne-Jules Marey. A leg length has been superimposed on the original image, approximately from the hip to an approximate center of pressure for heel-to-toe walking. Overall, this approximates a vaulting or pendular motion of the body about the foot.



**FIGURE 3.4.2** (left) A modified photograph of a passive walking mechanism, and (right) an illustration of an inverted pendulum model that represents a physics-based mathematical model. The original walking mechanism photograph is from [McGeer \(1990\)](#).

of legs). This characteristic walking pattern can also be exhibited by *bipedal* inverted pendulum models.

In biological and robot walking, when the *stance leg* is on the ground, the other leg, a *swing leg*, swings forward into position for *touch down*. Swing leg touchdown typically occurs before the current stance leg will *lift off* (also called take off), which leads to a double stance phase. This is followed by another single leg stance phase, and the overall pattern repeats. This process can be approximated in bipedal inverted pendulum models. For mathematical simplicity, in some inverted pendulum models the double stance phase is assumed to occur in an instant.

Overall, the basic walking movements of humans and other animals can be approximated by bipedal inverted pendulum locomotion, and can also be embodied physically in walking mechanisms. Though the concept of a bipedal inverted pendulum is dramatically simple when compared to walking humans or other animals, it can nonetheless help us understand and predict many aspects of walking.

### 3.4.2 PASSIVE WALKING MECHANISMS: PHYSICAL MODELS AND PHYSICS-BASED MATH MODELS

The notion of walking like an inverted pendulum can be investigated via experimental study of simple physical “inverted pendulum” walking mechanisms, such as walking toys and other *passive* dynamic walking mechanisms (e.g., [McGeer, 1990](#); [Coleman and Ruina, 1998](#); [Collins et al., 2001](#)). Many of these mechanisms are passive in the sense that they do not have active power elements such as motors to drive locomotion. Passive walkers generally maintain a steady gait by walking down an inclined plane, though some do use

small actuators to maintain nearly passive walking (e.g., [Collins et al., 2005](#); [Bhounsule et al., 2014](#)). In many of these walking mechanisms, one can directly observe a physical bipedal inverted pendulum mechanism in action ([Fig. 3.4.2](#)). In this way, a walking mechanism can be considered as a kind of *physical model* of human and animal locomotion that strongly demonstrates a role that passive dynamics can play in locomotion. Mathematical models of bipedal inverted pendulum walking can be closely associated with physical walking mechanisms, via application of the laws of mechanics, or they can be developed in more direct relationship to empirical studies of biological locomotion, via motion capture, ground reaction forces and other techniques.

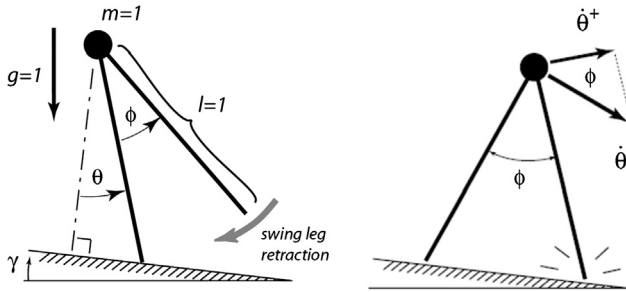
### 3.4.3 MATHEMATICAL EQUATIONS GOVERNING A BIPEDAL INVERTED PENDULUM (IP) MODEL

Now that we have discussed many of the foundational concepts of walking that are embodied in bipedal inverted pendulum models, here we explicitly present a mathematical model for a bipedal inverted pendulum. In particular, we present the mathematical equations that describe the mechanics, hybrid dynamics, and control of a bipedal inverted pendulum. We present one particular inverted pendulum model of walking in order to provide a simple example of explicit mathematical governing equations. Please see [Subchapter 3.6](#) for an overview of multiple established Inverted Pendulum models, including models that are more complex than the one presented here.

Here we present a highly simplified version of the inverted pendulum model based closely on a previous study by [Wisse et al. \(2006\)](#). Specifically, we assume that the swing leg has negligible mass, that the swing leg is controlled to touch down with a prescribed angle and leg retraction speed, that the time spent in double stance phase is negligible compared to the total stride, and that stance leg liftoff and swing leg touchdown are instantaneous. Other inverted pendulum models have relaxed some of these assumptions. For example, several models have included the effects of swing leg mass (e.g., [Coleman and Ruina, 1998](#) and others reviewed in [Subchapter 3.6](#)), and more recent extensions of the inverted pendulum model have included leg compliance, enabling a substantial double stance phase as well as more accurate prediction of ground reaction forces (e.g., [Geyer et al., 2006](#)). The particular and highly simplified Inverted Pendulum model presented here, along with its corresponding equations and figures, are reproductions of the particular model and results presented by [Wisse et al. \(2006\)](#).

#### 3.4.3.1 Behavior Within a Single Stance Phase

The mathematical equations governing stance for a simple inverted pendulum model can be derived by applying laws of physics. Common approaches include



**FIGURE 3.4.3** A simple Inverted Pendulum model of walking. The figure is reproduced from [Wisse et al. \(2006\)](#). The model is simplified via fixing mass, gravity, and leg length in the governing equations to be equal to one. Reductions in the total number of system parameters can also be gained through formal nondimensionalization techniques.

applying Newton's Second Law of Mechanics to a Free Body Diagram of forces acting on the system, or applying the Euler–Lagrange Equation to a description of the system's energy. Here, we use the Euler–Lagrange approach: Looking at the simple bipedal IP system modeled in [Fig. 3.4.3](#), where the system rotates with an angle  $\theta$ , and where the mass of the swing leg is assumed to be negligible, we can describe the kinetic energy  $T$ , potential energy  $V$ , and resulting Lagrangian  $L$  as follows:

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl \cos(\theta - \gamma).$$

Here  $m$  is the mass,  $l$  is the leg length,  $g$  is gravity, and  $\gamma$  is the angle of the inclined plane. We then apply the Euler–Lagrange Equation of mechanics to yield the following differential equation governing motion of the system, in terms of the angle  $\theta$ :

$$\ddot{\theta} = gl^{-1} \sin(\theta - \gamma).$$

This equation, along with the initial conditions at the beginning of stance, determine the motion of the inverted pendulum during stance. For the model shown in [Fig. 3.4.3](#), the model analysis was simplified by taking the leg length, gravity, and mass to be equal to one. More explicit nondimensionalization could yield a similar simplification. Note also that the angle  $\theta$  used here has a different reference than that used for the Spring-Loaded Inverted Pendulum model presented in [Subchapter 3.3](#).

### 3.4.3.2 Stance Leg Liftoff and Swing Leg Touchdown

The termination of the stance phase of a given leg is often defined as when the foot loses contact with the ground (which can also be related to the ground

reaction force). Realistically, for normal walking this would occur sometime after the current swing leg touches down and so a substantial double stance phase would occur. However, for simple inverted pendulum models the double stance phase is assumed to be infinitesimal in duration. Therefore, here we allow the termination of stance to occur approximately at the same instant as the swing leg touches down. By determining when the swing leg touches down in such a simplified model, we automatically determine the time when the stance leg terminates.

The touchdown of the swing leg can be defined as when the foot of the swing leg reaches the ground, or when the distance between the swing leg foot and ground reaches zero. Further, to avoid counting glancing contacts as a swing leg touchdown, one can also require that the velocity of the foot is pointing into the ground when this distance reaches zero. The distance between the swing foot and ground will generally depend on complicated dynamics and control of the swing leg. For the simplified model discussed here, it is assumed that there is a swing leg controller that maintains a prescribed trajectory of the swing leg, with a prescribed retraction angular velocity near the time of touchdown. In this scenario, the swing leg actually first swings past what will become the touchdown angle and then *retracts* towards the desired angle for touchdown. If touchdown were delayed or occurred early, it would result in a different touchdown angle. In this simple model, the stability of walking can be influenced by this effect.

### 3.4.3.3 The Mechanics of Switching from One Stance Leg to the Next

During the infinitesimal double stance phase of motion of this simple walking model, the current stance leg lifts off and the swing leg touches down at the same time. During this period, even if it occurs over an infinitesimal period as assumed in simple walking models, there is a change in momentum of the system such that the velocity which was heading downwards at the end of one stance will change and head upwards in order to vault over the next stance leg. In order to model this process, an impulse–momentum equation can be used. We assume the simplified model presented here is entirely passive, so the leg lifting off is not able to apply an impulse during this sequence (other inverted pendulum models include active toe-off impulses which have been shown to help reduce overall energetic cost, [Kuo, 2002](#)). We are left to assume that for the system to have a velocity direction consistent with the circular arc of the new stance leg; there must be a net impulse that makes it so. It is reasonable to assume that much of this impulse occurs along the length of the touchdown leg, and so we assume a touchdown impulse entirely aligned with the touchdown leg that cancels all momentum in that direction. The remaining momentum is per-

pendicular to the new stance leg. From the application of impulse–momentum equations it is worked out that the angular velocity just after the leg switching is less than that just before the leg switching, depending on the angle  $\phi$  between the two legs:

$$\dot{\theta}^+ = \cos(\phi)\dot{\theta}^-,$$

where  $\dot{\theta}^+$  is the angular speed of the inverted pendulum the instant just after the leg switching process, and  $\dot{\theta}^-$  is just before.

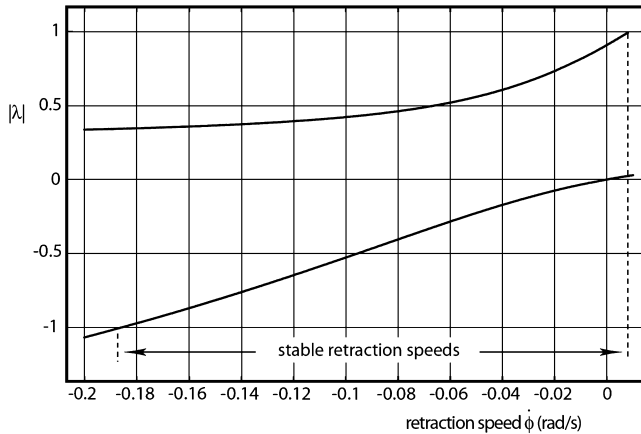
#### 3.4.3.4 Stride Maps: Behavior Investigated Step-by-Step

One of the key methods currently used to investigate walking behavior is to see how the states of the system, such as positions and velocities, change at discrete intervals from step-to-step. This creates a mapping of the system states from one stride to the next, and can be constructed using a Poincaré Return Map. In less precise terms, this is like taking a snapshot of the system at either a set interval of time, or alternatively, every time a well-defined event occurs (e.g., every time a foot touches down, or every time the trunk mass reaches a maximum height apex). The mapping that results is often referred to as a *stride map*. For systems that are integrable, a return map (or stride map) can be written in closed-form mathematical expressions. However, it is also common to numerically integrate governing equations to produce a stride map, especially for more complex models of locomotion.

#### 3.4.3.5 Stability of Locomotion

The dynamic stability of locomotion is often of interest when studying Inverted Pendulum models of walking. Surprisingly, such systems can exhibit stable locomotion even when no active control is present. Many physical parameters of the walking system could potentially affect stability in important ways, and understanding the underlying passive dynamics can also benefit the design of controllers that can be added to the system. Here we present only one simple example: We investigate how stability of walking depends on the swing leg retraction speed, as an example to highlight how stability is studied for such a simple walking model. For more general discussion of stability and analysis methods, please see [Full et al. \(2002\)](#), [Strogatz \(1994\)](#), or [Guckenheimer and Holmes \(1983\)](#).

A common approach to measure stability is to use the stride map to find periodic walking motions and then determine whether small deviations to the periodic motion will lead to the system diverging away from the periodic motion or returning to it. This is done systematically by analytically or numerically



**FIGURE 3.4.4** Example result of a stability analysis of inverted pendulum models of walking. The result shown here is reproduced from [Wisse et al. \(2006\)](#) for this example. It shows how the stability of the system can be influenced by the speed of swing leg retraction. In particular, this study found that there is a range of swing leg retraction speeds for which stable walking motions were found (where the magnitude of the two eigenvalues of the system are both less than one).

calculating a linearization of the stride map, evaluated with respect to the periodic solution being investigated. This yields a linear discrete dynamic system that approximates the stride map. The eigenvalues of this linear system will indicate the kind of local stability that occurs in the neighborhood of the periodic locomotion being investigated (where if the magnitude of all eigenvalues is less than one there exists asymptotic stability; if greater than one, unstable; if equal to one, further analysis is needed). For example, for the model presented above it was found that asymptotically stable periodic walking exists for a range of the leg retraction speeds: This result is reproduced here in [Fig. 3.4.4](#) from [Wisse et al. \(2006\)](#).

### 3.4.4 SOME INSIGHTS INTO WALKING AIDED BY INVERTED PENDULUM MODELS

Walking mechanisms, and the associated study of the inverted pendulum model of walking, have been influential and have provided insights regarding the mechanics and control of walking. They have led to theory about the flow of energy during the walking cycle, both what is physically possible and insights on what may be happening in biological systems. Empirical study of these walking mechanisms and related theoretical study of inverted pendulum models have also suggested that the regulation of walking could rely in part on its passive dynamics. One such insight was that passive dynamic walking is a process that is statically unstable about a given resting point, but yet has dynamic stability over



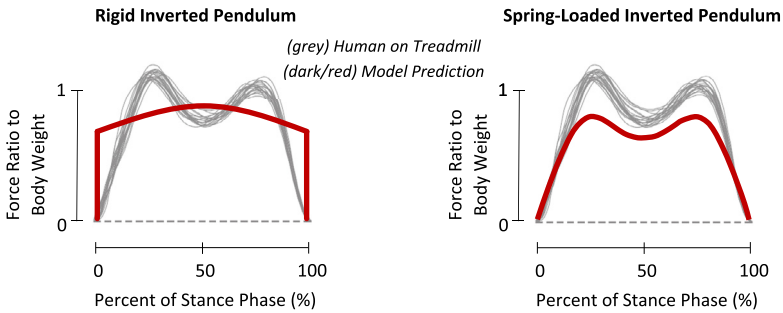
a walking cycle (see, for example, the work by [Coleman and Ruina, 1998](#)). Following are a few concepts about walking that have been influenced by simple walking models. These concepts are largely presented using a mechanics perspective of walking, though they have implications for control. These concepts are complementary to those presented for the Spring-Loaded Inverted Pendulum model in Subchapter 3.3.

#### 3.4.4.1 Walking Includes a Pendular Flow of Energy

One basic discovery, inspired by mechanical analysis of inverted pendulum motion, is that kinetic energy and potential energy are exchanged during walking. For a passive inverted pendulum system, energy is conserved during stance, and so we have the basic theoretical mechanics result: Kinetic Energy + Potential Energy = Constant. Since the total system energy is constant, any change of kinetic energy corresponds with an equal and opposite change in potential energy. This means that as the mass center rises the speed slows, corresponding to a decrease in kinetic energy that is equal to the increase in potential energy. This occurs until apex, at the highest point where the mass center is directly above the pivot. Assuming that the initial kinetic energy (and corresponding mass center speed) is enough for the inverted pendulum to reach and pass through apex, then after apex the mass center falls. As the mass center falls, the speed increases, corresponding to an increase in kinetic energy that is equal to the decrease in potential energy. Not all levels of energy in the system will permit a “gait” or motion of the inverted pendulum. For a passive inverted pendulum walker, the initial value of kinetic energy needs to be larger than the increase in potential energy needed to reach apex, in order for the system to pass through apex with a nonzero speed.

#### 3.4.4.2 Walking Includes the Catching of Repeated Falls

Walking can also be likened to controlled falling, where the body falls and pivots about the stance foot, only to be caught by the next stance leg. This view is consistent with the energetic concepts of inverted pendulum motion during stance and adds to it the importance of the *placement of the swing leg* to switch from one leg to the next. This requires a bipedal inverted pendulum walking system that is able to transition from one leg to another. While one leg is in stance, the other is in a swing phase. The swing leg is often assumed to follow a passive dynamic trajectory under the influence of its own weight, though in reality it is also likely regulated (e.g., [Coleman and Ruina, 1998](#)). A simplified modeling approach is to assume the swing leg is controlled to follow a trajectory based on time, phase, or system states relative to the main body and/or ground (e.g., [Wisse et al., 2006](#)).



**FIGURE 3.4.5** Walking vertical ground reaction force predicted by the IP and SLIP models. Human experiment and model traces reproduced from [Geyer et al. \(2006\)](#).

### 3.4.4.3 Momentum is Exchanged During Double Stance

Walking generally includes a significant “double stance” phase where the body is supported by two legs. For inverted pendulum models, it is often assumed that double support happens over a negligible period of time. In inverted pendulum models the legs are often assumed to be rigid, and this requires that double support phases vanish since movement is otherwise kinematically restricted. Despite this simplification, physical insights can still be gained regarding how the two legs act during the transition from one stance leg to the next. For example, it makes an energetic difference what the order is of the different possible impulses from the two legs (e.g., a toe-off impulse occurring before the touchdown leg impulse can reduce energetic costs, as presented in [Kuo, 2002](#)). However, a more accurate analysis could be gained by relaxing the rigid leg assumption and enabling a substantial-duration double support phase. For example, [Geyer et al. \(2006\)](#) showed that a bipedal Spring-Loaded Inverted Pendulum model can more accurately predict the ground reaction forces of human walking, including the portion during which double-support phases occur; see [Fig. 3.4.5](#). This approach has an added benefit of utilizing the SLIP modeling framework already associated with running.

## 3.4.5 INTEGRATION OF WALKING AND RUNNING MODELS

Though we have so far studied walking as separate from running, walking and running can be viewed as expressions of the same legged locomotion system, whether of humans, other animals, or robotic systems. It is also possible to predict both walking and running in a single mathematical model without adding much additional complexity compared with the IP and SLIP models. This can be achieved by blending the bipedal nature of the IP model and the effective leg spring of SLIP, into a bipedal SLIP model, as demonstrated in [Geyer et al.](#)

(2006). There it was also demonstrated that many predictions of human walking are improved, such as predictions of ground reaction forces: For example, as reproduced in Fig. 3.4.5, it is apparent that the bipedal SLIP model captures well the characteristic shape of human walking ground reactions. Other extensions to the SLIP model have also demonstrated how two gaits might arise from one simple SLIP-based mechanism. For example, a clock-torqued SLIP model was inspired by the robot RHex, as well as cockroaches, and demonstrates that a range of walking and running behaviors result from simple clock parameter adjustments (Seipel and Holmes, 2007). Though the simplest walking models and the simplest running models may continue to have many uses and may be preferred for the sake of simplicity, a more integrated modeling framework of walking and running also has many potential advantages and uses. Regardless of the model used, it is likely helpful to remember that walking and running are behaviors that can arise from similar underlying processes, and that the study of one can often inform the other.

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## Chapter 3.5

# Locomotion as an Oscillator

Shai Revzen and Matthew Kvalheim

*University of Michigan, Ann Arbor, MI, United States*

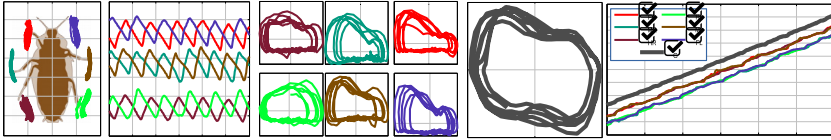
## 3.5.1 LOCOMOTION AS AN OSCILLATOR

Virtually all animals, and even more so bipeds such as humans, move in a rhythmic way at moderate to high speeds of their available range of speeds. We refer to these as “rhythmic”, rather than the more mathematically strict term “periodic”, which is reserved for systems that have a precisely defined period within which motions repeat exactly. We define a rhythmic system as a stochastic system whose underlying deterministic part (the “drift” in the language of Stochastic Differential Equations) has an exponentially stable periodic solution. The cycles of legged locomotion, known as “strides”, typically vary from each other in duration and geometry of motion. As animals move slightly faster or slower, their limbs follow similar trajectories at slightly higher or lower rates. Even at a given stride frequency animal motions exhibit variability. At least to casual observation, it seems this variability (normalized for body size) is greater in smaller animals, in animals using more legs for propulsion, and in animals moving more slowly.

Taking the “templates and anchors” perspective of Subchapter 3.2, we can rephrase this observation as a statement that the so-called “phase oscillator” is the simplest template of most moderate-speed legged locomotion. In other words, the simplest model of legged locomotion is the timely progression through a repeating sequence of body postures, which happens also to include interaction with the ground that produces propulsion. For this chapter, we will refer to this cycle as the “gait cycle.”

The phase oscillator template of locomotion can be modeled as a curve in the configuration space of the animal’s body, and a velocity associated with every point on that curve. Alternatively, it can be modeled as a periodic function of time, e.g., using a Fourier series model of the body configuration as a function of “phase.”

Under sufficiently small perturbations of the environment or body posture, animal motions recover to the gait cycle after few steps. This suggests that the slightly richer structure of an “asymptotically stable oscillator” (“oscillator” for short) applies just as universally. From a mathematical perspective, an oscillator is the differential equation that governs motion within the stability basin of a cycle, i.e., the gait cycle and all bodily states that allow for the gait cycle to be recovered.

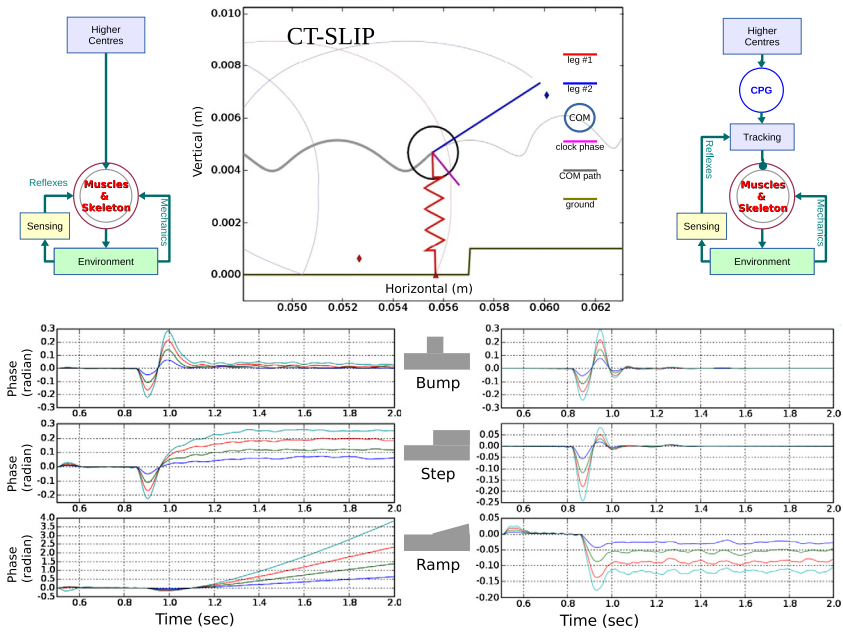


**FIGURE 3.5.1** Obtaining a prediction of future motion using a phase estimate. (Left to right) Starting with cockroach foot motions in the body frame of reference (first plot), we focused on the fore-aft motions as a function of time (second plot), and computed the velocity as a function of position (middle six plots; nondimensionalized by z-scoring). We combined these linearly to a single “foot oscillation state” with +1 coefficients for one tripod and  $-1$  for the other. This gave rise to a state which collectively describes the phase of the gait as a whole (large oval plot). The polar angles of the plots of each leg and of the combined state follow a linear trend over multiple strides (rightmost plot). All plots shown are from the same time segment of a single experiment tracking a *Blaberus* cockroach running at moderate speed.

There is a rich mathematical literature on the structure of oscillators. If we restrict our attention in that literature to those oscillators that are “structurally stable” and “generic”, i.e., oscillators which are physically observable and whose dynamics would change only a little if properties of the body and environment change slightly, all smooth oscillators share several properties. One of the most important of these is that oscillators have a phase coordinate for the entire stability basin. This phase specializes to, and is therefore consistent with, the phase oscillator phase on the gait cycle itself. Any perturbations of the animal away from the gait cycle will typically result in a phase shift that will persist after the animal returns to the gait cycle. Since all observables of a rhythmic system must themselves be rhythmic, we may aim to estimate the phase of an animal’s phase oscillator template (“dynamical phase” hereon, called “asymptotic phase” in Subchapter 3.2 – not to be confused with “dynamical phase” in geometric mechanics) by observing neuromechanical quantities such as body configurations, speeds of various body parts, forces and torques, and EMG or other neuronal measurements.

Once a method of phase estimation is available, predicting phase as a function of time should produce a linear trend if the phase oscillator template is compatible with the observations (see Fig. 3.5.1). By subtracting this linear trend from the instantaneous phase estimate we can obtain the “residual phase” which can be used to identify how oscillations change under the influence of external perturbations (Revzen et al., 2009).

Dynamical phase is the only dynamical variable of the phase oscillator template. The study of how that phase responds to the body and environment allows us to eliminate possible neuromechanical control architectures, e.g., by separating out responses that could be achieved only with changes to descending neural signals, and responses that could occur for solely mechanical reasons (Revzen et al., 2009). An example of such an analysis is shown in Fig. 3.5.2.



**FIGURE 3.5.2** Clocked, Torqued SLIP model (CT-SLIP, Seipel and Holmes, 2007) set to parameters of a *Blaberus* cockroach running gait (upper center) with two different control architectures (upper left & right diagrams), was subjected to an assay of three perturbations (lower center). In each case, the magnitude of the perturbation is varied, producing qualitatively different residual phase response curves (plots lower left & right) for the two architectures. Results show that phase alone can be used to differentiate the neuromechanical control architecture (Revzen et al., 2009).

### 3.5.2 STRIDE REGISTRATION AS PHASE ESTIMATION

Estimating dynamical phase can also be seen as a way of representing methods of “stride registration.” Whenever we observe a rhythmically moving animal, we encounter the problem of stride registration: which samples of stride  $n$  represent “the same” state in the gait cycle as which samples of stride  $n + 1$ ? Whenever investigators construct a notion of a gait cycle, they implicitly define such a registration method. In each such class of states which are “the same” in this sense, there is one distinguished representative which lies on the gait cycle itself. Because it lies on the cycle, it is a state of the phase oscillator template of that animal motion. Thus we see that any stride registration method corresponds to a choice of assigning phase to data samples.

Typical stride registration methods in the literature include linearly interpolating once-per-stride events in time, e.g., heel strike (Jindrich and Full, 2002; Ting et al., 1994) or anterior extreme position of a limb (Cruse and Schwarze, 1988). Some work in robot control has attempted to parameterize a target gait

using a hip-to-heel angle, or other combination of internal angles (Chevallereau et al., 2003; Sreenath et al., 2011). By construction, these driving variables are a form of step registration as well.

The advantage of phase-based stride registration becomes clear if we assume a state independent measurement noise, and that observed motions are perturbations around a core phase oscillator template. Estimating the phase oscillator's phase and using it for binning and averaging the measurements ensures that all equal sized bins have (asymptotically) the same number of samples. Thus the bin average estimates provided are homoscedastic and standard statistical hypothesis testing tools can be used to test for treatment effects. If any other stride registration method is used the bin averages will be heteroscedastic, and require much more refined statistical techniques.<sup>5</sup>

Let us compare the process of naïve stride registration and a dynamical phase-based one. For the former, we define an event detector function which has positive zero crossings when the desired event occurs, e.g., for heel-strike based stride registration we take the time and force pairs  $(t_i, f_i)$  from a force plate under the running human and renormalize to  $(t_i, 1 - f_i/(mg))$ . We then detect the positive crossing times  $\{c_k\}$  and form the piecewise linear function of time  $p(\cdot)$  such that  $p(c_k) = k$  are its knot points. We now select a number of bins  $N_b$  and put the (multidimensional) data sample  $(t_i, d_i)$  in the bin  $b_i := \lfloor N_b(p(t_i) - \lfloor p(t_i) \rfloor) \rfloor$ . We estimate the period of the gait cycle  $\tau$  by taking a central statistic such as the median of  $\{c_{k+1} - c_k\}$ . Taking a representative such as sample average of the data in each bin in an appropriate way for the data itself, we obtain the model that at time  $t$  the gait cycle places the animal at body configuration given by the representative of the bin  $\lfloor N_b(t \bmod \tau) / \tau \rfloor$ .

A dynamical phase-based stride registration would consist of first training or deriving a phase estimator that gives a phase  $p_i$  for every data sample  $(t_i, d_i)$ . Using that phase estimate instead of  $p(t_i)$ , i.e., by taking  $b_i := \lfloor N_b(p_i - \lfloor p_i \rfloor) \rfloor$ , we proceed with the same approach to obtain bin representatives.

It should be noted that in many cases, producing the gait cycle model at a given phase does not require binning, and can instead be done by building a Fourier series model of animal properties  $d(t)$  as a function of phase using a Fourier series of some order  $N_f$ :

$$x(\varphi) = \sum_{k=-N_f}^{N_f} a_k e^{i2\pi k \varphi}, \quad (3.5.1)$$

---

5. Phase defines a measure on the cycle which is flow invariant, and thus averaging a function of state with respect to the phase measure along trajectories does not introduce additional variance due to the dynamics—only the preexisting measurement noise.

where

$$a_k := \int_{\text{all } t} e^{-2\pi i kt/\tau} d(t) \frac{dp}{dt}(t) dt. \quad (3.5.2)$$

### 3.5.3 RECOVERY FROM PERTURBATIONS

The structurally stable, generic oscillators that we use as models of locomotion share an additional property: they can be “linearized exactly.” The core insight dates to the late 19th century, when Gaston Floquet showed that linear time periodic (LTP) differential equations can be solved by writing their solutions as a periodic part multiplying the solutions for a linear time invariant part (Floquet, 1883). This insight extends from LTP systems to oscillators because one may view the dynamics of the oscillator as a perturbation of the dynamics of its phase oscillator template, which is time periodic. The theory of “Normal Forms” (Bronstein and Kopanskii, 1994; Lan and Mezić, 2013) shows that Floquet’s result does in fact extend to the entire stability basin of the oscillator.

In other words, the oscillators that appear in locomotion problems can be rewritten with respect to appropriately chosen coordinates such that they are linear time invariant (LTI) systems in the new coordinates. In these linearizing coordinates, the tools of linear systems theory and control theory can be brought to bear, telling us that the long-term dynamics are governed by a single “system matrix”  $A$  which describes the LTI equation of motion. For a gait cycle with period  $\tau$ , the matrix norm of  $e^{\tau A}$  provides a bound on how quickly perturbations decay back to the unperturbed gait, with the magnitude decreasing by at least a factor of  $|e^{\tau A}|$  every stride. It is important to note that in the linearizing coordinates, the results apply to both large and small perturbations; if return map Jacobians are used without a full coordinate change, the result only applies to small perturbations.

The Floquet Normal Form provides even more detailed insight. Every perturbation to the state of the animal can be rewritten in terms of a linear combination  $\{\xi_k\}$  of the eigenvectors  $\{v_k\}$  of  $A$ ,  $x(0) = \sum_k \xi_k v_k$ , and will thus evolve as

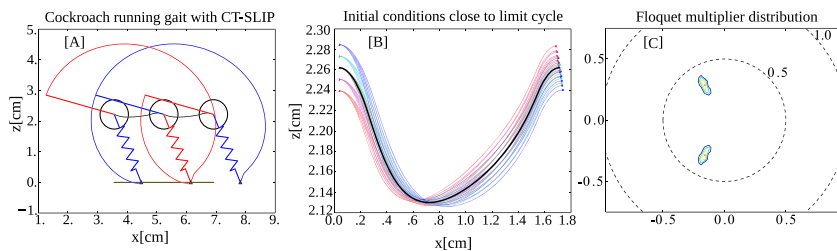
$$x(t) = \sum_k e^{\lambda_k t} \xi_k v_k. \quad (3.5.3)$$

The “Floquet Multipliers”  $e^{\lambda_k \tau}$  are invariant to the choice of coordinates,<sup>6</sup> and can therefore be computed in the original coordinates we use to obtain our measurements. Computing Floquet Multipliers is thus the method of choice for determining the stability of smooth oscillators (see Fig. 3.5.3).

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6. This follows because the matrices involved in different coordinate representations are similar (conjugate) to each other and thus have the same eigenvalues.





**FIGURE 3.5.3** A CT-SLIP (Seipel and Holmes, 2007) model of a running *Blaberus* cockroach alternates between right and left foot touchdown events [A]. The center of mass bounces vertically every step, exhibiting a limit cycle [B], heavy black line). At “apex”, with vertical velocity zero and going negative, it is convenient to define a Poincaré section. This section is 2D, consisting of height ( $z$ ) and horizontal velocity ( $v_x$ ) of the center of mass. An ensemble of initial conditions at apex, varying in both  $z$  and  $v_x$  [B], colored lines) can be integrated to the next apex [B], colored dots). Using linear regression, the affine map taking apex states to the next apex can be estimated, and its eigenvalues—the Floquet Multipliers—computed. Bootstrap analysis can further be used to get a distribution of eigenvalues and produce confidence bounds for the estimate [C], 1000 bootstrap computations from the ensemble in [B]). For this gait the eigenvalues are a complex conjugate pair, of magnitude less than 0.5. This tells us that the oscillator is very robustly stable, and (small) perturbations decay in magnitude by better than a factor of 2 every step. (For interpretation of the references to color in this figure, the reader is referred to the web version of this chapter.)

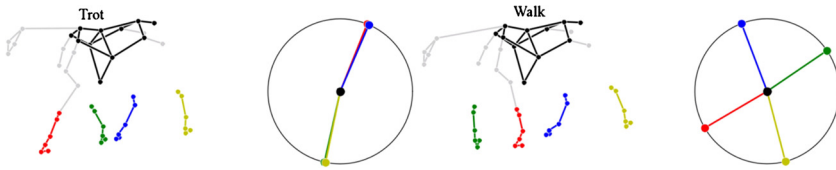
### 3.5.4 SUBSYSTEMS AS COUPLED OSCILLATORS

The entire argument presented above for treating a phase oscillator as a template for animal locomotion applies equally well to parts of an animal’s body. The partitioning of the animal into subsystems can be physiological, e.g., viewing the nervous system as one or more oscillators as well as viewing the musculoskeletal system as one or more oscillators. It can also follow morphology, e.g., treating each limb as an oscillator. In all such cases, one ends up with a notion of “subsystem phases” (Revzen et al., 2009), and of an animal locomotion template consisting of coupled phase oscillators.

Which gait an animal is employing at any given time can be ascertained from the relative phases of the legs (see Fig. 3.5.4).

### 3.5.5 LEGGED LOCOMOTION OSCILLATORS ARE HYBRID DYNAMICAL SYSTEMS

The theory of oscillators, as described hereto, was developed for “smooth” dynamical systems—ones for which the equations of motion are at least continuously differentiable. Unfortunately, the models used for legged locomotion rarely satisfy this requirement. Typically, the equations of motion of a legged system depend strongly on which legs are in ground contact. Indeed, the very dimension of the system or the number of mechanical degrees of

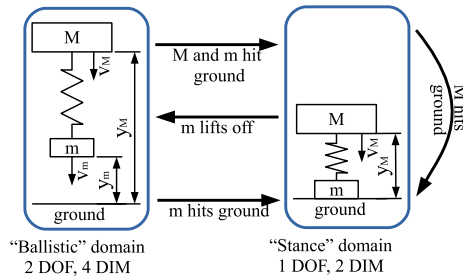


**FIGURE 3.5.4** Coupled oscillator description of horse gaits. A body reference frame was fixed to the horse (black dots) and the motion of six markers on each leg was described as a phase oscillator and thereby reduced to a single phase per leg. The phases of all four legs are shown (radial lines in circles; color corresponds to animal leg) for both trotting (left) and walking (right) [figure from Yu et al., 2016].

freedom may change as contact varies. For legged systems, we must extend our scope to the study of “Hybrid Dynamical Systems.” Several subtly different definitions of Hybrid Systems exist in the literature (Alur et al., 1993; Burden et al., 2015, 2016; Goebel et al., 2009; Holmes et al., 2006; Nerode and Kohn, 1993), but all share several features: (1) the solutions of the Hybrid System are referred to as “executions”, rather than “trajectories”; (2) dynamics are defined over several “domains” and are smooth within each domain; (3) “reset maps” link domains to each other, and an execution may go through a reset map by taking its value in one domain, applying the map and using the image as the initial condition in the new domain; (4) the sets of points in each domain over which reset maps may be applied are called “guards.” As a concrete example which is also of interest to legged locomotion, assume we have two masses linked by a vertical spring and constrained to bounce in the vertical direction in earth gravity above level ground. While both masses are in the air, the dynamics are the smooth ballistic motion of the two masses, with the additional internal force of the connecting spring. The flight domain is four-dimensional, with two mechanical degrees of freedom (DOF). Assume further that when a mass hits the ground, it loses all kinetic energy in a plastic collision. Thus, with the lower mass on the ground, we may use a two-dimensional, one DOF model. Adding the assumption that at length 0 the spring exerts enough force to lift the top mass from the ground, we have a hybrid system with 2 domains and 4 reset maps (see Fig. 3.5.5).

One may readily envision that with the addition of a periodic actuation force applied by the spring, the system may enter a range of persistent hopping at some constant amplitude which balances the energy lost by  $m$  colliding with the ground with the energy injected by the actuator.

While the core results of oscillator theory and Floquet theory do not apply to this system as stated, since it is not a smooth oscillator, recent results (Burden et al., 2015) show that after two cycles this system becomes restricted to a 2D surface in the 4D ballistic domain, such that the motions in the stance domain



**FIGURE 3.5.5** An example of a hybrid dynamical system model. A vertically bouncing pair of masses ( $M$ ,  $m$ ) connected by a spring can be modeled with two domains (rounded frames) and four reset maps (labeled arrows). In the ballistic flight domain, the system is 4-dimensional since the state contains position and velocity for each mass. In the stance domain, the lower mass ( $m$ ) is stationary on the ground, and the state is two dimensional, consisting only of position and velocity of one mass. Reset maps take states in which masses collide with ground to the associated stance state, and take states in which the lower mass would detach from the ground, from stance into ballistic motion.

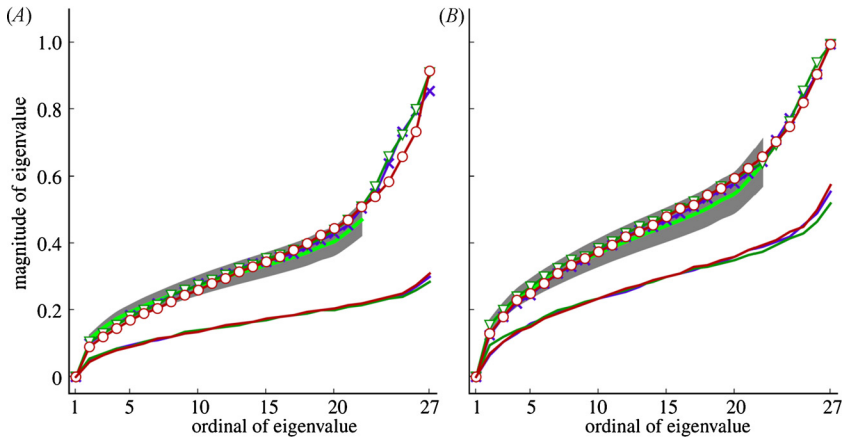
and in the ballistic domain can be “stitched together” using a function that is smooth everywhere except the guards, and leading to dynamics that are smooth in the new coordinates. In fact, this equivalence<sup>7</sup> to smooth systems extends to multilegged locomotion gaits in which many legs hit the ground at once—a class of models which was only analyzed recently (Burden et al., 2016). Thus, we find that once some technical complications are addressed, the long-term behavior of hybrid oscillator models that arise in legged locomotion is the same as that of the more familiar smooth oscillators.

### 3.5.6 ADVANCED APPLICATION: DATA DRIVEN FLOQUET MODELS

One of the strengths of the oscillator perspective on locomotion is the ability to identify properties of feasible locomotion models from observational data (Revzen, 2009; Revzen and Kvalheim, 2015; Wang, 2013). This approach has been called “Data Driven Floquet Analysis (DDFA)” and consists of a collection of numerical methods that attempt to reconstruct the oscillator dynamics of the putative legged locomotion oscillator directly from observational data.

One application of DDFA is the identification of plausible dimensions for template models. As described in Subchapter 3.2, multiple models with varying levels of detail may exist for a given legged locomotion behavior. Viewed as an oscillator, the same behavior has a set of Floquet multipliers, the magnitudes of which define a set of decay rates. Each Floquet multiplier is associated with a “Floquet mode”—a specific phase-dependent way of the motions being offset

7. Formally, a piecewise smooth and everywhere continuous conjugacy.

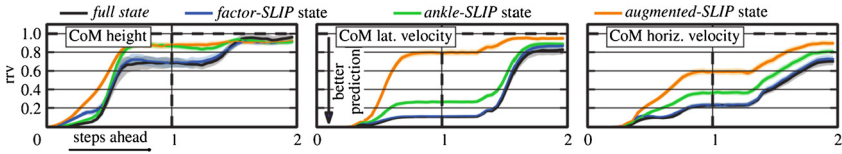


**FIGURE 3.5.6** (Reproduced from Fig. 8 of [Revzen and Guckenheimer, 2008](#)) Comparison of Floquet multiplier magnitude distributions obtained from running cockroaches. Since this analysis is done at a specific phase in the cycle, magnitudes are plotted for three different phases (0.79, 1.57, and 3.14 radians in red, green, and blue, respectively). Experimental motion data is marked with markers; unmarked lines come from surrogates—randomly paired crossings of the surface on which the Floquet multipliers are computed—and offer a null hypothesis which demonstrates that meaningful cycle-to-cycle dynamics exist. A 21-dimensional random effects model selected by the algorithm of [Revzen and Guckenheimer \(2008\)](#) (gray confidence band with green center-line) shows the portion of the Floquet multiplier magnitudes that can be explained by random effects. In this 27-dimensional dataset, the template dynamics are therefore at most 6-dimensional. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this chapter.)

from the limit cycle. For example, a Floquet multiplier of magnitude 0.5 would be associated with a mode that decays by a factor of two every cycle. Floquet modes evolve independently of each other, and thus any subset of modes is, in principle at least, a reduced-dimension model of the dynamics.

By the very requirement that they describe the long-term dynamics of locomotion, templates will thus comprise modes that correspond to the larger Floquet multipliers. This observation allows Floquet multipliers computed from experimental data to be sorted by magnitude and compared with the Floquet multipliers of a null (random effect) model ([Revzen and Guckenheimer, 2011](#)). The multipliers that cannot be accounted for by random effects may be counted, and provide an upper bound on the dimension of a template model that can reasonably be supported with those data (see [Fig. 3.5.6](#)).

The Floquet models obtained from DDFA may be used to extend existing models of locomotion by identifying additional states that improve prediction. In the case of human running, while the SLIP model has an excellent fit to observations ([Ludwig et al., 2012](#)), it fails to predict stability properties, and is in fact



**FIGURE 3.5.7** (Reproduced from Fig. 5, Maus et al., 2014) ability of various models to explain observed quantities in human running data, plotted as “relative remaining variance (rrv)”: the ratio of residual variance to data variance. An rrv of 1 means no predictive ability; rrv of 0 is perfect prediction. The “full state” DDFA model, and the “factor-SLIP” model derived from it are better predictors than the “augmented-SLIP” model which is itself slightly more powerful than classical Spring-Loaded Inverted Pendulum models. The structure of the data driven factor-SLIP suggested adding ankle states to the system, leading to the physically meaningful ankle-SLIP model and capturing most of the potential prediction gains of the DDFA full state model.

unstable at some of the range of running gait parameters humans use. In attempting to predict step-to-step running dynamics, Maus et al. (2014) showed linear feedback using an augmented SLIP model whose state consists of SLIP state variables and all SLIP parameters was less effective at predicting future states than a DDFA-derived linear model. By subjecting the DDFA model to factor analysis, five governing linear factors were obtained for a state with nearly 200 dimensions. Examination of the weights in these factors suggested that adding an ankle state could extend SLIP and give large improvements in prediction (see Fig. 3.5.7). This showed that DDFA modeling may be used to incrementally extend existing analytical models for specific goals, e.g., maximizing predictive ability.

### 3.5.7 SUMMARY

At intermediate speeds, limit cycle oscillators are a useful reduced model of legged locomotion. The rich theory and tools available for analysis of oscillator dynamics provide a uniform language for expressing and understanding gaits.

Future work includes the substantial space for improvement in the numerical algorithms used for DDFA and development of algorithms that require shorter time series. Better algorithms for identifying parameters of coupled oscillator models of locomotion are needed, as most of the coupled oscillator methods from the physics literature (Pikovsky et al., 2003) assume far weaker coupling and far lower phase noise. New directions from Koopman Theory (Budišić et al., 2012) suggest a reframing of DDFA in terms of decomposition of oscillator dynamics into Koopman modes, although numerical algorithms for accomplishing this goal are in their infancy. Finally, little to no work exists on the identification and numerical analysis of hybrid oscillators, as the theory of such oscillators is a recent addition to the field.

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## Chapter 3.6

# Model Zoo: Extended Conceptual Models

Maziar A. Sharbafi\* and André Seyfarth†

\*University of Tehran, Tehran, Iran †Lauflabor Locomotion Laboratory, TU Darmstadt, Germany

In the previous chapters we explained how abstractions and simplification can help understand locomotion principles. For this, several locomotion models with reduced representation of the human body were introduced. In general, the description of legged systems can be based on

- **Highly simplified models** (e.g., template models) which focus on the principal dynamics of the movement using only few parameters, or
- **More detailed simulation models** (e.g., muscle–skeletal models) like OpenSim (<http://opensim.stanford.edu>) and AnyBody (<http://www.anybodytech.com/>) with a high number of degrees of freedom (DOF) and with many model parameters.

In this chapter we will describe how simplified models can be subsequently extended in order to increase the level of more detail of the simulation models.

Whereas complex simulation models are often directly related to the structure of the human body (body segments corresponding to bones, muscles, tendons and other soft tissues), the design of conceptual simplified models highly depends on mechanical intuition like in the inverted pendulum (IP) model (Cavagna et al., 1963), the lateral leg spring (LLS) model (Schmitt and Holmes, 2000), or the spring-loaded inverted pendulum (SLIP) model (Blickhan, 1989; McMahon and Cheng, 1990). These models are focusing on describing the *axial leg function* as a simple telescopic leg spring, with either a constant leg length during stance (IP model) or a leg force proportional to the amount of leg compression (LLS or SLIP model). The assumption of spring-like leg function can be found (in approximation) experimentally both in animals (Blickhan and Full, 1993) and humans (Lipfert, 2010) during steady state locomotion. However, there are also clear deviations in the locomotion dynamics that are not well described by these simple models.

The key limitations of both the IP model and the SLIP model as the most common template models for legged locomotion are summarized in Table 3.6.1. Corresponding model extensions that are suitable to overcome these limitations are also presented. It is important to note that we only select elementary extensions of the model, however, also combinations of the model extensions



**TABLE 3.6.1** Extensions of the Template Models to Resolve the Limitations in Explaining Locomotion Features

Limitation of model	Extension	Extended model	Description
Focus on a single leg	Second leg	Bipedal SLIP (B-SLIP) rimless Wheel, IP with swing leg dynamics	Required to study different gaits (e.g., walking and running)
	More legs	Quadrupedal SLIP (Q-SLIP)	For animal or infant locomotion
Focus on axial leg function	Rigid trunk	SLIP with trunk (T-SLIP)	Enables control of body posture
	Foot segment	IP with trunk (e.g., bisecting) SLIP with rigid flat / curved foot (F-SLIP) IP with rigid flat /curved foot	Enables roll-over function of foot with shift in center of pressure (COP) during contact
	Leg segments	2-segment leg with thigh and shank 3-segment leg with thigh, shank and foot	Leg geometry influences transfer between joint torque and leg force
	Hip spring model	Hip spring between both legs	Tuning of leg swing with stance leg
Prescribed axial leg function	Varying leg parameters in stance phase	E-SLIP	Permits energy stability with change in leg spring parameters in midstance
		VLS-SLIP	Permits energy stability with variable leg spring parameters during stance
		LIP	Permits changes in leg length and leg force
Mass-less leg	Add leg masses	M-SLIP Passive dynamic walker or Acrobot model	Considers leg masses in stance leg

**TABLE 3.6.1** (continued)

Limitation of model	Extension	Extended model	Description
Focus on sagittal plane	Lateral movements of COM	3D SLIP	Permits 3D running and walking with lateral leg placements
		3D LIP	Permits 3D running and walking with lateral leg placements
Purely mechanical description	Add muscle dynamics	Leg with muscle model	Actuation of the leg through muscle forces with optional reflex pathways

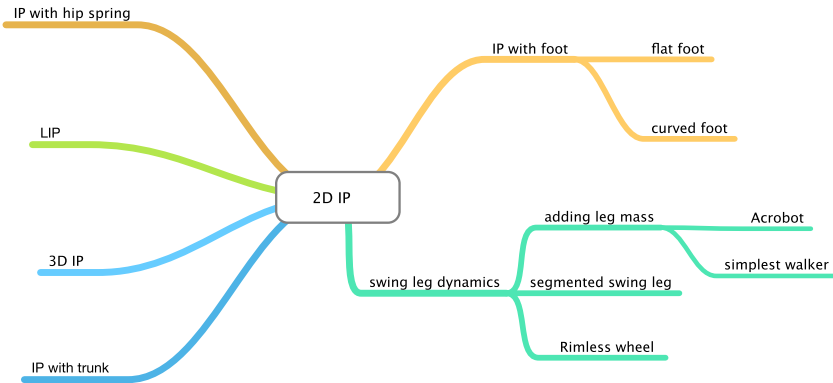
are possible to consider, like XT-SLIP (Sharbafi et al., 2013a) which is an extended SLIP model with trunk (T-SLIP), and added leg mass (M-SLIP) or the ballistic walking model presented of Mochon and McMahon (1980). Model extensions can address either mechanics or control of the system. Another class of model extensions comprises muscles (e.g., single-joint and two-joint muscles with muscle fiber-tendon dynamics) and neural circuits (e.g., sensory feedback pathways) describing muscle stimulation and integration of sensory signals. A sophisticated extension of the SLIP model including muscles, reflex pathways, and segmented legs is the gait model of Geyer and Herr (2010), which originates on the neuro-muscular model introduced by Geyer et al. (2003).

The extensions of IP and SLIP models described in this subchapter (Table 3.6.1) are shown in Fig. 3.6.1 and Fig. 3.6.2, respectively. The reasoning of the different extensions in both templates is often similar. In the following, we will describe selected model extensions in more detail. We will start with model extensions regarding the leg structure, followed by models describing the dynamics of the trunk and finally models including lateral leg placements and locomotion in 3D.

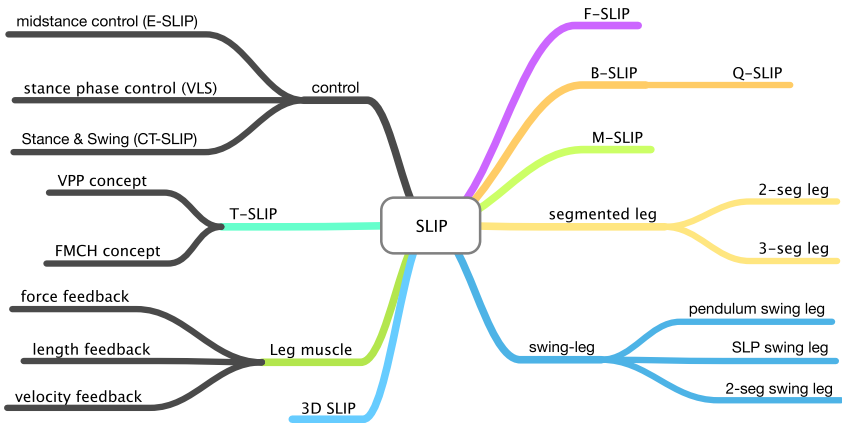
### 3.6.1 MORE DETAILED REPRESENTATIONS OF THE LEG

In the aforementioned template models, a point mass sits on top of a rigid or compliant massless leg. The focus of these models is on CoM movement, which considers the stance leg movement as the first locomotion subfunction and partially leg swinging (the second locomotion subfunction). The following extensions in the leg structure addressing more features in each of these two subfunctions will be presented:

- **Stance leg:** (a) adding leg mass, inertia and damping, (b) adaptation of leg parameters during motion and (c) increasing number of segments



**FIGURE 3.6.1** Extensions of the sagittal inverted pendulum (2D IP) model with selected added model features: hip spring between both legs, foot (flat or curved) attached to the lower end of the IP, swing leg dynamics by adding leg masses, segmented swing leg or rimless wheel model, linear inverted pendulum (LIP) with leg force law, including lateral movements (3D IP), and adding a trunk. Different control policies can be applied to each of these model extensions, e.g., the capture point concept for LIP model (Pratt et al., 2006).



**FIGURE 3.6.2** Extensions of the sagittal SLIP model with selected added model features: foot segment (F-SLIP); the number of legs (bipedal B-SLIP, quadrupedal Q-SLIP, etc.); leg masses (M-SLIP); segmented legs, 2 and 3 leg segments; swing leg dynamics as a pendulum, spring loaded pendulum SLP or two-segmented swing leg; control for varying leg spring properties during stance (VLS-SLIP), at mid-stance (E-SLIP) or continuous control during step (CT-SLIP); added trunk (T-SLIP); muscle-like leg function (leg muscle) and different reflex pathways (force, length, and velocity feedback) and lateral movements (3D SLIP). Each of these model extensions can be considered as a separate or in combination with others, e.g., BT-SLIP. Gray color indicates control features of SLIP based models.

- **Swing leg:** (a) addition of one or more legs (b) increasing number of segments in swing leg, (c) adding leg mass

The simplest way to extend the locomotion template models regarding the leg function is adding additional massless legs to the body. With this more advanced swing leg adjustment approaches (e.g., leg retraction; [Wisse et al., 2006](#)) can be developed. Second groups of extensions can be from the control point of view. For example adaptation of the stance leg parameters during gait (E-SLIP, [Ludwig et al., 2012](#)) or a continuous unified controller for swing and stance phase (CT-SLIP, [Seipel and Holmes, 2007](#)) which enhances the model abilities in reproducing more biological and also more stable gaits while keeping the model complexity. Third, the legs can be represented in a more physical way by considering leg mass, inertia or damping. Finally, the number of segments can approach the numbers in human/animal legs. In the following, we explain some of these extensions in related models.

### 3.6.1.1 Extending the Number of Limbs (B-SLIP, Q-SLIP)

Inspired by the work on the SLIP model, [Herr et al. \(2002\)](#) developed a quadrupedal SLIP model to describe trotting and galloping in several animals (chipmunk, dog, goat, and horse). The model was extended with a compliant trunk (described by neck and back stiffness). Hip and shoulder were actively powered resulting in a running pattern that was similar in kinematics (e.g., limb angles) and kinetics (e.g., peak force, limb stiffness) to experimental data. Interestingly, swing leg retraction was identified as a key feature required obtaining stable running in the model ([Seyfarth et al., 2003](#)). A very similar quadrupedal SLIP model with rigid trunk was created following the design of the Scout II robot ([Poulakakis et al., 2005](#)). The predicted stable bounding gait is in close agreement with the behavior observed in the robot. Later on, the model and robot dynamics were extended to galloping ([Smith and Poulakakis, 2004](#)).

[Geyer et al. \(2006\)](#) extended the sagittal SLIP model to a bipedal version (B-SLIP), which was the first model capable of predicting walking and running gait within the same model setup. In this model, the only parameter change required to achieve different gaits was the system energy. For moderate speeds (around 1 m/s) walking patterns with double humped profiles of the ground reaction force are found. In contrast, with higher locomotion speed, a running gait with single-humped patterns of the ground reaction force is observed. Surprisingly, the model predicts further walking gaits with more than two humps for lower energies. Such gaits (e.g., with three-humped force patterns) can indeed be found in human locomotion like in amputees' gaits or in the development of gait during early childhood ([Gollhofer et al., 2013](#)).

### 3.6.1.2 Rimless Wheel

The inverted pendulum model is developed to explain walking. Because of the rigidity of the stance leg, no flight phase exists to represent running or hopping. Therefore, the minimum number of legs in this model is 2. The addition of the number of legs for this model is not common because it is usually applied to understand human gaits or developing passive dynamic robots except for increasing stability in 3D, e.g., McGeer passive walker (McGeer, 1990). However, rimless wheel model can be considered as an extension of inverted pendulum model with adding more legs, which are coupled with a fixed angle between limbs (McGeer, 1990). More explanations about the passive dynamic walking model and rimless wheel can be found in Section 4.4.

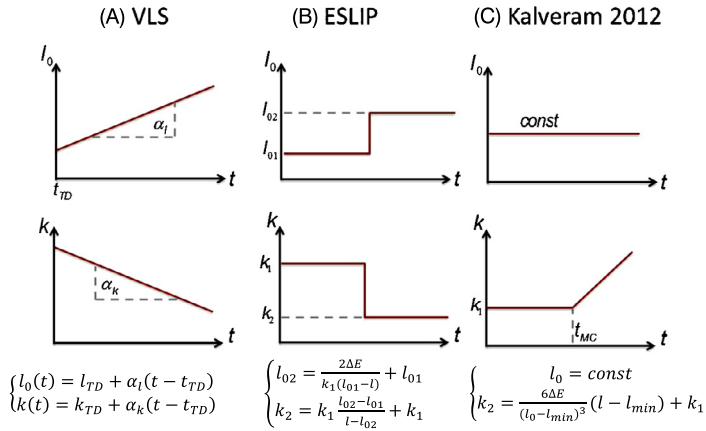
### 3.6.1.3 Stance Leg Adaptation (VLS and E-SLIP)

In the SLIP model, stance leg parameters like leg stiffness and angle of attack are often set to a specific value. This usually represents the steady-state (average) gait pattern during locomotion. However, leg function varies from step to step (e.g., in response to ground level changes, Daley et al., 2007; Müller et al., 2010, and also during the stance phase Riese et al., 2013). Such variations in leg parameters can be represented in extensions of the SLIP model in order to better match experimental data. With this also deviations from the conservative spring-like leg function can be described which may lead to also energetically stable gait patterns.

During human locomotion, there is a tendency towards higher leg stiffness during leg loading (leg shortening) compared to unloading (leg lengthening). Additionally, the leg length is often larger at takeoff compared to touchdown (Lipfert et al., 2012). There are several possible explanations for that, e.g., eccentric force enhancement during leg compression or the role of leg segmentation (Maykranz et al., 2009; see F-SLIP model below).

Based on the SLIP model, two simple approaches were introduced to address changes in leg parameters during stance phase:

- 1) In the variable leg spring (VLS) model a continuous change of leg parameters over time is assumed (Riese and Seyfarth, 2011). For stable hopping, a decrease in leg stiffness and a continuous increase in rest length of the leg spring (Fig. 3.6.3A) were required in the model unless sufficient leg damping is provided. This is in line with experimental findings on changes in stance leg parameters during human locomotion (Lipfert et al., 2012).
- 2) In the E-SLIP model, a sudden change in leg parameters at midstance is considered (Fig. 3.6.3B) without a sudden drop or increase in leg force. This model permits to consider step-to-step changes in system energy as found in human running (Ludwig et al., 2012).



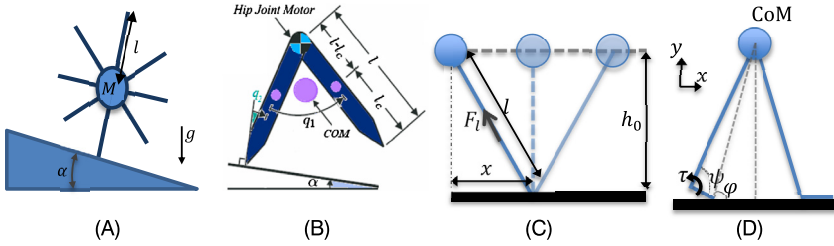
**FIGURE 3.6.3** Extensions of the SLIP model with adjustable leg springs parameters (stiffness  $k$  and rest length  $l_0$ ). In the variable leg spring VLS approach, leg stiffness and rest length change linearly with time (Riese and Seyfarth, 2011). In the two other approaches (ESLIP and Kalveram et al., 2012), a fixed amount of energy  $\Delta E$  is added during stance phase after maximum leg compression ( $t_{MC}$ ) when the leg length reaches its minimum value ( $l_{min}$ ). The equations show the parameters used in the figure.

- 3) Following the approach of Kalveram et al. (2012), leg stiffness can be changed during leg extension (leg unloading) such that a defined amount of energy is injected to the leg. This approach inspired the control of Marco hopper robot as well as the Marco-2 hopper robot with segmented leg (Oehlke et al., 2016).

Changes in leg parameters in steady-state movements were observed experimentally at a global (leg) level (Lipfert et al., 2012; Lipfert, 2010 for walking and running; Riese et al., 2013 for hopping; Seyfarth et al., 1999 for take-off phase in long jump) as well as at local (joint or muscle) level (Peter et al., 2009 AMS for running). So far, it is still unclear whether and how limb stiffness is adjusted at a global (leg) level or a local (joint, muscle) level. It remains for future research to investigate in more detail how changes in state variables (angles, angular velocities) and environmental changes (e.g., changed ground properties) effect these adjustments of leg parameters during stance phase.

### 3.6.1.4 Clock-Torque SLIP (CT-SLIP)

In order to keep the simplicity of the SLIP model and increasing the ability to predict more features of legged animal and robot locomotion dynamics, the CT-SLIP model was developed by Seipel and Holmes (2007). In this model, a reference clock drives the leg movement (using a PD controller for stance leg) while damping is added to the stance leg. In this model, the same mechanism



**FIGURE 3.6.4** Extensions of IP model (A) rimless wheel, (B) Acrobot, (C) LIP, and (D) actuated ankle.

(continuous leg rotation) is utilized to control the leg in both stance and swing phases. This model, which is inspired by RHex robot (Saranli et al., 2001), can address hip actuation, more realistic take off and touch down (with leg retraction) and more importantly, a more robust gait compared to the SLIP model, without increasing dimension of the model (Seipel and Holmes, 2007).

### 3.6.1.5 Linear Inverted Pendulum Mode (LIPM)

The original IP model does not consider displacements in axial leg direction during stance and thus forces the CoM to move on a circular arch. Introducing a prismatic joint in stance leg converts the IP into a SLIP model if the generated force is proportional to the leg length. In that respect SLIP can be considered as an extension of IP with additional leg spring.

For a long time, many studies in walking were related to the CoM movement described by the inverted pendulum paradigm and the six determinants of gait (Kuo, 2007). Based on minimization of CoM displacement, the six determinants of gait theory (Saunders et al., 1953) result in no vertical CoM excursion in walking. In 1991, the linear inverted pendulum model was introduced by Kajita and Tani (1991), in which the leg force is determined to compensate gravity, resulting in zero vertical acceleration. The ground reaction force can only act along the leg axis (CoP–CoM line) and the vertical element of the leg force should be equal to the body weight ( $Mg$ ). According to parameters shown in Fig. 3.6.4C, the required leg force ( $F_l$ ) to achieve the CoM height ( $h_0$ ) when the horizontal distance between CoM and CoP equals  $x$  is computed as follows:

$$F_l = \frac{Mg}{h_0} l = \frac{Mg \sqrt{x^2 + h_0^2}}{h_0}. \tag{3.6.1}$$

Following this approach, leg force is predicted to increase with leg lengthening, which is opposite to experimental findings (Lipfert et al., 2012) and the concept of spring-like leg function.

The LIPM model was used to develop capture point concept (Pratt et al., 2006) as a method for leg adjustment to reach zero forward speed at vertical leg configuration within one step (see Section 2.2 for details). Some versions of this model consider a rotation around the CoM by using an upper body (e.g., Kajita and Tani, 1991 with a constant angular velocity) or a flywheel with torque control (Pratt et al., 2006).

### 3.6.1.6 Addition of Leg Mass to IP (Acrobot, Simplest Walking Model)

The “pure” IP model with massless legs (Alexander, 1976; Hemami and Golli-day, 1977; Wisse et al., 2006) is rarely utilized in walking analysis. Addition of a point mass to each leg can simplify control (e.g., based on passive swing leg movement) and also makes the model more realistic. The resulting model was called “the simplest walking model” (Garcia et al., 1998) or the “compass gait model” (Goswami et al., 1996). The compass gait concept was already pointed out by Borelli (1680) in his famous book “De Motu Animalium.” This popular model is able to represent walking without the need for active control of the swing leg. The stability of the predicted gait was well analyzed (Goswami et al., 1996). Investigation of the limit cycle stability for walking on slope with this model versus parameter variations (slope, normalized leg mass, and leg length) demonstrate that a wide range of solutions gradually evolves through a regime of bifurcations from stable symmetric gaits to asymmetric gaits and eventually arriving at an apparently chaotic gait where no two steps are identical (Goswami et al., 1998).

Different leg mass locations are considered, like the leg’s CoM position (at about the center of the leg) like in the passive dynamic walking model (McGeer, 1990) or small masses at tip toes (Garcia et al., 1998). A very similar model compared to the simplest walking model is the Acrobot model (Westervelt et al., 2007). In this model the mass is distributed along the leg and not concentrated at the hip. In general, addition of leg mass (i) can simplify control, (ii) enables passive walking down a shallow slope, (iii) permits describing leg swinging (another locomotion subfunction), but at the same time it (iv) requires control, e.g., of hip torques when walking on flat terrain to stabilize the gait and to compensate for energy losses (energy management). Please find more explanations on passive dynamic walking models in Subchapter 4.6.

### 3.6.1.7 Addition of Mass to SLIP Leg (M-SLIP)

In the M-SLIP (Peucker et al., 2012) model, the leg is represented by a rigid leg segment and a prismatic spring attached at the distal part of this segment.



In stance phase, the spring is aligned in leg axis by a viscoelastic rotational coupling between the rigid segment and the leg spring. During swing phase, the prismatic leg spring is bent such that the leg segment can freely swing forward. The leg angle is adjusted by setting the rest angle of the rotational hip spring and this leg angle switches between two values depending on the state (one stiffness for swing phase and another one for stance phase).

With leg masses, the gait dynamics is more realistic but also more complex (e.g., landing impacts). Compared to the SLIP model, the predicted solutions for stable running of a one-legged system with leg masses (M-SLIP) are shifted towards flatter angles of attack (Peucker et al., 2012). In an alternating, bipedal M-SLIP model, however, the inertial effects of both legs are compensating each other such that the region of stable running is similar to the one observed in the SLIP model. This indicates that also the model with leg masses can inherit solutions of the SLIP model. At the same time, leg inertia of the leg with mass permits creating swing-leg trajectories (e.g., by introducing a hip torque) that were not represented by the original SLIP model. In this model, a PD (proportional, derivative) controller is used for hip torque control in walking. It is similar to the hip spring model developed by Dai Owaki for running (Owaki et al., 2008).

### 3.6.1.8 Extending SLIP with Leg Segments (F-SLIP, 2-SEG, 3-SEG)

Biological limbs are designed as a serial arrangement of leg segments with muscles spanning the leg joints. In the following, benefitting from muscle properties, we describe a number of extended SLIP models using muscle mechanics, which lead to a more detailed representation of leg segments and describe their effects on the gait dynamics.

In the **F-SLIP model** (Maykranz et al., 2009), the prismatic leg spring is extended distally by a rigid foot segment, which is attached by a rotational foot spring (similar to the ankle joint). This model permits to describe a shift of the center of rotation along the foot segment as found in heel to toe running or in human walking. Similar to the VLS model it can describe an increase in leg length from touchdown to take-off due to the asymmetric arrangement of the foot, pointing forward (Maykranz and Seyfarth, 2014). The resulting force-length curve of the leg indicates a drop in leg stiffness from early to late stance phase. This is a consequence of the mechanical action of the compliantly attached foot segment. In late stance the foot joint (ankle) is leaving the ground resulting in a more realistic representation of the push-off in human locomotion. Surprisingly, the F-SLIP model is able to predict running well, but has limited capability to generate walking patterns.

Another segmented model extending the SLIP model is the 2-segment model by [Rummel and Seyfarth \(2008\)](#). Here, the leg spring is replaced by a rotational (knee) spring attached between the two massless leg segments (upper and lower leg). The analysis of the model shows that running solutions can be observed for different rest angles of the knee spring with clear deformations in the predicted regions for stable locomotion. With extended rest angles (knee joint angles between 150–170 degrees), a larger region of angles of attack compared to the SLIP model result in stable running. In contrast to the SLIP model, knee stiffness needs to be increased for faster running. This increase of knee stiffness with speed was also found experimentally ([Rummel and Seyfarth, 2008](#); [Lipfert, 2010](#)). There has been a number of similar leg models with two segments presented with muscle-like joint function, e.g., for describing jumping ([Alexander, 1990a, 1992](#); [Seyfarth et al., 2000](#)) and hopping tasks ([Geyer et al., 2003](#)).

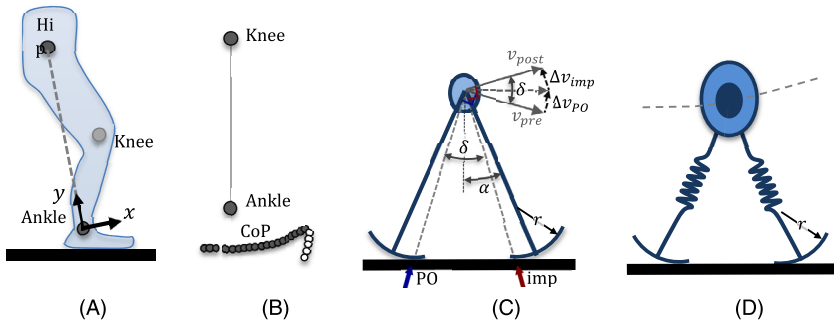
Finally, in a three-segment model including foot, shank and thigh, the adjustment of joint stiffness for spring-like leg function was investigated by [Seyfarth et al. \(2001\)](#). This simulation study shows that a shared loading of knee and ankle requires not only a proper distribution of knee and ankle stiffness but also additional mean to avoid joint buckling or overextension. The following means for achieving stable leg function could be identified ([Seyfarth et al., 2006](#)):

- 1) Elastic two-joint connection between ankle and knee (e.g., gastrocnemius muscle)
- 2) Asymmetric segment lengths with shorter foot and asymmetric joint configurations (extended knee, bent ankle)
- 3) Joint constraints (e.g., heel contact by calcaneous) prevents too large ankle bending and avoids knee overextension
- 4) Nonlinear progressive joint stiffness (with larger nonlinearity in knee compared to ankle)
- 5) Transition from a zig-zag mode to a bow configuration of the leg (like in spiders)

The transfer of this mechanical three-segment leg model to a muscle–skeletal model was presented by [Geyer and Herr \(2010\)](#).

### 3.6.1.9 Ankle Actuated IP

The position dependence of passive ankle joint mechanics was shown in [Weiss et al. \(1986\)](#). Considering a flat foot and elastic element to model ankle torque, [Ahn](#) developed an ankle actuated IP model ([Ahn, 2006](#)) (see [Fig. 3.6.4D](#)). In this model, the constraint of having instantaneous double support in IP is resolved. A rotational spring with rest angle equal to  $\pi$  starts working for the trailing leg



**FIGURE 3.6.5** Curved foot model (A) knee–ankle–foot (KAF) coordinate system (adopted from Hansen et al., 2004), (B) CoP motion in KAF coordinate system (adopted from Hansen et al., 2004), (C) IP with curved feet (adopted from Adamczyk et al., 2006). Push-off (PO) produces positive work which should be equal to the negative work consumed at impact (imp) for periodic gait. (D) SLIP with curved feet (Whittington and Thelen, 2009).

after touchdown of the other leg until take-off or reaching straight ankle angle:

$$\tau = \begin{cases} k(\pi - \psi) & \psi < \pi, \\ 0 & \psi \geq \pi. \end{cases} \quad (3.6.2)$$

This preloaded ankle spring (Fig. 3.6.4B) injects energy and supports push-off.

### 3.6.1.10 Curved Feet Model

In bipedal locomotion the center of pressure (CoP) is not fixed on the ground like assumed in gait template models (with point contact). Extending the model with flat feet is a possible solution for introducing a moving CoP during ground contact. This model extension needs additional ankle torque control (Ahn, 2006). A simpler solution, which can generate human-like CoP movement without requiring ankle torque control, is using curved feet (McGeer, 1990). In human walking, a circular path with a radius of curvature 30% of leg length represents the CoP trajectory in knee–ankle–foot (KAF) coordinate system shown in Fig. 3.6.5A, B (Hansen et al., 2004). Similar ratio between the foot curvature radius and the leg length was found in the case of different prosthetic legs (Curtze et al., 2009) and also agrees closely with McGeer’s robot (McGeer, 1990).

In line with these findings, extended IP and SLIP models with curved feet were used to investigate foot function in walking. Inspired by McGeer’s robot (McGeer, 1990), Kuo presented an IP model with arc-shaped feet, called “Anthropomorphic Model” (Kuo, 2001). This model was used to predict the preferred speed–step length relationship (Kuo, 2001) and later to predict the effects of changing the radius of curvature on cost of transport (Adamczyk et al., 2006). In the latter study, the mechanical work and metabolic activities of human body

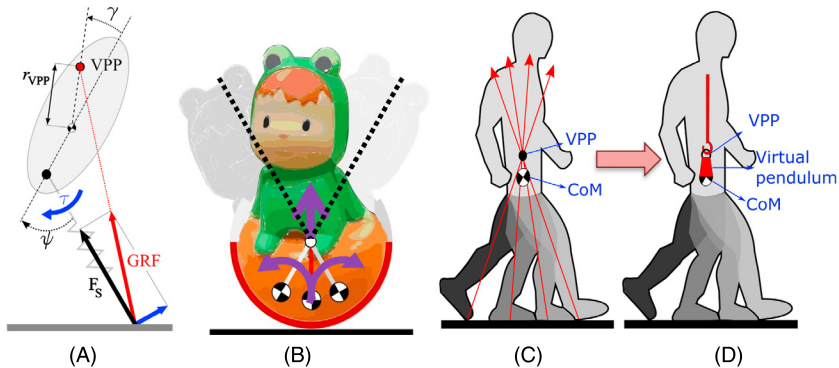
are measured in walking experiments with shoes having different curvatures showing that using 33% of the leg length as the curved foot radius results in the minimum mechanical work and metabolic rate activities (Adamczyk et al., 2006). Considering impulsive push-off as the energy source to compensate losses at impact (Fig. 3.6.5C), it is shown that having this curvature will appear energetically advantageous for plantigrade and human walking, partially due to decreased work for step-to-step transitions (losses is reduced with any nonzero radius (Fig. 3.6.5C, as  $\delta < 2\alpha$ ). In Whittington and Thelen (2009), a new SLIP model extended with curved feet (Fig. 3.6.5D) illustrates stable gait for an interval of feet curvature and shows that increase of foot radius up to one-third of the leg length decreases the maximum amount of the ground reaction force. All these studies show that extending template models with arc-shaped feet is useful for analyzing gait dynamics and energetics.

### 3.6.2 UPPER BODY MODELING

For posture control, the third locomotion subfunction, we need to extend the template models by adding an upper body, e.g., by a rigid trunk. With this additional degree of freedom, developing a controller for balancing is required. Therefore, several models were developed to address posture balance based on template models. In extended IP with a rigid torso, often traditional control engineering methods are used for keeping the torso upright (McGeer, 1988; Grizzle et al., 2001; Gregg and Spong, 2009). In Wisse et al. (2004) a passive model is presented, in which the upper body is aligned mechanically creating a bisecting angle between the two legs. In the following we explain bioinspired SLIP-based models for posture control based on by human/animal locomotion: VPP, compliant hip, and FMCH.

#### 3.6.2.1 Virtual Pivot Point (VPP)

In the SLIP model the body dynamics is described by a point-mass. This model can only describe leg force pointing to this point-mass which differs from GRF patterns in human (or other bipedal) gait. During locomotion, the forces acting on the body are not necessarily directed to the center of mass (COM). For instance, in human walking the stance leg forces point to a slightly above the COM. In order to describe such deviations of the leg force from the leg axis (from contact point at ground to COM), the point-mass needs to be replaced by an extended body, e.g., a rigid trunk (Fig. 3.6.6A). To study the control of a hopping robot, Poulakakis and Grizzle (2009) extended the SLIP model with a rigid upper body. They used the hybrid zero dynamics (HZD) approach to successfully control the system. Maus et al. (2010) applied the same extension

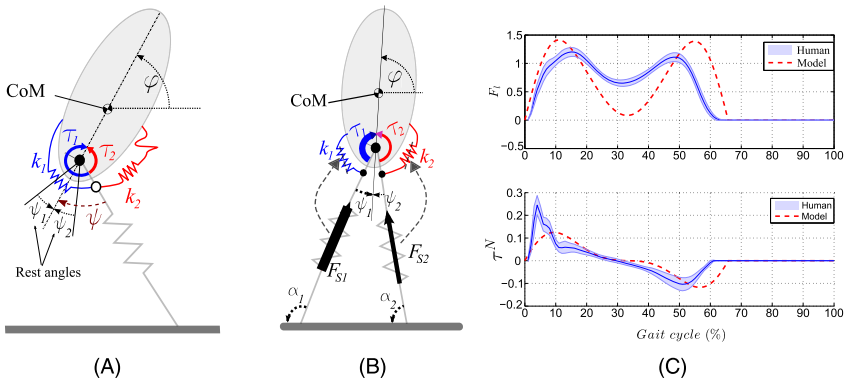


**FIGURE 3.6.6** (A) T-SLIP model and VPP control concept introduced by [Maus et al. \(2010\)](#). (B) Virtual pendulum point (VPP) concept shown illustrated in a Roly Poly toy, (C) GRF vectors and VPP concept in human walking. (D) Virtual pendulum concept in human locomotion.

(rigid upright trunk) to a bipedal SLIP model to implement the virtual pivot point (VPP) concept. This approach assumes leg forces to intersect at a fixed location above the body COM to keep postural balance like a Roly Poly toy ([Fig. 3.6.6B, C](#)). Both stable walking and running could be predicted by this model ([Fig. 3.6.6A](#)) and the predicted hip torques are similar to those observed in human walking. As a result, the inverted pendulum model of locomotion can be transferred to a periodic movement, modeled by a regular virtual pendulum (VP) as shown in [Fig. 3.6.6C, D](#).

### 3.6.2.2 Force Modulated Compliant Hip (FMCH)

The key idea behind the FMCH approach is to substitute the VPP concept with a structural model that has physical representation. With that we want to keep the basic idea of a virtual pendulum (VP) through adaptable hip compliance. Therefore, first, the hip joint (between trunk and leg) of the T-SLIP model was equipped with a passive spring ([Fig. 3.6.7A](#)) simulating the effects of extensor and flexor muscles resulting in stable walking ([Rummel and Seyfarth, 2010](#)). Stable running and hopping could be predicted by implementing the virtual pendulum (VP) concept using passive hip springs ([Sharbafi et al., 2013b](#)). In this study, the quality of posture control based on of the passive compliant hip was compared to the virtual pendulum posture controller (VPPC) and also the hybrid zero dynamics (HZD, [Westervelt et al., 2007](#), see Section 4.7). The robustness and control quality (e.g., settling time) with passive hip springs are worse than VPPC, but sufficient considering passivity of the control. Then, by applying the leg force to modulate hip compliance within FMCH model, a large improvement in balance control was achieved. It results in human-like posture balance and provides a mechanical explanation for the VPP concept



**FIGURE 3.6.7** (A) T-SLIP with passive compliant hip. (B) Force modulated compliant hip (FMCH) model for walking. (C) Comparison of the leg force and the hip torque in human walking and FMCH model at normal walking speed.

(Sharbafi and Seyfarth, 2014, 2015). In this force modulated compliant hip (FMCH) model (shown in Fig. 3.6.7B) the hip torque ( $\tau$ ) is a product of a constant ( $c$ ), the leg force ( $F_s$ ) and the difference between the hip to the leg angle ( $\psi$ ), and its rest angle ( $\psi_0$ ) as follows:

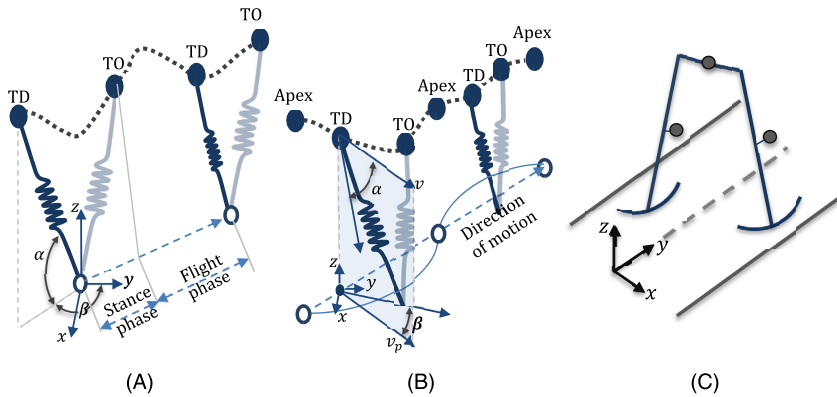
$$\tau = cF_s(\psi - \psi_0). \quad (3.6.3)$$

It is mathematically shown that the required torque in VPPC is precisely approximated by FMCH in a range of hip and leg movements which are representative for human gaits (Sharbafi and Seyfarth, 2015). Fig. 3.6.7C shows the leg force and hip torque developed by the FMCH model for stable walking at normal walking speed (1.4 m/s) compared to the human experimental results. In addition to explain human gaits, this concept can be utilized in bipedal robot control and assistive devices (e.g., exoskeleton).

This model can be considered as a candidate for neuro-mechanical template for posture control. The model suggests a sensory pathway originating at a force sensor of the leg extensor muscle (e.g., in the knee) and a gain factor (constant  $c$ ). In contrast to the neural system, no processing delays are considered in the FMCH model. Also, the muscle function is reduced to an activation-dependent tunable spring. These are clear simplifications compared to neuro-muscular processing of sensory data.

### 3.6.3 EXTENSION TO 3D

In order to extend the models to 3D, in addition to increasing the system degrees of freedom and enlargement of the state space the lateral leg placement is the main challenge. In the following the 3D SLIP and IP models are presented.



**FIGURE 3.6.8** (A) 3D SLIP model with leg adjustment w.r.t. world coordinate (Seipel and Holmes, 2005). The desired leg orientation is defined by  $\alpha$  and  $\beta$ . (B) 3D SLIP model with leg adjustment in body coordinate (Peucker et al., 2012). In this approach the velocity vector ( $v$ ) defines the body coordinate frame. The vector  $v_p$  is the projection of  $v$  in the horizontal plane ( $x, y$ ). The body lateral plane is defined by  $v$  and  $v_p$ . The desired leg orientation  $l$  is defined by  $\alpha$  and  $\beta$  the angle between  $v$  and projection of  $l$  in the frontal plane and the angle between  $v_p$  and the projection of  $l$  in the horizontal plane, respectively. (C) 3D passive dynamic walking model (McGeer, 1990).

### 3.6.3.1 3D SLIP

Like in 2D SLIP, stable gaits require a proper leg adjustment. Interestingly, stable gaits are not predicted with a given step length (or step frequency) but by adjusting the leg angle for landing (angle of attack) with respect to gravity. This indicates that the locomotion pattern is rather an outcome than a target of control. For instance, if a subject runs on a treadmill, the variability of the gait pattern increases when the very same preferred step length or step frequency is provided (markings on the belt, metronome) as targets for locomotion (Ludwig et al., 2010).

In 2005 Seipel and Holmes published a paper investigating running stability predicted by a new SLIP model extended to 3D by including a lateral leg placement at touchdown (Seipel and Holmes, 2005). The lateral leg angle was selected with alternating direction (left or right) with respect to a desired running direction (Fig. 3.6.8A). Surprisingly, no stable running patterns were predicted by this novel 3D SLIP model. Later, Peucker et al. (2012) introduced a velocity-based leg adjustment. Here, the leg angle was laterally adjusted within the plane spanned by the COM velocity vector and gravity vector. With this change in the coordinate frame for swing leg adjustment, stable running solutions were predicted for a huge range of angle of attack and lateral leg angles before landing.

In 2014, Maus and Seyfarth developed an extended bipedal SLIP model for 3D walking. The simulation results of this 3D walking model reveals that

changes in leg adjustments between the two legs can result in walking in curves (Maus and Seyfarth, 2014). However, there are combinations of leg parameter adjustments between the two legs which still results in straight walking (with a fixed direction of progression) for even asymmetric leg configurations regarding leg stiffness and angle of attack. The predicted asymmetric walking patterns can be neutrally stable. This means that the direction of walking will change if a sudden lateral push is applied to the body, however after the perturbation the walking direction remains constant. This outcome is similar to the predictions of the lateral leg spring (LLS) model of Schmitt and Holmes (2000) that operates in the horizontal plane only.

### 3.6.3.2 3D IP

The focus of conceptual model based gait analyses is on 2D motion in sagittal plane. Moving from side to side to modulate lateral foot placement and rotating about the vertical (yaw) axis at the ankles are the two observations in biological legged locomotion (humans). One of the first attempts for such extensions was 3D model of passive dynamic walking (Fig. 3.6.6C) incorporating both roll and yaw rotation (McGeer, 1993). However, they found that the model couldn't stably walk without control. Representing the theoretical stability of a walking machine that rocks side to side without yaw motion, Kuo could stabilize the passively unstable system by a simple control scheme inheriting much of the passive behavior (Kuo, 1999).

In Zijlstra and Hof (1997) a 3D inverted pendulum model was utilized to explain human walking in 3D space with a sinusoidal left-right movement of CoM. Using such a 3D compass gait model, Gregg and Spong (2009) extended the planar walking into directional 3D dynamic walking (e.g., moving on a circle) by controlled reduction approach. Other extensions like 3D LIPM (Kajita et al., 2001), 3D IP+torso (Gregg and Spong, 2009), the generalized 3D IP (Sakka et al., 2010), and 3-segmented IP based model with small actuation at ankle (Wisse et al., 2001) are instances of studies to build an anthropomorphic 3D model for stable walking based on inverted pendulum model. Recently, 3LP, a 3D linear IP-based model including torso and swing dynamics, was presented by Faraji and Ijspeert (2016) to represent all three subfunctions of legged locomotion with a IP based model. In addition, they could predict nonlinear speed frequency relationship as one optimality trends in human walking.

## 3.6.4 EXTENSION WITH MUSCLE MODELS

In the previous sections we focused on mechanical representations of legged locomotion. Compared to human locomotion, the complex interactions within the



biological (e.g., human) body were described based on highly simplified models with only a few lumped parameters. For instance, leg stiffness is such a common parameter. It summarizes the complex interaction of segmented body mechanics with active and passive compliant structures (e.g., muscles, ligaments, tendons, connective tissues, etc.) and the environment (e.g., compliant ground contact). In this section we present a number of gait model extensions, which additionally take muscle–tendon dynamics into account.

For spring-like leg operation, a concerted interplay between many components in the biological body is required including:

- Active muscle forces based on muscle properties (force–length and force–velocity relations) and muscle activation dynamics (see Subchapter 8.1),
- Connective tissues
- Titin filaments (see Subchapter 8.1)
- Muscle lever arm geometry at joints
- Tendon compliance
- Geometry of segmented legs
- Muscle arrangement in relation to joints (e.g., two-joint muscles)
- Interface mechanics to the environment (e.g., foot–ground interaction)

In a first approach, the geometry of the leg was represented by two leg segments (see above, e.g., [Rummel and Seyfarth, 2008](#)). To generate leg force, joint torque can be introduced by mechanical components (e.g., rotational spring; [Alexander, 1990b](#)) or by an extensor muscle spanning the joint ([Alexander, 1990a](#); [Seyfarth et al., 2000](#)). Due to the eccentric force enhancement in muscles, leg force becomes larger during leg compression (muscle lengthening) compared to leg extension (muscle shortening) if constant muscle activation is assumed. To generate continuous movements like in hopping or running, a modulation of muscle stimulation with time is required with lower muscle activation during eccentric phase, compared to concentric phase. This even holds if the leg geometry is ignored and the muscle is directly replacing the leg spring. Here, repulsive leg function (like in hopping and running) can be achieved by an appropriate muscle stimulation pattern (feedforward control; [Häufle et al., 2010](#)), by using sensory feedback pathways, or by a combination of both ([Häufle et al., 2012](#)). The combination of feedforward and feedback provides superior stability and perturbation rejection compared to feedforward and feedback schemes in isolation.

In the two-segment leg model with a knee extensor muscle and the neural control of [Geyer et al. \(2003\)](#) it was shown that similar leg function as described in the spring–mass model was predicted by a positive force feedback applied to the leg muscle. This spring-like leg function emerges after 1–2 hopping cycles and recovers quickly after perturbations (e.g., ground level changes).

With more than two leg segments different arrangements of muscles can be considered including one- and two-joint muscles (Seyfarth et al., 2001) spanning ankle, knee, and hip joints. There has been a long scientific debate about the specific role of two-joint (biarticular) muscles, including their ability to coordinate the action of adjacent joints ((Doorenbosch and van Ingen Schenau, 1995)), transfer of energy (Sharbafi et al., 2016), reduced energy and peak power requirements of joint actuation (Grimmer et al., 2012). Another suggested function of two-joint muscles is their ability to direct (orient) leg force (Doorenbosch and van Ingen Schenau, 1995; Sharbafi et al., 2016).

Muscle function is largely supported by the action of compliant structures arranged in series (e.g., tendon) and in parallel to the muscle fibers (e.g., titin). Serial elastic elements can reduce the muscle fiber displacement and speed during stretch-shortening-cycles (e.g., in jumping, running or walking; Seyfarth et al., 2000). In contrast, parallel elastic elements (Rode et al., 2009) help reduce the muscle fiber force, but keep the elongation and the speed of the fibers unchanged. Both elastic elements can largely reduce the energy and the peak power requirements of the muscle during movement.

The potential role of sensory feedback for achieving stable locomotion was demonstrated in a 7-segment neuromuscular human walking model presented by Geyer and Herr (2010). In this model, seven muscles were represented in each leg. The muscles were controlled by tuning the corresponding reflex parameters (sensor source, gain, delay). The model was continuously extended during the last years and can predict human-like walking and running at different speeds and in different environmental conditions (e.g., stairs, slopes, curves). This model is described in more detail in Subchapter 6.5.

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