Prova 2 Substitutiva - 4302303 - Eletromagnetismo I - 13/07/2023

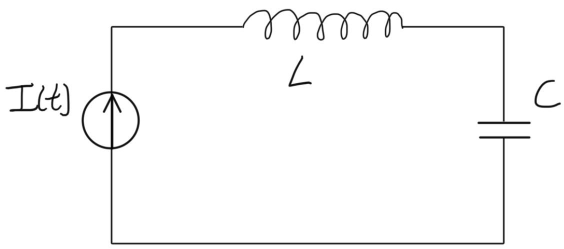
1) [6,0] Um indutor feito de N espiras densamente enroladas em um cilindro de comprimento ℓ e diâmetro $d \ll \ell$ é ligado em série a um capacitor de placas paralelas circulares de diâmetro Δ, separadas por uma distância β. Pelo circuito, circula uma corrente $I(t) = I_0\, sen(\omega t)$.

a) Qual é o valor do campo elétrico $\vec{E}(t)$ dentro do capacitor?

b) Qual é o valor do campo magnético $\vec{B}(t)$ dentro do indutor?

c) Calcule, pela lei de Ampère-Maxwell, qual o campo magnético dentro do capacitor.

d) Calcule, pela lei de Faraday, qual o campo elétrico dentro do indutor.

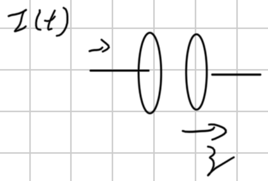


e) Qual a energia elétrica total dentro do capacitor? E qual a energia magnética total dentro do capacitor?

f) Qual a energia magnética total dentro do indutor? E qual a energia elétrica total dentro do indutor?

g) Para qual valor de frequência ω a energia magnética média dentro do indutor se iguala à energia elétrica média dentro do capacitor?

a) Lei de Gauss: $\vec{E}(t) = \frac{Q(t)}{\varepsilon_0 A}\hat{z}$; $A = \frac{\pi \Delta^2}{4}$

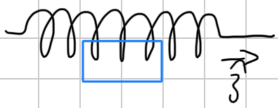


$$I(t) = \frac{dQ(t)}{dt}$$

$$Q(t) - Q(0) = \int_0^t I(t)\,dt$$

Tomando: $Q(0) = 0 \Rightarrow Q(t) = \int_0^t I_0 sen(\omega t)\,dt = -\frac{I_0}{\omega}\cos(\omega t)$

$$\boxed{\vec{E}(t) = -\frac{4 I_0}{\pi \varepsilon_0 \omega \Delta^2}\cos(\omega t)\,\hat{z}}$$

b) 

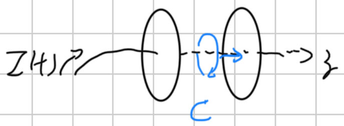
$$\nabla \times \vec{H} = \vec{J} + \cancel{\frac{\partial \vec{D}}{\partial t}} \sim 0$$

$$\oint_C \frac{\vec{B}}{\mu_0} \cdot d\vec{l} = \int \vec{J} \cdot d\vec{a}$$

$$\frac{B_z}{\mu_0}\, l = N \cdot I \Rightarrow \boxed{\vec{B} = \mu_0 \frac{N}{l} I_0 \,\mathrm{sen}(\omega t)\, \hat{z}}$$

c) Ampère-Maxwell:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t} \vec{D}$$



$$\int_A \left(\nabla \times \frac{\vec{B}}{\mu_0} \right) d\vec{a} = \frac{1}{\mu_0} \oint_C \vec{B} \cdot d\vec{l} = \frac{\partial}{\partial t} \int_A \vec{D} \cdot d\vec{a}$$

$d\vec{a} = da \cdot \hat{z}$

$d\vec{l} = r \cdot d\theta \, \hat{\theta}$

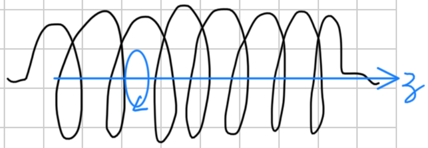
$$\Downarrow$$

$$\frac{B_\theta \cdot 2\pi r}{\mu_0} = \left(\frac{d}{dt} Q \right) \frac{\pi r^2}{A} \quad ; \quad A = \pi \frac{D^2}{4}$$

$$\Rightarrow \vec{B} = 2\mu_0 \frac{I(t)}{\pi D^2} r \hat{\theta} = \boxed{\frac{2\mu_0 I_0}{\pi D^2} \,\mathrm{sen}(\omega t)\, r \hat{\theta}}$$

d)

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$



$$\int_A (\nabla \times \vec{E})\, d\vec{a} = -\frac{d}{dt}\int_A \vec{B}\cdot d\vec{a}$$

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\, \mu_0 \frac{N}{l} I_0 \operatorname{sen}(\omega t)\, \pi r^2$$

$d\vec{a} = da\,\hat{z}$

$d\vec{l} = r\,d\theta\,\hat{\theta}$

$$E_\theta \cdot 2\pi r = -\mu_0 \frac{N}{l} I_0\, \omega \cos(\omega t)\, \pi r^2$$

$$\vec{E}(t) = -\mu_0 \frac{N \omega I_0}{2l} \cos(\omega t)\, r\, \hat{\theta}$$

e)

$$E_{ec} = \int \frac{\vec{E}\cdot\vec{D}}{2}\, dv = \frac{1}{2}\left(\frac{4 I_0}{\pi \varepsilon_0 \omega D^2}\cos(\omega t)\right)^2 \cdot \beta \cdot \frac{\pi D^2}{4}$$

$$E_{ec} = \frac{1}{2}\left(\frac{\beta}{\varepsilon_0}\,\frac{4}{\pi D^2}\right)\frac{I_0^2}{\omega^2}\cos^2(\omega t) = \frac{V_0^2 \cos^2(\omega t)}{2C}$$

$$E_{Bc} = \int \frac{\vec{B}\cdot\vec{H}}{2}\, dv = \frac{1}{2\mu_0}\left(\frac{2\mu_0 I_0}{\pi D^2}\operatorname{sen}(\omega t)\, r\right)^2 r\,dr\,d\theta\,dz$$

$$= \frac{1}{\cancel{2}\mu_0}\,\frac{4\mu_0^{\cancel{2}} I_0^2}{\pi^{\cancel{2}} D^4}\operatorname{sen}^2(\omega t) \cdot \cancel{2\pi}\,\beta \cdot \int_0^{D/2} r^3\, dr$$

$$= \mu_0 \cdot \frac{4\beta}{\cancel{D^4}}\,\frac{I_0^2}{\pi}\operatorname{sen}^2(\omega t) \cdot \frac{1}{4}\left(\frac{\cancel{D}}{2}\right)^4$$

$$E_{Bc} = \frac{\mu_0}{4\pi}\cdot \beta\, I_0^2 \operatorname{sen}^2(\omega t)$$

f) $E_{BL} = \int \frac{\vec{B}\cdot\vec{H}}{2}\,dv = \frac{1}{2\mu_0}\int \left(\mu_0 \frac{N}{\ell} I_0 \,sen\,(wt)\right)^2 dV$

$$E_{BL} = \frac{\mu_0}{2}\left(\frac{N}{\ell}\right)^2 I_0^{\,2}\,sen^2(wt)\,\left(\pi \cdot \frac{d^2}{4}\cdot \ell\right)$$

$$E_{BL} = \frac{\mu_0}{2}\,\frac{N^2}{\ell}\,\frac{\pi d^2}{4}\,I_0^{\,2}\,sen^2(wt) = \frac{L \cdot I_0^{\,2}\,sen^2(wt)}{2}$$

$$E_{CL} = \int \frac{\vec{E}\cdot\vec{D}}{2}\,dv = \frac{\varepsilon_0}{2}\int\left(\mu_0 \frac{N}{2\ell}\, w\, I_0 \cos(wt)\; r\right)^2 dV$$

$$= \frac{\varepsilon_0}{2}\,\mu_0^{\,2}\,\frac{N^2}{4\ell^2}\,w^2 I_0^2 \cos^2(wt)\cdot 2\pi\cdot\ell\cdot\int_0^{d/2} r^2\, r\,dr$$

$$= \pi\,\frac{\mu_0}{4c^2}\cdot\frac{N^2}{\ell}\,w^2 I_0^2 \cos^2(wt)\cdot\frac{d^4}{16}$$

$$= \frac{\pi}{64}\,\frac{\mu_0}{c^2}\,\frac{N^2 d^4}{\ell}\,w^2 I_0^2 \cos^2(wt)$$

g) $\langle E_{BL}\rangle = \langle E_{eC}\rangle$

$$\frac{\mu_0}{\cancel{2}}\,\frac{N^2}{\ell}\,\frac{\pi d^2}{4}\,\cancel{I_0^2}\langle \cancel{sen^2(wt)}\rangle = \frac{1}{\cancel{2}}\left(\frac{\beta}{\varepsilon_0}\,\frac{4}{\pi D^2}\right)\frac{\cancel{I_0^2}}{w^2}\langle\cancel{\cos^2(wt)}\rangle$$

$$w^2 = \left(\frac{\beta}{\varepsilon_0}\,\frac{4}{\pi D^2}\right)\cdot\left(\frac{1}{\mu_0}\,\frac{4\ell}{N^2 \pi d^2}\right) = \frac{1}{LC} \;!$$