# 5. The principle of virtual work

The principle of virtual work is the starting point for the formulation of the finite element method for solids and structures and we dedicate this chapter to its study.

We start by introducing the principle of virtual work for a 1-D model and by exploring the fundamental facts. Then, the principle is given for the 3-D elasticity model and for the remaining mathematical models discussed in Chapter 4.

In the last section, we introduce some energy concepts in 3-D analysis which provide alternative ways to formulate the mathematical models of Chapters 3 and 4. These energy based formulations will be useful in Chapters 6 and 8.

## 5.1 The principle of virtual work for the bar problem

We introduce in this section the principle of virtual work considering a one-dimensional problem, namely a bar subjected to distributed axial loading $f(x)$ and a concentrated load $R$ at its end as shown in Figure 5.1a.

The mathematical model for this problem was discussed in Section 4.2.1. However, for completeness, we recall below the differential formulation and stress the key steps.

### Equilibrium

As usual, we consider a differential element. The equilibrium requirement gives (see Figure 5.1b)

$$-\tau A + (\tau + d\tau)\,A + f dx = 0 \qquad \text{within the bar} \tag{5.1}$$

and so

$$A\frac{d\tau}{dx} + f = 0. \tag{5.2}$$

Also, at the right end of the bar we have the equilibrium condition

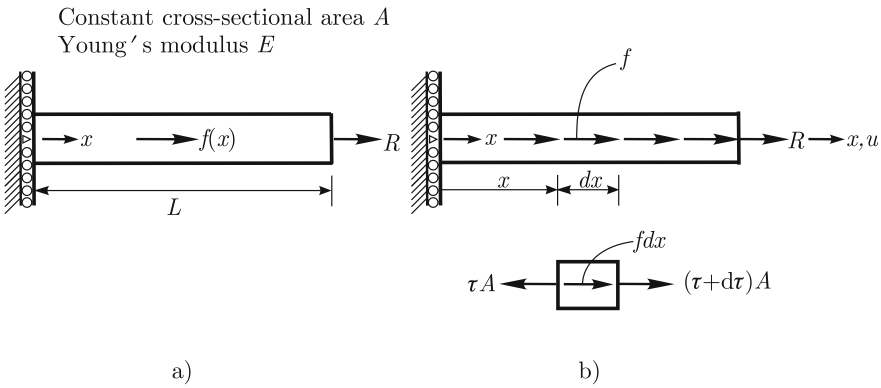


**Fig. 5.1.** a)Bar problem; b) Differential equilibrium for bar problem (plane sections remain plane)

$$\tau(L)A = R.$$

## Compatibility

The relevant strain for this model is $\varepsilon = \varepsilon_{xx}$. The strain compatibility condition is

$$\varepsilon = \frac{du}{dx} \tag{5.3}$$

where $u$ is the displacement field[1] with the condition $u(0) = 0$.

## Constitutive equation

We are considering a linear elastic material. Therefore

$$\tau = E\varepsilon \tag{5.4}$$

where $E$ is the Young's is modulus.

## Boundary conditions

The boundary conditions at the ends of the bar are listed as

$$u(0) = 0 \tag{5.5}$$

[1] In this chapter we suppose that the functions such as $u(x)$ are sufficiently smooth to allow the evaluation of all required integrations and differentiations. In Section 6.2 we will address this issue more rigorously

and

$$\tau(L)A = R. \tag{5.6}$$

Considering equations (5.2) to (5.6), which represent the basic equilibrium, compatibility and constitutive behavior requirements (including the boundary conditions at both ends), we can write the differential formulation for the problem in terms of the displacement field only.

*Differential formulation of the bar problem*

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$$EA\frac{d^2u}{dx^2} + f = 0 \qquad \text{within the bar} \tag{5.7}$$

$$u(0) = 0 \tag{5.8}$$

$$EA\frac{du}{dx}(L) = R \tag{5.9}$$

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Given a particular functional form for $f(x)$, an analytical solution can be calculated by integrating equation (5.7) and imposing the boundary conditions (5.8) and (5.9).

We would like to derive an alternative form, yet equivalent, to the differential formulation presented above. Let us consider the equilibrium equation given in (5.2). We recognize that this equation is valid for any section within the bar. Therefore we can write

$$\left(A\frac{d\tau}{dx} + f\right)\delta u = 0 \tag{5.10}$$

where $\delta u(x)$ is a continuous function defined in the interval $0 \leqslant x \leqslant L$ and zero at the point where the boundary condition in displacement of the differential formulation is prescribed, *i.e.,* $\delta u(0) = 0$. The function $\delta u(x)$ is called the virtual displacement field.

Hence, we also have

$$\int_0^L \left(A\frac{d\tau}{dx} + f\right)\delta u \; dx = 0 \tag{5.11}$$

or equivalently

$$-\int_0^L A\frac{d\tau}{dx}\delta u \; dx = \int_0^L f\delta u \; dx. \tag{5.12}$$

Integrating the left-hand side of equation (5.12) by parts we arrive at

$$-\int_0^L A\frac{d\tau}{dx}\delta u \; dx = -\left.(A\tau\delta u)\right|_0^L + \int_0^L \frac{d\delta u}{dx}A\tau \; dx$$

and substituting into equation (5.12), we obtain

$$\int_0^L \frac{d\delta u}{dx}A\tau \; dx = \int_0^L f\delta u \; dx + \left.(A\tau\delta u)\right|_0^L \tag{5.13}$$

Calling $\delta\varepsilon = \frac{d\delta u}{dx}$ the virtual strain field, since it is the strain field associated with the virtual displacement field, and using the force boundary condition $\tau(L)A = R$ we can re-write (5.13) as

$$\int_0^L \delta\varepsilon\,\tau A \; dx = \int_0^L f\delta u \; dx + R\,\delta u|_{x=L} \tag{5.14}$$

where we employed $\delta u|_{x=0} = 0$.

Using the notion of work from basic mechanics let us interpret the terms of equation (5.14):

- $R\delta u|_{x=L}$ is the work done by the force $R$ for the virtual displacement at $x = L$.
- $\int_0^L f\delta u \; dx$ is the work done by the distributed force $f$ for the virtual displacement field $\delta u$.
- $\int_0^L \delta\varepsilon\,\tau A \; dx$ is the internal work done by the stress field $\tau$, solution of the differential formulation, for the virtual strain field $\delta\varepsilon$ which – it is important to note – corresponds to $\delta u$ (see also Section 2.3.8).

We denote by $\delta W_i$ the *internal virtual work* given by

$$\delta W_i = \int_0^L \delta\varepsilon\,\tau A \; dx$$

and by $\delta W_e$ the *external virtual work* given by

$$\delta W_e = \int_0^L f\delta u \; dx + R\,\delta u|_{x=L}\,.$$

Of course,

$$\delta W_i = \delta W_e. \tag{5.15}$$

Equation (5.14) or equivalently (5.15) is the mathematical statement of the "Principle of Virtual Work" or the "Principle of Virtual Displacements" for the bar problem. We note that, for the stress field which satisfies equilibrium within the bar and at $x = L$, the external virtual work is equal to the internal virtual work for any virtual displacement field that is zero at $x = 0$ (where the displacement boundary condition is imposed).

Hence, we have shown that equilibrium implies the virtual work equation, but not yet the vice-versa. In order to prove the equivalence, we also need to show that the virtual work statement, equation (5.14), implies equilibrium, *i.e.*, equations (5.2) and (5.6).

Using integration by parts we have the following identity

$$\int_0^L \delta\varepsilon\tau A dx = \int_0^L \frac{d\delta u}{dx}\tau A dx = \delta u \tau A|_0^L - \int_0^L \delta u \frac{d\tau}{dx} A dx. \tag{5.16}$$

Substituting (5.16) into (5.14) we obtain

$$\int_0^L \left(A\frac{d\tau}{dx} + f\right)\delta u dx - \delta u\tau A|_0^L + R\delta u|_{x=L} = 0$$

and using that $\delta u|_{x=0} = 0$, we arrive at

$$\int_0^L \left(A\frac{d\tau}{dx} + f\right)\delta u \; dx + \left[(R - \tau A)\,\delta u\right]|_{x=L} = 0. \tag{5.17}$$

Equation (5.17) has to be valid for any $\delta u\,(x)$ with $\delta u\,(0) = 0$. Let us take

$$\delta u\,(x) = x\left(1 - \frac{x}{L}\right)\left(A\frac{d\tau}{dx} + f\right). \tag{5.18}$$

Although we do not explicitly know the functional form of $(A\frac{d\tau}{dx} + f)$, the above choice for $\delta u\,(x)$ is certainly allowed, since this function satisfies $\delta u\,(0) = 0$. Since for this choice $\delta u\,(L) = 0$, equation (5.17) becomes

$$\int_0^L \left(A\frac{d\tau}{dx} + f\right)^2 x\left(1 - \frac{x}{L}\right) dx = 0.$$

Considering that for $(0, L)$, $x > 0$, $\left(1 - \frac{x}{L}\right) > 0$ and $\left(A\frac{d\tau}{dx} + f\right)^2 \geq 0$, the only possibility for the above integral to vanish is to have

$$A\frac{d\tau}{dx} + f = 0 \qquad \text{within the bar} \tag{5.19}$$

*i.e.*, to satisfy the equilibrium equation (5.2).

Using (5.19), we can now return to equation (5.17) which then reads

$$\left[(R - \tau A)\,\delta u\right]|_{x=L} = 0.$$

Since the value of $\delta u\,(x)|_L$ is arbitrary, we conclude that

$$R = \tau A|_{x=L}\,.$$

In other words, the equilibrium at the boundary, *i.e.*, the force boundary condition is also implicitly contained in the virtual work statement.

Hence, we proved the equivalence of the equilibrium equations (5.2), (5.6) and the principle of virtual work equation (5.14). This is a fundamental result that we emphasize in Figure 5.2. Although we proved this equivalence for the 1-D bar problem, this is a general result which is valid for *every* mathematical model in solid and structural mechanics.

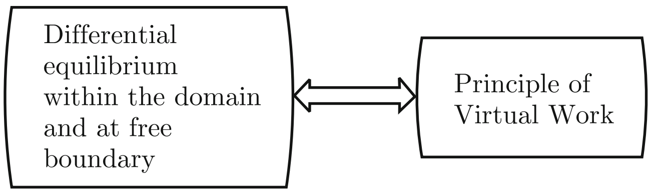


**Fig. 5.2.** Equivalence between equilibrium and the principle of virtual work

The above discussion shows that we may interpret the principle of virtual work to be an integral form of the equilibrium equations, and it is also referred to as a variational formulation because the virtual displacements can be thought of as a variation of the real displacements.

We summarize what we have discussed so far in Table 5.1. The repetition shown in this table is intentional, because we want to emphasize which equations are the same and which are not the same for the two formulations.

In equations (5.7), (5.8) and (5.9) we wrote the differential formulation in terms of the displacement field only. We write below the variational formulation in terms of displacements only.

*Variational formulation of the bar problem*

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Find $u\,(x)$, $u(0) = 0$ such that

$$\int_0^L EA\frac{d\delta u}{dx}\frac{du}{dx}dx = \int_0^L f\delta u dx + R\delta u|_{x=L} \tag{5.20}$$

for any $\delta u\,(x)$, $\delta u\,(0) = 0$.

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The relation (5.20) is the principle of virtual work in terms of displacements only, and also contains the compatibility and constitutive requirements.

**Table 5.1.** Summary of differential and variational formulations for bar problem

| **Differential Formulation** | **Variational Formulation** |
|---|---|
| Find $\tau(x), \varepsilon(x)$ and continuous $u(x)$ such that the following holds: | Find $\tau(x), \varepsilon(x)$ and continuous $u(x)$ such that the following holds: |
| **Displacement boundary condition**<br>$u(0) = 0$ | **Displacement boundary condition**<br>$u(0) = 0$ |
| **Differential equilibrium**<br>$A\frac{d\tau}{dx} + f = 0$<br>**Equilibrium at the boundary**<br>$\tau A = R \text{ for } x = L$ | **Principle of virtual work**<br>$\int_0^L \delta\varepsilon\tau A \, dx = \int_0^L f\delta u \, dx + R\delta u\vert_{x=L}$<br>for any $\delta u(x)$, $\delta u(0) = 0$<br>and $\delta\varepsilon = \frac{d\delta u}{dx}$ |
| **Compatibility**<br>$\varepsilon = \frac{du}{dx}$ | **Compatibility**<br>$\varepsilon = \frac{du}{dx}$ |
| **Constitutive equation**<br>$\tau = E\varepsilon$ | **Constitutive equation**<br>$\tau = E\varepsilon$ |

### Example 5.1

Considering the bar problem of Figure 5.1, show that the principle of virtual work contains the condition of global equilibrium of the bar.

### Solution

In Figure 5.3, we show the reaction at the support represented by $F$, which is given by

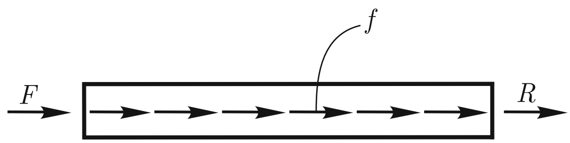


**Fig. 5.3.** Bar problem with reaction explicitly represented

$$F = -\tau\,(0)\,A.$$

In this case, the reaction can be calculated directly by imposing equilibrium in the $x$ direction to obtain

$$F = -\left(\int_0^L f dx + R\right).$$

Consider the problem described in Figure 5.3 with no support and let us apply the principle of virtual work taking as a virtual displacement field a rigid body translation in the $x$ direction of magnitude $\Delta$. Since it is a rigid body motion, the virtual strain field associated with it is identically zero (in fact, $\varepsilon = \frac{d\delta u}{dx} = \frac{d}{dx}\Delta = 0$). Therefore, the right-hand side of the virtual work statement is also zero. Then, the principle of virtual work yields

$$\int_0^L \delta\varepsilon\,\tau\,A\,dx = 0 = \int_0^L f\Delta dx + F\Delta + R\Delta$$

$$0 = \Delta\left(\int_0^L f dx + F + R\right)$$

Since $\Delta$ is arbitrary

$$\int_0^L f dx + F + R = 0$$

and therefore

$$F = -\left(\int_0^L f dx + R\right)$$

which is the global equilibrium equation of the bar.

□

Hence, we note that the principle of virtual work contains not only the local, *i.e.,* the differential equilibrium written in integral form as previously shown, but also the global equilibrium. Although we explored this fact in this very particular setting, this result is general and it will be detailed further for a 2-D case (see Example 5.6).

In other words, if we introduce the support reactions as external loads, suppress the kinematic restraints provided by the supports, and use as virtual displacement fields rigid body modes, the principle of virtual work yields the global equilibrium equations.

The condition that the equilibrium of every part of a solid implies global equilibrium was also exemplified for a truss structure (see Section 2.2). Hence, the above result that the principle of virtual work contains the global equilibrium condition must be expected, since this principle is equivalent to differential equilibrium, that is, equilibrium of every part of the bar.

**Example 5.2**

Consider the problem of a steel bar which supports a weight as described in Figure 5.4.

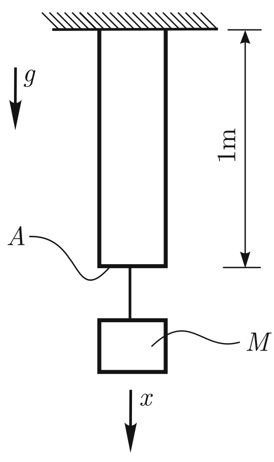


**Fig. 5.4.** Steel bar subjected to gravity. The density $\rho$[kg/m$^3$], the area $A$[m$^2$] and the mass $M$[kg]

(i) State and solve the differential formulation of this problem.
(ii) Show explicitly, considering the exact solution derived in (i), that the principle of virtual work is satisfied for the following virtual displacement patterns: $\delta u(x) = ax$ and $\delta u(x) = ax^2$, where $a$ is a constant.

## Solution

This problem is clearly a particular case of the bar problem studied above and its differential formulation can be written as

$$EA\frac{d^2u}{dx^2} + \rho g A = 0 \tag{5.21}$$

$$u(0) = 0 \tag{5.22}$$

$$EA \left.\frac{du}{dx}\right|_{x=L} = Mg \tag{5.23}$$

where $L = 1$ m is the bar length.

The solution can be found by substitution and integration and is given by

$$u(x) = \frac{g}{E}\left[-\frac{\rho x^2}{2} + \left(\frac{M}{A} + \rho L\right)x\right].$$

(ii) The exact solution for the stress field is

$$\tau(x) = E\frac{du}{dx} = g\left[-\rho x + \left(\frac{M}{A} + \rho L\right)\right].$$

Considering the virtual displacement $\delta u(x) = ax$, the associated virtual strain field is

$$\delta\varepsilon(x) = \frac{d\delta u(x)}{dx} = a.$$

For the principle of virtual work to be satisfied for this virtual displacement field, we need to have that

$$\int_0^L ag\left[-\rho x + \left(\frac{M}{A} + \rho L\right)\right] A dx = \int_0^L \rho g A a x dx + MgaL$$

which indeed is satisfied.

For the second virtual displacement field $\delta u(x) = ax^2$, we obtain

$$\delta\varepsilon(x) = 2ax$$

and the equation to be verified is

$$\int_0^L 2axg\left[-\rho x + \left(\frac{M}{A} + \rho L\right)\right] Adx = \int_0^L \rho g A a x^2 dx + MgaL^2$$

which also holds.

□

**Example 5.3**

Consider the bar problem with varying cross-section described in Figure 5.5.

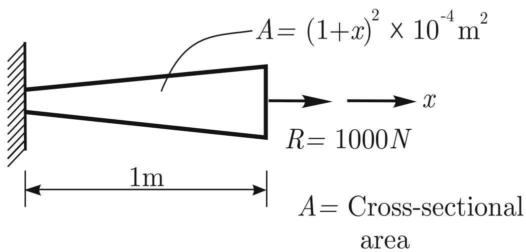


**Fig. 5.5.** Bar problem with varying bar cross-sectional area

(i) Establish the differential formulation and the principle of virtual work for this problem.

(ii) Find the exact solution for this problem.

(iii) Show that the principle of virtual work is satisfied for the exact stress field and a virtual displacement field $\delta u(x) = ax$.

**Solution**

(i) Since the cross-sectional area is not constant along the bar, we need to generalize equation (5.1) as,

$$-\tau A + (\tau + d\tau)(A + dA) + fdx = 0$$

$$d\tau A + \tau dA + d\tau dA + fdx = 0$$

and hence obtain

$$\frac{d}{dx}(\tau A) + f = 0. \tag{5.24}$$

Substituting the strain and constitutive equations into (5.24) we obtain the differential formulation for this problem.

Find $u(x)$ such that

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + f = 0 \qquad \text{within the bar} \tag{5.25}$$

$$u(0) = 0 \tag{5.26}$$

$$EA\frac{du}{dx}|_{x=L} = R. \tag{5.27}$$

(ii) For this example, $f = 0$, and so (5.25) becomes

$$\begin{aligned}\frac{E}{10^4}\frac{d}{dx}\left[(1+x)^2\frac{du}{dx}\right] &= 0\\ (1+x)^2\frac{d^2u}{dx^2} + 2(1+x)\frac{du}{dx} &= 0\end{aligned}$$

or

$$\frac{d^2u}{dx^2} + \frac{2}{(1+x)}\frac{du}{dx} = 0$$

leading to

$$u(x) = \frac{R\cdot 10^4}{E}\left(\frac{x}{1+x}\right).$$

(iii) To verify that the principle of virtual work holds for $\delta u(x) = ax$ we use

$$\delta\varepsilon(x) = a.$$

The stress field is given by

$$\tau(x) = E\frac{du}{dx} = R\cdot 10^4\frac{1}{(1+x)^2}$$

and the principle obviously holds

$$\int_0^L a\frac{R\cdot 10^4}{(1+x)^2}(1+x)^2\cdot 10^{-4}dx = RaL.$$

□

**Example 5.4**

Consider the problem described in Figure 5.6. The Young's modulus is constant and equal to $E_1$ for $0 < x < L/2$ and constant and equal to $E_2$ for $L/2 < x < L$ with $E_1 \neq E_2$.

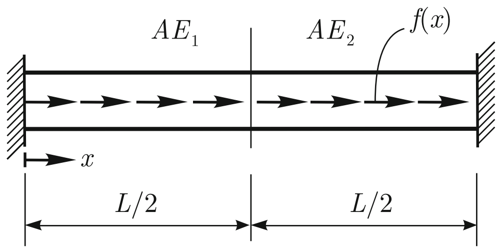


**Fig. 5.6.** Bar problem with varying material properties

(i) Establish the differential formulation for this problem.

(ii) Establish the principle of virtual work formulation for this problem.

**Solution**

(i) The differential equilibrium equation of the bar in terms of displacements when both $E$ and $A$ may vary was derived in Example 5.3

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) + f = 0$$

However in this problem, $E$ changes discontinuously at $x = \frac{L}{2}$ therefore we need to use the differential formulation for subdomain 1

$$E_1 A\frac{d^2u_1}{dx^2} + f = 0 \qquad\qquad 0 < x < \frac{L}{2}$$

and for subdomain 2

$$E_2 A\frac{d^2u_2}{dx^2} + f = 0 \qquad\qquad \frac{L}{2} < x < L.$$

where $u_1(x)$ and $u_2(x)$ represent the axial displacement for subdomains 1 and 2 respectively. The displacement boundary conditions are

$$u_1(0) = 0$$

$$u_2(L) = 0$$
$$u_1|_{(\frac{L}{2})} = u_2|_{(\frac{L}{2})}.$$

Also, the normal force $N$ at $x = \frac{L}{2}$ should be the same when we approach $x = \frac{L}{2}$ from the right side or from the left side of the section. From the left side

$$N|_{(\frac{L}{2})} = E_1 A \frac{du_1}{dx}|_{(\frac{L}{2})}$$

and from the right side

$$N|_{(\frac{L}{2})} = E_2 A \frac{du_2}{dx}|_{(\frac{L}{2})}.$$

Hence we have the force boundary condition

$$E_1 A \frac{du_1}{dx}|_{(\frac{L}{2})} = E_2 A \frac{du_2}{dx}|_{(\frac{L}{2})} .$$

(ii) The principle of virtual work can however be directly written for the whole bar

$$\int_0^L \delta\varepsilon \tau A dx = \int_0^L \delta u f dx$$

or

$$\int_0^{L/2} \frac{d\delta u}{dx} E_1 \frac{du}{dx} A dx + \int_{L/2}^{L} \frac{d\delta u}{dx} E_2 \frac{du}{dx} A dx = \int_0^L \delta u f dx$$

because the discontinuous change of $E$ does not introduce a difficulty in this equation.

□

## 5.2 The principle of virtual work in 2-D and 3-D analyses

In the previous section we introduced the principle of virtual work for the one-dimensional bar problem. That setting, due to its simplicity, allowed us to concentrate on the fundamental facts.

We now plan to state the principle for the other mathematical models discussed in Chapters 3 and 4. We choose to first present the principle of virtual work for the three-dimensional elasticity mathematical model and then discuss its specialization to the remaining mathematical models.

Recalling the discussion of the principle of virtual work for the bar problem, we concluded that satisfying the principle of virtual work is equivalent to satisfying local (and therefore also global) equilibrium:

**(I)** If a stress field $\tau$ for which the virtual work of the stresses is equal to the virtual work of the externally applied forces for any virtual displacement field $\delta u$, zero at the fixed boundary, then this stress field also satisfies the differential equilibrium (equation (5.2)) and the stress equilibrium at the free boundary, *i.e.*, the force boundary condition (equation (5.6)).

**(II)** If a stress field $\tau$ satisfies differential equilibrium (equation (5.2)) and the force boundary condition (equation (5.6)) then for any virtual displacements $\delta u$, zero at the fixed boundary, the internal virtual work of this stress field is equal to the virtual work of the externally applied forces (the reverse of **(I)**).

### 5.2.1 The principle of virtual work for 3-D elasticity

In Figure 3.59 of Chapter 3, we described the 3-D elasticity problem. For a given choice of body forces $\mathbf{f}^B$, surface tractions $\mathbf{f}^S$ on $S_f$ and displacement boundary conditions $\mathbf{u} = \hat{\mathbf{u}}$ on $S_u$, let $\tau_{xx}$, $\tau_{yy}$, $\tau_{zz}$, $\tau_{xy}$, $\tau_{xz}$, $\tau_{yz}$ be the stress field that corresponds to the solution of this elasticity problem. Therefore, this stress field satisfies differential equilibrium in the volume of the body and at the boundary, *i.e.*, the force boundary conditions on $S_f$.

The above statements **(I)** and **(II),** which were proved in the context of the one-dimensional bar problem, are also valid for 2-D and 3-D problems. However, we need to precisely define, in each case, the internal virtual work.

The internal virtual work for the 3-D problem is a generalization of the internal virtual work for the 1-D bar problem for which the stress and strain states were fully represented by only one stress ($\tau$) and one strain component ($\varepsilon$). The virtual work of the 3-D stress field $\tau_{xx}, \tau_{yy}, \tau_{zz}, \tau_{xy}, \tau_{xz}, \tau_{yz}$ considering a virtual strain field $\delta\varepsilon_{xx}, \delta\varepsilon_{yy}, \delta\varepsilon_{zz}, \delta\gamma_{xy}, \delta\gamma_{xz}, \delta\gamma_{yz}$ (derived from the virtual displacements) is given by

$$\delta W_i = \int (\delta\varepsilon_{xx}\tau_{xx} + \delta\varepsilon_{yy}\tau_{yy} +$$

$$+\delta\varepsilon_{zz}\tau_{zz} + \delta\gamma_{xy}\tau_{xy} + \delta\gamma_{xz}\tau_{xz} + \delta\gamma_{yz}\tau_{yz}) \; dV \tag{5.28}$$

which can be re-written using the stress and strain column matrices as

$$\delta W_i = \int_V \delta\varepsilon^T \tau \; dV = \int_V \bar{\varepsilon}^T \tau \; dV.$$

As in the expression above, we now denote the virtual quantities by an overbar.

The external virtual work is given by

$$\delta W_e = \int_V \bar{\mathbf{u}}^T \mathbf{f}^B \; dV \; + \; \int_{S_f} \bar{\mathbf{u}}^T \; \mathbf{f}^S \; dS$$

where

$$\bar{\mathbf{u}} = \begin{bmatrix} \bar{u}(x,y,z) \\ \bar{v}(x,y,z) \\ \bar{w}(x,y,z) \end{bmatrix}$$

is the column matrix which represents the virtual displacement field. Note that by the definition of the virtual displacement field $\bar{\mathbf{u}} = \mathbf{0}$ on $S_u$.

Therefore the virtual work statement reads[2]

$$\int_V \bar{\varepsilon}^T \tau \; dV = \int_V \bar{\mathbf{u}}^T \mathbf{f}^B \; dV \;+\; \int_{S_f} \bar{\mathbf{u}}^T \; \mathbf{f}^S \; dS. \tag{5.29}$$

In the following example we demonstrate an important result which was already shown for the 1-D bar problem in Section 5.1.

**Example 5.5**

In the context of a 3-D body, show that if a stress field satisfies differential equilibrium and the force boundary conditions then the virtual work statement is satisfied by this stress field.

**Solution**

The differential equilibrium equations for a 3-D body are given by equations (3.114) and the force boundary conditions by equations (3.117) which are assumed to be satisfied by the stress field $\tau_{xx}$, $\tau_{yy}$, $\tau_{zz}$, $\tau_{xy}$, $\tau_{xz}$, $\tau_{yz}$ considered.

Let $\bar{u}(x,y,z), \bar{v}(x,y,z)$ and $\bar{w}(x,y,z)$ be an arbitrarily chosen displacement field such that

$$\bar{u} = 0, \; \bar{v} = 0, \; \bar{w} = 0 \quad \text{on } S_u. \tag{5.30}$$

Hence $\bar{u}$, $\bar{v}$ , $\bar{w}$ are a virtual displacement field.

Using the equilibrium equations, we write

$$\left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x^B \right) \bar{u} = 0 \tag{5.31}$$

$$\left( \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y^B \right) \bar{v} = 0 \tag{5.32}$$

$$\left( \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + f_z^B \right) \bar{w} = 0. \tag{5.33}$$

[2] It is implicitly understood that for the integral over $S_f$ the virtual displacements $\bar{\mathbf{u}}$ in the integrand represent the virtual displacements evaluated on $S_f$. We could have used $\bar{\mathbf{u}}^{S_f}$ to emphasize this fact

Using the derivative product rule, we have

$$\begin{aligned}\frac{\partial}{\partial x}\left(\tau_{xx}\bar{u}+\tau_{yx}\bar{v}+\tau_{zx}\bar{w}\right) &= \frac{\partial \tau_{xx}}{\partial x}\bar{u}+\frac{\partial \tau_{yx}}{\partial x}\bar{v}+\frac{\partial \tau_{zx}}{\partial x}\bar{w}\\ &\quad +\tau_{xx}\frac{\partial \bar{u}}{\partial x}+\tau_{yx}\frac{\partial \bar{v}}{\partial x}+\tau_{zx}\frac{\partial \bar{w}}{\partial x}\end{aligned}$$

and re-arranging terms, we obtain

$$\begin{aligned}\frac{\partial \tau_{xx}}{\partial x}\bar{u}+\frac{\partial \tau_{yx}}{\partial x}\bar{v}+\frac{\partial \tau_{zx}}{\partial x}\bar{w} &= \frac{\partial}{\partial x}\left(\tau_{xx}\bar{u}+\tau_{yx}\bar{v}+\tau_{zx}\bar{w}\right)\\ &\quad -\left(\tau_{xx}\frac{\partial \bar{u}}{\partial x}+\tau_{yx}\frac{\partial \bar{v}}{\partial x}+\tau_{zx}\frac{\partial \bar{w}}{\partial x}\right).\end{aligned}$$

Now, using analogous expressions for the derivatives with respect to $y$ and $z$, and equations (5.31) to (5.33) we can write

$$\begin{aligned}&\int_V \Bigg[\frac{\partial}{\partial x}\left(\tau_{xx}\bar{u}+\tau_{yx}\bar{v}+\tau_{zx}\bar{w}\right)+\frac{\partial}{\partial y}\left(\tau_{xy}\bar{u}+\tau_{yy}\bar{v}+\tau_{zy}\bar{w}\right)+\\ &+\frac{\partial}{\partial z}\left(\tau_{xz}\bar{u}+\tau_{yz}\bar{v}+\tau_{zz}\bar{w}\right)+\\ &-\left(\tau_{xx}\frac{\partial \bar{u}}{\partial x}+\tau_{yy}\frac{\partial \bar{v}}{\partial y}+\tau_{zz}\frac{\partial \bar{w}}{\partial z}+\tau_{yx}\frac{\partial \bar{v}}{\partial x}+\tau_{xy}\frac{\partial \bar{u}}{\partial y}+\right.\\ &\left.\tau_{zx}\frac{\partial \bar{w}}{\partial x}+\tau_{xz}\frac{\partial \bar{u}}{\partial z}+\tau_{zy}\frac{\partial \bar{w}}{\partial y}+\tau_{yz}\frac{\partial \bar{v}}{\partial z}\right)\\ &+f_x^B\bar{u}+f_y^B\bar{v}+f_z^B\bar{w}\ \Bigg]\ dV=0.\end{aligned}\tag{5.34}$$

The following mathematical identity follows from the use of the divergence theorem[3]

$$\begin{aligned}&\int_V \Bigg[\frac{\partial}{\partial x}\left(\tau_{xx}\bar{u}+\tau_{yx}\bar{v}+\tau_{zx}\bar{w}\right)+\frac{\partial}{\partial y}\left(\tau_{xy}\bar{u}+\tau_{yy}\bar{v}+\tau_{zy}\bar{w}\right)+\\ &+\frac{\partial}{\partial z}\left(\tau_{xz}\bar{u}+\tau_{yz}\bar{v}+\tau_{zz}\bar{w}\right)\Bigg]\ dV\\ &=\int_S \left[\left(\tau_{xx}\bar{u}+\tau_{yx}\bar{v}+\tau_{zx}\bar{w}\right)n_x+\left(\tau_{xy}\bar{u}+\tau_{yy}\bar{v}+\tau_{zy}\bar{w}\right)n_y+\right.\\ &\left.\left(\tau_{xz}\bar{u}+\tau_{yz}\bar{v}+\tau_{zz}\bar{w}\right)n_z\right]\ dS\end{aligned}\tag{5.35}$$

where $n_x, n_y$ and $n_z$ are the components of the outward unit normal to the surface $S$. But

[3] The divergence theorem states: Let $\mathbf{F}$ be a vector field in the volume $V$; then $\int_V\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right)\,dV=\int_S \mathbf{F}\cdot\mathbf{n}\,dS$ where $\mathbf{n}$ is the unit outward normal on the surface $S$ of $V$

$$\int_S [(\tau_{xx}n_x + \tau_{xy}n_y + \tau_{xz}n_z)\,\bar{u} + (\tau_{yx}n_x + \tau_{yy}n_y + \tau_{yz}n_z)\,\bar{v} +$$

$$(\tau_{zx}n_x + \tau_{zy}n_y + \tau_{zz}n_z)\,\bar{w}]\ dS$$

$$= \int_{S_f} \left[\bar{u}f_x^S + \bar{v}f_y^S + \bar{w}f_z^S\right]\ dS \tag{5.36}$$

where we used (3.117) and (5.30).

Next, using the symmetry of the stress tensor and substituting (5.35) and (5.36) into (5.34), we arrive at

$$\int_V (\bar{\varepsilon}_{xx}\tau_{xx} + \bar{\varepsilon}_{yy}\tau_{yy} + \bar{\varepsilon}_{zz}\tau_{zz} + \bar{\gamma}_{xy}\tau_{xy} + \bar{\gamma}_{yz}\tau_{yz} + \bar{\gamma}_{xz}\tau_{xz})\ dV =$$

$$\int_V \left[\bar{u}f_x^B + \bar{v}f_y^B + \bar{w}f_z^B\right]\ dV + \int_{S_f} \left[\bar{u}f_x^S + \bar{v}f_y^S + \bar{w}f_z^S\right]\ dS. \tag{5.37}$$

We note that the strains with the overbar are obtained from the virtual displacement field using the strain-displacement relations (3.115) and therefore are virtual strains. Re-writing (5.37) using matrix quantities gives

$$\int_V \bar{\varepsilon}^T\tau\ dV = \int_V \bar{\mathbf{u}}^T\mathbf{f}^B\ dV + \int_{S_f} \bar{\mathbf{u}}^T\mathbf{f}^S\ dS \tag{5.38}$$

which is the virtual work statement (5.29).

□

Hence we proved statement **(II)** for the 3-D problem. The proof of statement **(I)** is obtained in the same way as for the 1-D case, by starting from equation (5.38) – the virtual work statement – and perform the mathematical operations backwards.

Regarding the principle of virtual work equation, we emphasize once more:

- The virtual strains $\bar{\varepsilon}$ are calculated by the differentiations given in (3.115) from the assumed virtual displacements $\bar{\mathbf{u}}$.
- The virtual displacements $\bar{\mathbf{u}}$ must be smooth enough to allow the evaluation of the integral in (5.29), with $\bar{\mathbf{u}}$ equal to zero on $S_u$.
- All integrations are performed over the original volume and surface area of the body, unaffected by the imposed virtual displacements.

The principle of virtual work in terms of the displacement variables only is the variational formulation which contains all requirements of equilibrium, compatibility and constitutive behavior.

*Variational formulation of the 3-D elasticity model*

---

Find $\mathbf{u}$, $\mathbf{u} = \hat{\mathbf{u}}$ on $S_u$ such that[4]

$$\int_V (\partial_\epsilon \bar{\mathbf{u}})^T \mathbf{C}\, \partial_\epsilon \bar{\mathbf{u}}\, dV = \int_V \bar{\mathbf{u}}^T \mathbf{f}^B\, dV + \int_{S_f} \bar{\mathbf{u}}^T \mathbf{f}^S\, dS \tag{5.39}$$

for any $\bar{\mathbf{u}}$ which is zero on $S_u$

---

Note that we have used equations (4.330) and (4.331) to enforce the strain-displacement and constitutive requirements.

Since all field and boundary conditions of the differential formulation are represented in the above variational formulation and vice-versa, these formulations are equivalent.

Of course, we can similarly derive the principle of virtual work for all differential formulations considered in Chapter 4 and all the results above are valid for these models. We detail below the derivation of the principle of virtual work for some of these models.

### 5.2.2 The principle of virtual work for the plane stress model

Figure 4.5 describes a plane stress situation. The non-zero stress components are $\tau_{xx}, \tau_{yy}$ and $\tau_{xy}$, and the non-zero strain components are $\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}, \varepsilon_{zz}$. Starting from the general expression of the internal virtual work of the 3-D elasticity problem and introducing the plane stress assumptions we have

$$\int_V \bar{\varepsilon}^T \tau \; dV = \int_A \int_{-h/2}^{+h/2} \left(\bar{\varepsilon}_{xx}\tau_{xx} + \bar{\varepsilon}_{yy}\tau_{yy} + \bar{\varepsilon}_{zz} \cdot 0 + \bar{\gamma}_{xy}\tau_{xy}\right) \; dz \; dA$$

$$= h \int_A \left(\bar{\varepsilon}_{xx}\tau_{xx} + \bar{\varepsilon}_{yy}\tau_{yy} + \bar{\gamma}_{xy}\tau_{xy}\right) \; dA$$

where we used $h$ to be constant.

Since both the body forces and surface tractions are assumed to have non-zero components only on the $xy$ plane and do not depend on $z$, the external virtual work reads

$$\int_V \mathbf{u}^T \mathbf{f}^B \; dV + \int_{S_f} \mathbf{u}^T \; \mathbf{f}^S \; dS =$$

---

[4] Throughout the discussion, we assume that $\mathbf{u}$ and $\bar{\mathbf{u}}$ are continuous

$$= \int_A \int_{-h/2}^{+h/2} \left(\bar{u}\ f_x^B + \bar{v}\ f_y^B\right)\ dz\ dA + \int_{L_f} \int_{-h/2}^{+h/2} \left(\bar{u}\ f_x^S + \bar{v}\ f_y^S\right)\ dz\ dL$$

$$= h \int_A \left(\bar{u}\ f_x^B + \bar{v}\ f_y^B\right) dA + h \int_{L_f} \left(\bar{u}\ f_x^S + \bar{v}\ f_y^S\right) dL.$$

with $V = h \times A$ and $S_f = h \times L_f$.

Using the column matrix quantities defined in Chapter 4 for the plane stress and plane strain models, we are led to the principle of virtual work for plane stress

$$h \int_A \bar{\varepsilon}^T \tau\ dA = h \int_A \bar{\mathbf{u}}^T\ \mathbf{f}^B dA + h \int_{L_f} \bar{\mathbf{u}}^T\ \mathbf{f}^S\ dL \tag{5.40}$$

for all $\bar{\mathbf{u}}$, $\bar{\mathbf{u}} = \mathbf{0}$ on $L_u$ and $\bar{\varepsilon} = \partial_\varepsilon \bar{\mathbf{u}}$.

Formally, the same kind of expression for the virtual work principle as for the 3-D case is obtained, that is,

$$\int_A \bar{\varepsilon}^T \tau\ dA = \int_A \bar{\mathbf{u}}^T\ \mathbf{f}^B dA + \int_{L_f} \bar{\mathbf{u}}^T\ \mathbf{f}^S\ dL \tag{5.41}$$

for all $\bar{\mathbf{u}}$, $\bar{\mathbf{u}} = \mathbf{0}$ on $S_u$, with $S_u = h \times L_u$ and $\bar{\varepsilon} = \partial_\varepsilon \bar{\mathbf{u}}$.

We note that in the above expressions $\mathbf{f}^B$ is given per unit of volume and $\mathbf{f}^S$ is given per unit of surface area.

Sometimes, it is more convenient to use (5.40) instead of (5.41). For example, when considering concentrated loads applied at the boundary (see the example below).

**Example 5.6**

Consider the plane stress problem defined in Figure 5.7.

Calculate the reactions by simple statics, and calculate the reactions using the principle of virtual work.

**Solution**

Referring to Figure 5.7, the equilibrium equations of statics are

$$\sum F_x = 0,\ \sum F_y = 0,\ \sum M_A = 0$$

which leads to

$$X_A = 0,\ Y_A + Y_B + t \int_A f_y^B dA + t \int_0^L f_y^S dx = 0$$

and

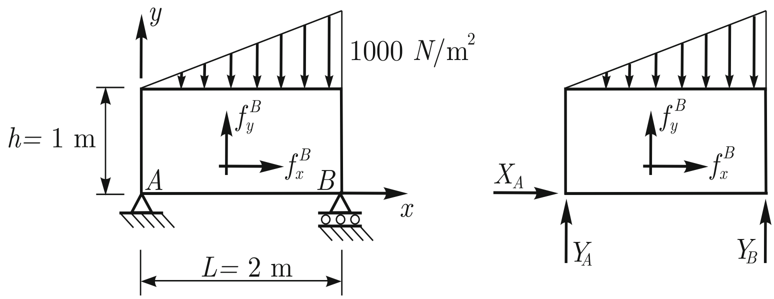


**Fig. 5.7.** a) Definition of the plane stress problem, $f_x^B = 0$, $f_y^B = -2400\ N/m^3$, $t = 0.1\,m$ (thickness); b) Equilibrium of the plate including reactions

$$Y_B L + t\int_A f_y^B x dA + t\int_0^L f_y^S x dx = 0$$

from which we obtain $Y_A = 273.33\,N$, $Y_B = 306.67\,N$.

We use equation (5.40) since we have an extra term on the right-hand side corresponding to the virtual work of the concentrated forces given by

$$\sum_{i=1}^{N_c} F_i\,\delta d_i$$

where $F_i$, $i = 1, \cdots, N_C$ are the magnitudes of the $N_C$ concentrated forces and $\delta d_i$ are the virtual displacements in the direction and the same orientation as the concentrated loads $F_i$.

Note that if we considered equation (5.41) , we would need to divide the magnitude of the concentrated load by the thickness $t$ since for this equation we divided by the constant thickness.

To calculate the reactions we use 3 independent virtual displacement fields corresponding to virtual rigid body mode displacements. Then, the internal virtual work is zero.

Choosing first, as virtual displacement, a translation in the direction $x$ of magnitude $\Delta_x$, we obtain

$$\begin{aligned} 0 &= t\int_A \begin{bmatrix} \Delta_x & 0 \end{bmatrix} \begin{bmatrix} 0 \\ f_y^B \end{bmatrix} dA + t\int_0^L \begin{bmatrix} \Delta_x & 0 \end{bmatrix} \begin{bmatrix} 0 \\ f_y^S \end{bmatrix} dL + X_A \Delta_x \\ 0 &= X_A \end{aligned}$$

which is the equilibrium equation in the $x$ direction.

Considering now a translation in the $y$ direction of magnitude $\Delta_y$, we obtain

$$0 = t\int_A \begin{bmatrix} 0 & \Delta_y \end{bmatrix} \begin{bmatrix} 0 \\ f_y^B \end{bmatrix} dA + t\int_0^L \begin{bmatrix} 0 & \Delta_y \end{bmatrix} \begin{bmatrix} 0 \\ f_y^S \end{bmatrix} dL + Y_A\Delta_y + Y_B\Delta_y$$

or

$$0 = Y_A + Y_B + t\int_A f_y^B dA + t\int_0^L f_y^S dx$$

which is the equilibrium equation in the $y$ direction.

As the last virtual displacement field, we choose an infinitesimal rigid rotation about point A which is given by

$$\bar{\mathbf{u}} = \begin{bmatrix} -d\theta\ y \\ d\theta\ x \end{bmatrix}.$$

The virtual work expression becomes

$$\begin{aligned} 0 &= Y_B(Ld\theta) + t\int_A [-d\theta\, y \quad d\theta\, x] \begin{bmatrix} 0 \\ f_y^B \end{bmatrix} dA + t\int_0^L [-d\theta\, y \quad d\theta\, x] \begin{bmatrix} 0 \\ f_y^S \end{bmatrix} dx \\ 0 &= Y_B L + t\int_A f_y^B x dA + t\int_0^L f_y^S x dx \end{aligned}$$

which corresponds to the moment equilibrium condition. Since we obtained exactly the same equilibrium equations, the reactions are those already calculated. □

### 5.2.3 The principle of virtual work for the plane strain model

The plane strain model was discussed in Section 4.1.1. Due to the kinematic assumptions the strains $\varepsilon_{zz} = \gamma_{yz} = \gamma_{zx} = 0$ and the non-zero components are, therefore, $\varepsilon_{xx}$, $\varepsilon_{yy}$ and $\gamma_{xy}$. The stresses $\tau_{xx}$, $\tau_{yy}$, $\tau_{xy}$ and $\tau_{zz}$ are, in general, non-zero.

Since in plane strain we have for every section the same stress and strain fields, and loading, we can take a unit thickness to write the principle of virtual work. Analogously to what we did for the plane stress model, we can use the matrix quantities defined in Chapter 4. Following the derivations of the plane stress case and considering a unit thickness, the principle of virtual work reads

$$\int_A \bar{\varepsilon}^T \tau \ dA = \int_A \bar{\mathbf{u}}^T \mathbf{f}^B dA + \int_{L_f} \bar{\mathbf{u}}^T \mathbf{f}^S \ dL. \tag{5.42}$$

for all $\bar{\mathbf{u}}$, $\bar{\mathbf{u}} = \mathbf{0}$ in $S_u$ $(i.e.\ L_u)$ and $\bar{\varepsilon} = \partial_\varepsilon \bar{\mathbf{u}}$.

### 5.2.4 The principle of virtual work for the axisymmetric model

Considering the discussion of the axisymmetric model in Section 4.1.3, besides the usual stress and strain components $\tau_{xx}, \tau_{yy}, \tau_{xy}$ and $\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}$ for the plane models, we have the hoop stress and strain components $\tau_{zz}$ and $\varepsilon_{zz}$ which are also non-zero.

In Figure 4.12 a generic cross-section of the solid of revolution is shown. The $y$ axis is the symmetry axis and the $x$ axis corresponds to the radial direction. We need to define a volume to apply the principle of virtual work. It is usual to take the volume corresponding to an angle of one radian centered at the symmetry axis. Then, for the evaluation of the internal virtual work, we have

$$\int_V \bar{\varepsilon}^T \tau \; dV = \int_A \left( \int_0^1 \bar{\varepsilon}^T \tau \; x \; d\theta \right) \; dA = \int_A \bar{\varepsilon}^T \tau \; x \; dA$$

where $\theta$ is the circumferential angle coordinate of the cylindrical system used (refer to Section 4.1.3 for details) and we adopt the column matrices of the axisymmetric model. Using the same integration procedure for the external work terms, we arrive at the principle of virtual work

$$\int_A \bar{\varepsilon}^T \tau \; x \; dA = \int_A \bar{\mathbf{u}}^T \mathbf{f}^B \, x \; dA + \int_{L_f} \bar{\mathbf{u}}^T \mathbf{f}^S x \; dL.$$

for all $\bar{\mathbf{u}}$, $\bar{\mathbf{u}} = \mathbf{0}$ in $S_u$ (*i.e.* $L_u$) and $\bar{\varepsilon} = \partial_\varepsilon \bar{\mathbf{u}}$ .

### 5.2.5 The principle of virtual work for the Bernoulli-Euler beam model

Let us derive the principle of virtual work for a structural mathematical model. We choose as a typical model, the Bernoulli-Euler beam model which we discussed in Section 4.2.2. Recognizing that the stress and strain components that do work are $\tau_{xx}$ and $\varepsilon_{xx}$, the internal virtual work reads

$$\delta W_i = \int_V \bar{\varepsilon}_{xx} \tau_{xx} \; dV.$$

Referring to equation (4.145) re-written below

$$\tau_{xx} = -\frac{M}{I} z$$

and to equations (4.124) and (4.141) leading to

$$\varepsilon_{xx} = -z \frac{d^2 w}{dx^2}$$

we obtain

$$\delta W_i = \int_0^L \int_A \left(-z\frac{d^2\bar{w}}{dx^2}\right)\left(-\frac{M}{I}z\right)\, dA\, dx = \int_0^L \frac{d^2\bar{w}}{dx^2} M\, dx$$

Recall that $\kappa = \frac{d^2w}{dx^2}$ is the curvature of the axis of the beam for infinitesimally small displacements. Considering only a transverse distributed force $p$ as the external load, the principle of virtual work for a beam, for example, clamped at both ends reads

$$\int_0^L \bar{\kappa} M\, dx = \int_0^L \bar{w} p\, dx$$

for all $\bar{w}$, $\bar{w}(0) = \bar{w}(L) = 0 = \frac{d\bar{w}}{dx}(0) = \frac{d\bar{w}}{dx}(L)$ and $\kappa = \frac{d^2\bar{w}}{dx^2}$.

This result was anticipated since for this model $\tau = [M]$ and $\varepsilon = [\kappa]$ (note that, if one end of the beam is free and subjected to an externally applied force or moment, their virtual work contribution should be included in $\delta W_e$ ).

## 5.3 Strain and potential energy in 3-D

In this section we discuss the concept of strain and potential energy for 3-D elastic solids.

### 5.3.1 Strain energy

Let us extend, to a 3-D linear elastic solid, the concept of *strain energy* introduced in Chapter 2 for a truss bar, *i.e.*, for a one-dimensional state of stress and strain.

Recall that the internal work per unit of volume in a 1-D state of stress and strain for a linear elastic material is given by

$$W(\varepsilon) = \int_0^\tau \tau(\varepsilon)\, d\varepsilon = \frac{1}{2}\tau\varepsilon = \frac{1}{2}E\varepsilon^2$$

which gives the total work per unit of volume at a point performed by the stress throughout the deformation, that is, from a stress-free state to the current state. Then, the total strain energy is defined as

$$\mathcal{U}(\varepsilon) = \int_V W(\varepsilon)\, dV.$$

Also recall that the strain energy depends only on the current state of deformation and gives the energy stored as elastic deformation.

Considering a 3-D elastic solid, we can generalize the above concepts defining the internal work per unit of volume or the strain energy density by

$$W(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}) = \frac{1}{2}(\tau_{xx}\varepsilon_{xx} + \tau_{yy}\varepsilon_{yy}$$

$$+\tau_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}) = \frac{1}{2}\varepsilon^T\tau$$

$$= \frac{1}{2}\varepsilon^T\mathbf{C}\varepsilon$$

and the total strain energy by

$$\mathcal{U}(\varepsilon) = \int_V W(\varepsilon)\, dV = \frac{1}{2}\int_V \varepsilon^T\mathbf{C}\varepsilon\, dV.$$

Sometimes we use the notation

$$a(\mathbf{v},\mathbf{v}) = 2\mathcal{U}(\varepsilon(\mathbf{v})) = \int_V (\,\partial_\varepsilon \mathbf{v}\,)^T\mathbf{C}\partial_\varepsilon\mathbf{v}\; dV \tag{5.43}$$

which gives twice the strain energy associated with the displacement field $\mathbf{v}$.

### 5.3.2 The total potential energy

The *total potential* of an elastic solid subjected to a field of body forces $\mathbf{f}^B$ in $V$ and a field of surface tractions $\mathbf{f}^S$ on $S_f$ is defined as

$$\Pi(\mathbf{v}) = \mathcal{U}\left(\varepsilon\left(\mathbf{v}\right)\right) - P\left(\mathbf{v}\right)$$

where $\mathbf{v}$ is a *compatible displacement field*, that is, a sufficiently smooth displacement field (the strains $\varepsilon(\mathbf{v}) = \partial_\varepsilon\mathbf{v}$ can be evaluated) which satisfies the kinematic boundary conditions on $S_u$, and $P(\mathbf{v})$ is the potential of the external loads, that is,

$$P\left(\mathbf{v}\right) = \int_V \mathbf{v}^T\mathbf{f}^B\, dV + \int_{S_f} \mathbf{v}^T\mathbf{f}^S\, dS.$$

There are two important results regarding the total potential energy which we discuss below.

(i) *The stationarity condition $\delta\Pi = 0$ is equivalent to the principle of virtual work and the displacement field $\mathbf{u}$ which makes $\Pi$ stationary is the solution of the elasticity problem.*

*Proof*

Let us consider a variation $\delta\mathbf{u}$ from a compatible displacement field $\mathbf{u}$. Let $\delta\varepsilon$ be the strain corresponding to $\delta\mathbf{u}$. Then

$$\delta\Pi = \frac{1}{2}\int_V \left[\delta\varepsilon^T\mathbf{C}\,\varepsilon\left(\mathbf{u}\right) + \varepsilon^T\left(\mathbf{u}\right)\mathbf{C}^T\delta\varepsilon\right]\, dV$$

$$-\int_V \delta\mathbf{u}^T\mathbf{f}^B \, dV - \int_{S_f} \delta\mathbf{u}^T\mathbf{f}^S \, dS = 0.$$

Since $\mathbf{C}$ is symmetric, we obtain from the above equation

$$\int_V \delta\varepsilon^T\mathbf{C}\,\varepsilon\,(\mathbf{u})\;dV = \int_V \delta\mathbf{u}^T\mathbf{f}^B \, dV + \int_{S_f} \delta\mathbf{u}^T\mathbf{f}^S \, dS. \tag{5.44}$$

The variation $\delta\mathbf{u}$ is arbitrary provided that $\mathbf{u}+\delta\mathbf{u}$ is a compatible displacement field. Hence, $\delta\mathbf{u} = \mathbf{0}$ on $S_u$, since $\mathbf{u}$ satisfies the displacement boundary conditions on $S_u$.
The variation $\delta\mathbf{u}$ can be interpreted as a virtual displacement field since it is an arbitrary displacement field which is zero on $S_u$. Since (5.44) holds for an arbitrary virtual displacement, it corresponds to the principle of virtual work in terms of displacements and, hence, the displacement $\mathbf{u}$ is the solution of the elasticity problem.

(ii) *The total potential energy attains its minimum value for the solution* $\mathbf{u}$ *of the elasticity problem.*

*Proof*

Let us evaluate the potential energy for the displacement field $\mathbf{u}+\delta\mathbf{u}$ where $\mathbf{u}$ is the solution of the elasticity problem and $\delta\mathbf{u}$ an arbitrary variation. Also, let $\varepsilon$ and $\delta\varepsilon$ be the strain corresponding to the displacements $\mathbf{u}$ and $\delta\mathbf{u}$, respectively.

$$\begin{aligned}
\mathit{\Pi}(\mathbf{u}+\delta\mathbf{u}) &= \frac{1}{2}\int_V (\varepsilon+\delta\varepsilon)^T\,\mathbf{C}\;(\varepsilon+\delta\varepsilon)\;dV \\
&- \int_V \left(\mathbf{u}^T+\delta\mathbf{u}^T\right)\mathbf{f}^B \, dV - \int_{S_f} \left(\mathbf{u}^T+\delta\mathbf{u}^T\right)\mathbf{f}^S \, dS \\
&= \frac{1}{2}\int_V \varepsilon^T\mathbf{C}\,\varepsilon\,dV - \int_V \mathbf{u}^T\mathbf{f}^B \, dV - \int_{S_f} \mathbf{u}^T\mathbf{f}^S \, dS \\
&\quad + \int_V \delta\varepsilon^T\mathbf{C}\,\varepsilon\,dV - \int_V \delta\mathbf{u}^T\mathbf{f}^B \, dV - \int_{S_f} \delta\mathbf{u}^T\mathbf{f}^S \, dS \\
&\quad + \frac{1}{2}\int_V \delta\varepsilon^T\mathbf{C}\,\delta\varepsilon\,dV
\end{aligned} \tag{5.45}$$

Considering that $\delta\mathit{\Pi} = 0$, equation (5.45) becomes

$$\mathit{\Pi}(\mathbf{u}+\delta\mathbf{u}) = \mathit{\Pi}\;(\mathbf{u}) + \frac{1}{2}\int_V \delta\varepsilon^T\mathbf{C}\,\delta\varepsilon\,dV.$$

Note that

$$\mathcal{U}\,(\delta\mathbf{u}) = \frac{1}{2}\int_V \delta\varepsilon^T \mathbf{C}\,\delta\varepsilon\,dV$$

is the strain energy corresponding to the displacement variation. The strain energy is always greater than or equal to zero and when the body is properly supported the strain energy is strictly positive for any compatible displacement field. Therefore, we can conclude that the total potential energy attains its minimum at the solution of the elasticity problem.

□

We note that the definition of the total potential energy

$$\Pi(\mathbf{u}) = \frac{1}{2}\int_V (\,\partial_\varepsilon \mathbf{u}\,)^T \mathbf{C} \partial_\varepsilon \mathbf{u}\,dV - \int_V \mathbf{u}^T \mathbf{f}^B\,dV - \int_{S_f} \mathbf{u}^T \mathbf{f}^S\,dS$$

is quite general and is valid for all mathematical models discussed in Chapters 3 and 4 provided, of course, that the variables are those defined in Tables 4.3 and 4.4. Hence the above results (i) and (ii) are also valid for all these models.