TRANSFORMADAS INTEGRAIS NA RESOLUÇÃO DE EQUAÇÕES DIFERENCIAIS

VII

- TRANSFORMADAS INTEGRAIS SÃO RELAÇÕES DO TIPO:

$$\hat{F}(s) = \int_a^b K(s,x)\, f(x)\, dx$$

ONDE $K(s,x)$ É CHAMADO DE "CAROÇO".

EXEMPLOS:

1) TRANSFORMADA DE LAPLACE

$$\hat{F}(s) = \int_0^\infty e^{-sx} f(x)\, dx$$

2) TRANSFORMADA DE FOURIER

$$\hat{F}(k) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx$$

TRANSFORMADA DE LAPLACE

$$\boxed{\mathcal{L}\{f(x)\} = \hat{F}(s) = \int_0^\infty e^{-sx} f(x)\, dx}$$

S ⇒ PARÂMETRO REAL OU COMPLEXO

EXEMPLOS

$\boxed{a = \text{CTE}}$

$$\int_0^\infty x^n e^{-sx}\, dx = \frac{n!}{s^{n+1}} \qquad \boxed{s>0}$$

$$\hat{F}_1(s) = \mathcal{L}\{x^2\} = \int_0^\infty x^2 e^{-sx}\, dx = \boxed{\frac{2}{s^3}}$$

$$\int_0^\infty e^{(a-s)x}\, dx = \left.\frac{e^{(a-s)x}}{a-s}\right|_0^\infty = \frac{-1}{a-s} \qquad \boxed{s>a}$$

$$\hat{F}_2(s) = \mathcal{L}\{e^{ax}\} = \int_0^\infty e^{ax} e^{-sx}\, dx = \int_0^\infty e^{(a-s)x}\, dx = \boxed{\frac{1}{s-a}}$$

$$\hat{F}_3(s) = \mathcal{L}\{a\} = a\int_0^\infty e^{-sx}\, dx = \boxed{\frac{a}{s}}$$

$$\int_0^\infty e^{-sx}\, dx = -\left.\frac{e^{-sx}}{s}\right|_0^\infty = -\left(\frac{-1}{s}\right) = \frac{1}{s} \qquad \boxed{s>0}$$

TABELA: ALGUMAS TRANSFORMADAS DE LAPLACE

$f(x)$	$\mathcal{L}\{f(x)\}=\hat{F}(S)$		$f(x)$	$\mathcal{L}\{f(x)\}=\hat{F}(S)$
x^n	$\frac{n!}{s^{n+1}}$	$\boxed{n=0,1,2\cdots}$	e^{-ax}	$\frac{1}{s+a}$
e^{ax}	$\frac{1}{s-a}$	$\boxed{s>a}$	a	$\frac{a}{s}$
$\operatorname{sen} ax$	$\frac{a}{s^2+a^2}$	$\boxed{s>0}$		
$x \operatorname{sen} ax$	$\frac{2as}{(s^2+a^2)^2}$			
$x^n e^{ax}$	$\frac{n!}{(s-a)^{n+1}}$	$\boxed{s>a}$		
$\cos ax$	$\frac{s}{s^2+a^2}$	$\boxed{s>0}$		
$x \cos ax$	$\frac{s^2-a^2}{(s^2+a^2)^2}$			

INVERTENDO TRANSFORMADAS DE LAPLACE: PARA REENCONTRAR A FUNÇÃO $f(x)$ ORIGINAL: $\boxed{f(x)=\mathcal{L}^{-1}\{\hat{F}(S)\}}$

EX: $\hat{F}_1(S)=\boxed{\frac{1}{s-1}}$ $\quad \mathcal{L}^{-1}\{\hat{F}_1(S)\}=\boxed{e^x}=f_1(x)$

$\hat{F}_2(S)=\boxed{\frac{2}{s^3}}$ $\quad \mathcal{L}^{-1}\{\hat{F}_2(S)\}=\boxed{x^2}=f_2(x)$

IGNORAR

PROPRIEDADES:

1 SE $\boxed{\mathcal{L}\{f(x)\}=\hat{F}(S)}$ ENTÃO $\boxed{\mathcal{L}\{e^{ax}f(x)\}=\hat{F}(S-a)}$

2 PARA

$$\boxed{g(x)=\begin{cases} 0 & 0\leq x<a \\ f(x-a) & x\geq a \end{cases}}$$

ENTÃO: $\boxed{\mathcal{L}\{g(x)\}=e^{-as}\hat{F}(S)}=\boxed{e^{-as}\mathcal{L}\{f(x)\}}$ 2

EX: CALCULE PROP. 1

$$\mathcal{L}\{e^{-2x}\cos\omega x\} = \hat{F}(s-(-2)) = \hat{F}(s+2)$$

$$\hat{F}(s) = \mathcal{L}\{\cos\omega x\} = \frac{s}{s^2+\omega^2}$$

ENTÃO $(s = s+2)$:

$$\boxed{\mathcal{L}\{e^{-2x}\cos\omega x\}} = \frac{(s+2)}{(s+2)^2+\omega^2} = \boxed{\frac{s+2}{s^2+4s+4+\omega^2}}$$

1. $\mathcal{L}\{a f(x)\} = a\,\mathcal{L}\{f(x)\}$ $\boxed{a \Rightarrow \text{CONSTANTE}}$
2. $\mathcal{L}\{f(x)+g(x)\} = \mathcal{L}\{f(x)\} + \mathcal{L}\{g(x)\}$
3. $\mathcal{L}^{-1}\{a\hat{F}(s)\} = a\,\mathcal{L}^{-1}\{\hat{F}(s)\}$ $\boxed{a \Rightarrow \text{CONSTANTE}}$
4. $\mathcal{L}^{-1}\{\hat{F}_1(s)+\hat{F}_2(s)\} = \mathcal{L}^{-1}\{\hat{F}_1(s)\} + \mathcal{L}^{-1}\{\hat{F}_2(s)\}$

PARA $f(x)$ CONTÍNUA PARA $x \geq 0$ E DE ORDEM EXPONENCIAL QUANDO $x \to \infty$ E $f'(x)$ CONTÍNUA EM INTERVALOS E DE ORDEM EXPONENCIAL QUANDO $x \to \infty$

⑤ $$\boxed{\mathcal{L}\left\{\frac{d^n f}{dx^n}\right\} = s^n \hat{F}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - \underbrace{s^0 f^{(n-1)}(0)}_{f^{(n-1)}(0)}}$$

SE AS CONDIÇÕES INICIAIS FOREM DADAS:

$$f(0) = C_0 \quad f'(0) = C_1 \cdots f^{(n-1)}(0) = C_{n-1}$$

VEJA QUE:

$$f^n = \frac{d^n f}{dx^n}$$

ENTÃO:

$$\boxed{\mathcal{L}\left\{\frac{d^n f}{dx^n}\right\} = s^n \hat{F}(s) - C_0 s^{n-1} - C_1 s^{n-2} - \cdots - C_{n-2} s - C_{n-1}}$$

EX: $\boxed{n=1}$

$$\mathcal{L}\{f'(x)\} = s\hat{F}(s) - s^0 f(0) = s\hat{F}(s) - C_0$$

$\boxed{n=2}$

$$\mathcal{L}\{f''(x)\} = s^2 \hat{F}(s) - s^1 f(0) - s^0 f'(0) = s^2 \hat{F}(s) - C_0 s - C_1$$

- PODEMOS APLICAR TRANSFORMADAS DE LAPLACE NA RESOLUÇÃO DE EDOS LINEARES COM COEFICIENTES CONSTANTES (HOMOGÊNEAS OU NÃO) COM CONDIÇÕES INICIAIS:

$$\boxed{a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)}$$

$a_i \Rightarrow$ SÃO CONSTANTES

$\boxed{EX1}$ RESOLVA A SEGUINTE EDÓ PELO MÉTODO DA TRANSFORMADA DE LAPLACE

$$y''(x) + y(x) = \cos x \quad \text{DADOS } y(0)=1 \text{ E } y'(0)=0$$

1º VAMOS TOMAR A TRANSFORMADA DE LAPLACE DA EQUAÇÃO:

$$\mathcal{L}\{y''(x)\} + \mathcal{L}\{y(x)\} = \mathcal{L}\{\cos x\} = \frac{s}{s^2+1}$$

MAS $\mathcal{L}\{y(x)\} = \hat{Y}(s)$:

$$\underbrace{\mathcal{L}\{y''(x)\}}_{\text{PROP. 4}} + \hat{Y}(s) = \frac{s}{s^2+1}$$

$$\underbrace{s^2\hat{Y}(s) - s\,y(0) - s^0 y'(0)}_{\text{COND. INIC.}} + \hat{Y}(s) = \frac{s}{s^2+1}$$

$$s^2\hat{Y}(s) - s + \hat{Y}(s) = \frac{s}{s^2+1}$$

2º ISOLANDO $\hat{Y}(s)$:

$$(s^2+1)\hat{Y}(s) = \frac{s}{s^2+1} + s$$

$$\hat{Y}(s) = \frac{s}{(s^2+1)^2} + \frac{s}{s^2+1}$$

3º INVERTENDO A TRANSFORMADA DE LAPLACE:

$$\boxed{\mathcal{L}^{-1}\{\hat{Y}(s)\} = y(x)}$$

$\boxed{\text{OPERADOR LINEAR}}$:

$$y(x) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2} + \frac{s}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

ENTÃO:

$$\boxed{y(x) = \frac{1}{2} x \operatorname{sen} x + \cos x}$$

POIS:

$$\boxed{\mathcal{L}\left\{\frac{x}{2} \operatorname{sen} x\right\}} = \frac{1}{2} \frac{2s}{(s^2+1)^2} = \boxed{\frac{s}{(s^2+1)^2}}$$

$\frac{1}{2}\mathcal{L}\{x \operatorname{sen} x\}$

EX.2 RESOLVA A SEGUINTE EDO PELO MÉTODO DA TRANSFORMADA DE LAPLACE:

$$y'(x) - 5y(x) = 0 \quad \text{DADO } y(0) = 2.$$

$$\underbrace{\mathcal{L}\{y'(x)\}} - 5\underbrace{\mathcal{L}\{y(x)\}} = \mathcal{L}\{0\} = 0$$

$$s\hat{Y}(s) - s^0 y(0) - 5\hat{Y}(s) = 0$$

$$s\hat{Y}(s) - 1 \cdot 2 - 5\hat{Y}(s) = 0$$

$$\hat{Y}(s)(s-5) = 2$$

$$\hat{Y}(s) = \frac{2}{s-5}$$

- MAS: $\boxed{\mathcal{L}^{-1}\{\hat{Y}(s)\} = y(x)}$

- ENTÃO:

$$\boxed{y(x)} = \mathcal{L}^{-1}\left(\frac{2}{s-5}\right) = 2\mathcal{L}^{-1}\left(\frac{1}{s-5}\right) = \boxed{2e^{5x}}$$

$\boxed{\text{EX. 3}}$ RESOLVA A SEGUINTE EDO PELO METODO DA TRANSFORMADA DE LAPLACE:

$$\frac{dI}{dt} + 50 I = 5 \quad \text{SABENDO QUE} \quad I(0)=0$$

$$\mathcal{L}\left\{\frac{dI(t)}{dt}\right\} + 50\mathcal{L}\{I(t)\} = 5\mathcal{L}\{1\}$$

MAS $\boxed{\hat{I}(s) = \mathcal{L}\{I(t)\}}$

E $\boxed{\mathcal{L}\left\{\frac{dI(t)}{dt}\right\} = s\hat{I}(s) - s^0 I(0) = s\hat{I}(s) - I(0)}$

$$s\hat{I}(s) - I(0) + 50\hat{I}(s) = 5 \cdot \frac{1}{s}$$

$$(s+50)\hat{I}(s) = \frac{5}{s} + I(0) = \frac{5}{s}$$

$$\hat{I}(s) = \frac{5}{s}\frac{1}{(s+50)} = 5 \cdot \frac{1}{s(s+50)}$$

$$\boxed{I(t) = \mathcal{L}^{-1}\{I(s)\}} = 5\mathcal{L}^{-1}\left\{\frac{1}{s(s+50)}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{50s} - \frac{1}{50(s+50)}\right\}$$

$$\frac{1}{50s} - \frac{1}{50(s+50)} = \frac{50(s+50) - 50s}{50.50.s(s+50)} = \frac{\cancel{50}.50}{\cancel{50}.50.s(s+50)} = \frac{1}{s(s+50)}$$

$$I(t) = 5\left(\mathcal{L}^{-1}\left\{\frac{1}{50s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{50(s+50)}\right\}\right) = \frac{5}{50}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{5}{50}\mathcal{L}^{-1}\left\{\frac{1}{s+50}\right\}$$

$$\boxed{I(t)} = \frac{5}{50}\cdot 1 - \frac{1}{10}e^{-50t} = \boxed{\frac{1}{10} - \frac{1}{10}e^{-50t}}$$