PRO 5970 Métodos de Otimização Não Linear

Convexity and matrices

Celma de Oliveira Ribeiro
2023

Departmento de Engenharia de Produção
Universidade de São Paulo

Positive semi definite matrices

Definition

A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called positive semidefinite if

$$x^{'} A x \geq 0 \quad \forall x \in \mathbb{R}^n$$

It is called positive definite if

$$x^{'} A x > 0 \quad \forall x \in \mathbb{R}^n, x \neq 0$$

A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called negative semidefinite if

$$x^{'} A x \leq 0 \quad \forall x \in \mathbb{R}^n$$

It is called negative definite if

$$x^{'} A x < 0 \quad \forall x \in \mathbb{R}^n, x \neq 0$$

Definition

For an $n \times n$ matrix of A, a minor of order k is principal if it is obtained by deleting $n - k$ rows and the corresponding $n - k$ columns.

For instance, in a principal minor where you have deleted row 1 and 3, you should also delete column 1 and 3.

Definition

For a given $k \in \{1, 2, \ldots, n\}$ the dominant principal submatrix A_k of matrix A ($\in \mathbb{R}^n$) is given as

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \ldots & a_{kk} \end{bmatrix}$$

Definition

The $k-th$ leading principal minor of an $n \times n$ matrix is the determinant of the $k \times k$ matrix obtained by deleting the last $n-k$ rows and columns of the matrix.

The leading principal minors of a matrix A $n \times n$ are the determinants of the submatrices:

$$A_1 = [a_{11}]$$

$$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\vdots$$

$$A_n = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Observation

The matrix A is positive semidefinite if and only if $-A$ is negative semidefinite. Similarly a matrix A is positive definite if and only if $-A$ is negative definite.

Theorem

The following statements are equivalent:

- The symmetric matrix A is positive definite (semidefinite).
- All eigenvalues of A are stricly positive (non negative).
- There exists a non singular $B \in \mathbb{R}^{n \times k}$ such that $A = B^{'}B$. (B may be singular) [1]

Theorem

Let A be a symmetric $n \times n$ matrix. Then:

A is positive definite $\Leftrightarrow$ all leading principal minors are positive

A is positive semidefinite $\Leftrightarrow$ determinant of all minors are non negative ≥ 0

- In the first case, it is enough to check the inequality for all the leading principal minors (i.e. for $1 \leq k \leq n$).
- In the last case, we must check for all minors , i.e. for each $1 \leq k \leq n$ and for each of the $\begin{pmatrix} n \\ k \end{pmatrix}$ principal minors of order k.

Example - Positive definite

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$det(A_1) = 2 > 0 \quad det(A_2) = 3 > 0 \quad det(A_3) = 4 > 0$

Example - Indefinite

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Leading minors $det(A_1) = 0 \quad det(A_2) = 0 \quad det(A_3) = 0$

$k = 2 \qquad det(a_{22}) = 0$, $det(a_{33}) = 2$

$k = 1$

$$det \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = -1 \quad det \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = 0$$