14.3 SOLUÇÕES

Revisão técnica: Ricardo Miranda Martins – IMECC – Unicamp

1. $f(x, y) = x^3y^5 \Rightarrow f_x(x, y) = 3x^2y^5$, $f_x(3, -1) = -27$

2. $f(x, y) = \sqrt{2x + 3y} \Rightarrow$ $f_y(x, y) = \frac{1}{2}(2x + 3y)^{-1/2}(3)$, $f_y(2, 4) = \dfrac{3/2}{\sqrt{4 + 12}} = \dfrac{3}{8}$

3. $f(x, y) = xe^{-y} + 3y \Rightarrow \partial f/\partial y = x(-1)e^{-y} + 3$, $(\partial f/\partial y)(1, 0) = -1 + 3 = 2$

4. $f(x, y) = \text{sen}(y - x) \Rightarrow \partial f/\partial x = -\cos(y - x)$, $(\partial f/\partial x)(3, 3) = -\cos(0) = -1$

5. $z = \dfrac{x^3 + y^3}{x^2 + y^2} \Rightarrow$

$$\frac{\partial z}{\partial x} = \frac{3x^2(x^2 + y^2) - (x^3 + y^3)(2x)}{(x^2 + y^2)^2} = \frac{x^4 + 3x^2y^2 - 2xy^3}{(x^2 + y^2)^2},$$

$$\frac{\partial z}{\partial y} = \frac{3y^2(x^2 + y^2) - (x^3 + y^3)(2y)}{(x^2 + y^2)^2} = \frac{3x^2y^2 + y^4 - 2yx^3}{(x^2 + y^2)^2}$$

6. $z = x\sqrt{y} - \dfrac{y}{\sqrt{x}} \Rightarrow$

$$\frac{\partial z}{\partial x} = \sqrt{y} - y\left(-\tfrac{1}{2}\right)x^{-3/2} = \sqrt{y} + \frac{y}{2x^{3/2}},$$

$$\frac{\partial z}{\partial y} = x\left(\tfrac{1}{2}\right)y^{-1/2} - \frac{1}{\sqrt{x}} = \frac{x}{2\sqrt{y}} - \frac{1}{\sqrt{x}}$$

7. $z = \dfrac{x}{y} + \dfrac{y}{x} \Rightarrow \dfrac{\partial z}{\partial x} = \dfrac{1}{y} - \dfrac{y}{x^2}$

8. $z = (3xy^2 - x^4 + 1)^4 \Rightarrow$

$\partial z/\partial x = 4(3xy^2 - x^4 + 1)^3(3y^2 - 4x^3)$,

$\partial z/\partial y = 4(3xy^2 - x^4 + 1)^3(6xy) = 24xy(3xy^2 - x^4 + 1)^3$

9. $u = xy\sec(xy) \Rightarrow$

$\partial u/\partial x = y\sec(xy) + xy[\sec(xy)\,\text{tg}(xy)](y) = y\sec(xy)[1 + xy\,\text{tg}(xy)]$

10. $u = \dfrac{u}{x + t} \Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{1(x + t) - x(1)}{(x + t)^2} = \dfrac{t}{(x + t)^2}$,

$\dfrac{\partial u}{\partial t} = x(-1)(x + t)^{-2}(1) = -\dfrac{x}{(x + t)^2}$

11. $f(x, y, z) = xyz \Rightarrow f_y(x, y, z) = xz$, assim $f_y(0, 1, 2) = 0$.

12. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \Rightarrow$

$f_z(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2z)$, assim $f_z(0, 3, 4) = \frac{4}{\sqrt{0 + 9 + 16}} = \frac{4}{5}$.

13. $u = xy + yz + zx \Rightarrow u_x = y + z$, $u_y = x + z$, $u_z = y + x$

14. $u = x^2y^3t^4 \Rightarrow u_x = 2xy^3t^4$, $u_y = 3x^2y^2t^4$, $u_t = 4x^2y^3t^3$

15. $f(x, y) = x^3y^5 - 2x^2y + x \Rightarrow$

$f_x(x, y) = 3x^2y^5 - 4xy + 1$, $f_y(x, y) = 5x^3y^4 - 2x^2$

16. $f(x, y) = x^2y^2(x^4 + y^4) \Rightarrow$

$f_x(x, y) = 2xy^2(x^4 + y^4) + x^2y^2(4x^3) = 6x^5y^2 + 2xy^6$

e, por simetria, $f_y(x, y) = 6y^5x^2 + 2yx^6$.

17. $f(x, y) = x^4 + x^2y^2 + y^4 \Rightarrow f_x(x, y) = 4x^3 + 2xy^2$, $f_y(x, y) = 2x^2y + 4y^3$

18. $f(x, y) = \ln(x^2 + y^2) \Rightarrow$

$f_x(x, y) = \dfrac{1}{x^2 + y^2}(2x) = \dfrac{2x}{x^2 + y^2}$, $f_y(x, y) = \dfrac{2y}{x^2 + y^2}$

19. $f(x, y) = e^x\,\text{tg}(x - y) \Rightarrow$

$f_x(x, y) = e^x\,\text{tg}(x - y) + e^x\sec^2(x - y) = e^x[\text{tg}(x - y) + \sec^2(x - y)]$,

$f_y(x, y) = e^x[\sec^2(x - y)](-1) = -e^x\sec^2(x - y)$

20. $f(s, t) = \dfrac{s}{\sqrt{s^2 + t^2}} \Rightarrow$

$$f_s(s, t) = \frac{(1)\sqrt{s^2 + t^2} - s\left(\frac{1}{2}\right)(s^2 + t^2)^{-1/2}(2s)}{\left(\sqrt{s^2 + t^2}\right)^2} = \frac{|s^2 + t^2| - s^2}{|s^2 + t^2|\sqrt{s^2 + t^2}} = \frac{t^2}{(s^2 + t^2)^{3/2}},$$

$$f_t(s, t) = s\left(-\tfrac{1}{2}\right)(s^2 + t^2)^{-3/2}(2t) = -\frac{st}{(s^2 + t^2)^{3/2}}$$

21. $g(x, y) = y\,\text{tg}(x^2y^3) \Rightarrow$

$g_x(x, y) = [y\sec^2(x^2y^3)](2xy^3) = 2xy^4\sec^2(x^2y^3)$,

$g_y(x, y) = \text{tg}(x^2y^3) + [y\sec^2(x^2y^3)](3x^2y^2) = \text{tg}(x^2y^3) + 3x^2y^3\sec^2(x^2y^3)$

22. $g(x, y) = \ln(x + \ln y) \Rightarrow$

$g_x(x, y) = \dfrac{1}{x + \ln y}(1) = \dfrac{1}{x + \ln y}$,

$g_y(x, y) = \dfrac{1}{x + \ln y}\left(\dfrac{1}{y}\right) = \dfrac{1}{y(x + \ln y)}$

23. $f(x, y) = e^{xy}\cos x\,\text{sen}\,y \Rightarrow$

$f_x(x, y) = ye^{xy}\cos x\,\text{sen}\,y + e^{xy}(-\text{sen}\,x)\,\text{sen}\,y = e^{xy}\,\text{sen}\,y(y\cos x - \text{sen}\,x)$,

$f_y(x, y) = xe^{xy}\cos x\,\text{sen}\,y + e^{xy}\cos x\cos y = e^{xy}\cos x(x\,\text{sen}\,y + \cos y)$

24. $f(s,t) = \sqrt{2 - 3s^2 - 5t^2} \quad \Rightarrow$

$$f_s(s,t) = \tfrac{1}{2}\left(2 - 3s^2 - 5t^2\right)^{-1/2}(-6s) = -\frac{3s}{\sqrt{2 - 3s^2 - 5t^2}},$$

$$f_t(s,t) = \tfrac{1}{2}\left(2 - 3s^2 - 5t^2\right)^{-1}(-10t) = -\frac{5t}{\sqrt{2 - 3s^2 - 5t^2}}$$

25. $z = \operatorname{senh}\sqrt{3x + 4y} \quad \Rightarrow$

$$\frac{\partial z}{\partial x} = \left(\cosh\sqrt{3x + 4y}\right)\left(\tfrac{1}{2}\right)(3x + 4y)^{-1/2}(3) = \frac{3\cosh\sqrt{3x + 4y}}{2\sqrt{3x + 4y}},$$

$$\frac{\partial z}{\partial y} = \left(\cosh\sqrt{3x + 4y}\right)\left(\tfrac{1}{2}\right)(3x + 4y)^{-1/2}(4) = \frac{2\cosh\sqrt{3x + 4y}}{\sqrt{3x + 4y}}$$

26. Uma vez que $z = \log_x y$, $x^z = y$ e $z \ln x = \ln y$. Então

$$\frac{\partial z}{\partial x}\ln x + z\left(\frac{1}{x}\right) = 0, \text{ então } \frac{\partial z}{\partial x} = -\frac{z}{x\ln x} = -\frac{\ln y}{x(\ln x)^2}.$$

Além disso, $(\ln x)\dfrac{\partial z}{\partial y} = \dfrac{1}{y}$, então $\dfrac{\partial z}{\partial y} = \dfrac{1}{y\ln x}$.

27. $f(u,v) = \operatorname{tg}^{-1}\left(\dfrac{u}{v}\right) \quad \Rightarrow$

$$f_u(u,v) = \frac{1}{1 + (u/v)^2}\left(\frac{1}{v}\right) = \frac{1}{v}\left(\frac{v^2}{u^2 + v^2}\right) = \frac{v}{u^2 + v^2},$$

$$f_v(u,v) = \frac{1}{1 + (u/v)^2}\left(-\frac{u}{v^2}\right) = -\frac{u}{v^2}\left(\frac{v^2}{u^2 + v^2}\right) = -\frac{u}{u^2 + v^2}$$

28. $f(x,t) = e^{\operatorname{sen}(t/x)} \quad \Rightarrow$

$$f_x(x,t) = e^{\operatorname{sen}(t/x)}\cos\left(\frac{t}{x}\right)\left(-\frac{t}{x^2}\right) = -t\cos\left(\frac{t}{x}\right)\frac{e^{\operatorname{sen}(t/x)}}{x^2},$$

$$f_t(x,t) = e^{\operatorname{sen}(t/x)}\cos\left(\frac{t}{x}\right)\left(\frac{1}{x}\right) = \frac{e^{\operatorname{sen}(t/x)}}{x}\cos\left(\frac{t}{x}\right)$$

29. $z = \ln\left(x + \sqrt{x^2 + y^2}\right) \quad \Rightarrow$

$$\frac{\partial z}{\partial x} = \frac{1}{x + \sqrt{x^2 + y^2}}\left[1 + \tfrac{1}{2}\left(x^2 + y^2\right)^{-1/2}(2x)\right] = \frac{\left(\sqrt{x^2 + y^2} + x\right)\Big/\sqrt{x^2 + y^2}}{\left(x + \sqrt{x^2 + y^2}\right)} = \frac{1}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x + \sqrt{x^2 + y^2}}\left(\frac{1}{2}\right)\left(x^2 + y^2\right)^{-1/2}(2y) = \frac{y}{x\sqrt{x^2 + y^2} + x^2 + y^2}$$

30. $z = x^{x^y}$, logo $\ln z = x^y \ln x$ e

$$\frac{1}{z}\frac{\partial z}{\partial x} = yx^{y-1}\ln x + x^y\left(\frac{1}{x}\right) \quad \Leftrightarrow$$

$$\frac{\partial z}{\partial x} = z\left[yx^{y-1}\ln x + x^{y-1}\right] = x^{y-1}x^{x^y}(1 + y\ln x),$$

$$\frac{\partial z}{\partial y} = \left(x^{x^y}\right)(\ln x)\frac{\partial}{\partial y}(x^y) = \left(x^{x^y}\right)(\ln x)x^y\ln x = x^{x^y + y}(\ln x)^2$$

31. $f(x,y) = \displaystyle\int_x^y e^{t^2}\,dt$. Pelo TFC1,

$\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ para f contínua. Então,

$$f_x(x,y) = \frac{\partial}{\partial x}\int_x^y e^{t^2}\,dt = \frac{\partial}{\partial x}\left(-\int_y^x e^{t^2}\,dt\right) = -e^{x^2}$$

e $f_y(x,y) = \dfrac{\partial}{\partial y}\displaystyle\int_x^y e^{t^2}\,dt = e^{y^2}$.

32. $f(x,y) = \displaystyle\int_y^x \frac{e^t}{t}\,dt$. Se 0 não estiver no intervalo $[y, x]$, então pelo TFC1, $f_x(x,y) = \dfrac{e^x}{x}$ e $f_y(x,y) = -\dfrac{e^y}{y}$.

33. $f(x,y,z) = x^2yz^3 + xy - z \quad \Rightarrow$
$f_x(x,y,z) = 2xyz^3 + y$, $f_y(x,y,z) = x^2z^3 + x$,
$f_z(x,y,z) = 3x^2yz^2 - 1$

34. $f(x,y,z) = x\sqrt{yz} \quad \Rightarrow \quad f_x(x,y,z) = \sqrt{yz}$,
$f_y(x,y,z) = x\left(\tfrac{1}{2}\right)(yz)^{-1/2}(z) = \dfrac{xz}{2\sqrt{yz}}$, e por simetria, $f_z(x,y,z) = \dfrac{xy}{2\sqrt{yz}}$.

35. $f(x,y,z) = x^{yz} \quad \Rightarrow \quad f_x(x,y,z) = yzx^{yz-1}$. Pelo Teorema 3.4.5, $f_y(x,y,z) = x^{yz}\ln(x^z) = zx^{yz}\ln x$ e por simetria, $f_z(x,y,z) = yx^{yz}\ln x$.

36. $f(x,y,z) = xe^y + ye^z + ze^x \quad \Rightarrow$
$f_x(x,y,z) = e^y + ze^x$, $f_y(x,y,z) = xe^y + e^z$,
$f_z(x,y,z) = ye^z + e^x$

37. $u = z\operatorname{sen}\left(\dfrac{y}{x+z}\right) \quad \Rightarrow$

$$u_x = z\cos\left(\frac{y}{x+z}\right)\left[-y(x+z)^{-2}\right] = \frac{-yz}{(x+z)^2}\cos\left(\frac{y}{x+z}\right),$$

$$u_y = z\cos\left(\frac{y}{x+z}\right)\left(\frac{1}{x+z}\right) = \frac{z}{x+z}\cos\left(\frac{y}{x+z}\right),$$

$$u_z = \operatorname{sen}\left(\frac{y}{x+z}\right) + z\cos\left(\frac{y}{x+z}\right)\left[-y(x+z)^{-2}\right] = \operatorname{sen}\left(\frac{y}{x+z}\right) - \frac{yz}{(x+z)^2}\cos\left(\frac{y}{x+z}\right)$$

38. $u = xy^2z^3 \ln(x + 2y + 3z) \Rightarrow$

$u_x = y^2z^3 \ln(x + 2y + 3z) = xy^2z^3 \left(\frac{1}{x + 2y + 3z}\right)$

$= y^2z^3 \left[\ln(x + 2y + 3z) + \frac{x}{x + 2y + 3z}\right],$

$u_y = 2xyz^3 \ln(x + 2y + 3z) + xy^2z^3 \left(\frac{1}{x + 2y + 3z}\right)(2)$

$= 2xyz^3 \left[\ln(x + 2y + 3z) + \frac{y}{x + 2y + 3z}\right],$

e, por simetria,

$u_z = 3xy^2z^2 \left[\ln(x + 2y + 3z) + \frac{z}{x + 2y + 3z}\right].$

39. $u = x^{y^z} \Rightarrow u_x = y^z x^{y^z - 1},$

$u_y = x^{y^z} \ln x \cdot zy^{z-1} = x^{y^z} y^{z-1} z \ln x,$

$u_z = x^{y^z} \ln x\,(y^z \ln y) = x^{y^z} y^z \ln x \ln y$

40. $f(x, y, z, t) = \frac{x - y}{z - t} \Rightarrow f_x(x, y, z, t) = \frac{1}{z - t},$

$f_y(x, y, z, t) = -\frac{1}{z - t},$

$f_z(x, y, z, t) = (x - y)(-1)(z - t)^{-2} = \frac{y - x}{(z - t)^2}$, e

$f_t(x, y, z, t) = (x - y)(-1)(z - t)^{-2}(-1) = \frac{x - y}{(z - t)^2}.$

41. $f(x, y, z, t) = xy^2z^3t^4 \Rightarrow f_x(x, y, z, t) = y^2z^3t^4,$

$f_y(x, y, z, t) = 2xyz^3t^4, f_z(x, y, z, t) = 3xy^2z^2t^4$, e

$f_t(x, y, z, t) = 4xy^2z^3t^3.$

42. $xy + yz = xz \Rightarrow \frac{\partial}{\partial x}(xy + yz) = \frac{\partial}{\partial x}(xz) \Leftrightarrow$

$y + y\frac{\partial z}{\partial x} = z + x\frac{\partial z}{\partial x} \Leftrightarrow (y - x)\frac{\partial z}{\partial x} = z - y$, logo

$\frac{\partial z}{\partial x} = \frac{z - y}{y - x}. \ \frac{\partial}{\partial y}(xy + yz) = \frac{\partial}{\partial y}(xz) \Leftrightarrow$

$x + z + y\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial y} \Leftrightarrow (y - x)\frac{\partial z}{\partial y} = -(x + z)$, logo

$\frac{\partial z}{\partial y} = \frac{x + z}{x - y}.$

43. $xyz = \cos(x + y + z) \Rightarrow$

$\frac{\partial}{\partial x}(xyz) = \frac{\partial}{\partial x}[\cos(x + y + z)] \Leftrightarrow$

$yz + xy\frac{\partial z}{\partial x} = [-\text{sen}(x + y + z)]\left(1 + \frac{\partial z}{\partial x}\right),$

$[xy + \text{sen}(x + y + z)]\frac{\partial z}{\partial x} = -[yz + \text{sen}(x + y + z)],$

então, $\frac{\partial z}{\partial x} = -\frac{yz + \text{sen}(x + y + z)}{xy + \text{sen}(x + y + z)},$

$\frac{\partial}{\partial y}(xyz) = \frac{\partial}{\partial y}(\cos(x + y + z))$, e então, por simetria,

$\frac{\partial z}{\partial y} = -\frac{xz + \text{sen}(x + y + z)}{xy + \text{sen}(x + y + z)}.$

44. $x^2 + y^2 - z^2 = 2x(y + z) \Leftrightarrow$

$\frac{\partial}{\partial x}\left(x^2 + y^2 - z^2\right) = \frac{\partial}{\partial x}[2x(y + z)] \Leftrightarrow$

$2x - 2z\frac{\partial z}{\partial x} = 2(y + z) + 2x\frac{\partial z}{\partial x} \Leftrightarrow$

$2(x + z)\frac{\partial z}{\partial x} = 2(x - y - z)$, logo $\frac{\partial z}{\partial x} = \frac{x - y - z}{x + z}.$

$\frac{\partial}{\partial y}\left(x^2 + y^2 - z^2\right) = \frac{\partial}{\partial y}[2x(y + z)] \Leftrightarrow$

$2y - 2z\frac{\partial z}{\partial y} = 2x\left(1 + \frac{\partial z}{\partial y}\right) \Leftrightarrow$

$2(x + z)\frac{\partial z}{\partial y} = 2(y - x)$, logo $\frac{\partial z}{\partial y} = \frac{y - x}{x + z}.$

45. $xy^2z^3 + x^3y^2z = x + y + z \Rightarrow$

$\frac{\partial}{\partial x}\left(xy^2z^3 + x^3y^2z\right) = \frac{\partial}{\partial x}(x + y + z) \Leftrightarrow$

$y^2z^3 + 3xy^2z^2\frac{\partial z}{\partial x} + 3x^2y^2z + x^3y^2\frac{\partial z}{\partial x} = 1 + \frac{\partial z}{\partial x}$, logo

$\left(3xy^2z^2 + x^3y^2 - 1\right)\frac{\partial z}{\partial x} = 1 - y^2z^3 - 3x^2y^2z$ e

$\frac{\partial z}{\partial x} = \frac{1 - y^2z^3 - 3x^2y^2z}{3xy^2z^2 + x^3y^2 - 1}.$

$\frac{\partial}{\partial y}\left(xy^2z^3 + x^3y^2z\right) = \frac{\partial}{\partial y}(x + y + z) \Leftrightarrow$

$2xyz^3 + 3xy^2z^2\frac{\partial z}{\partial y} + 2x^3yz + x^3y^2\frac{\partial z}{\partial y} = 1 + \frac{\partial z}{\partial y}$, logo

$\left(3xy^2z^2 + x^3y^2 - 1\right)\frac{\partial z}{\partial y} = 1 - 2xyz^3 - 2x^3yz$ e

$\frac{\partial z}{\partial y} = \frac{1 - 2xyz^3 - 2x^3yz}{3xy^2z^2 + x^3y^2 - 1}.$

46. $z = f(ax + by)$. Seja $u = ax + by$.

Então, $\frac{\partial u}{\partial x} = a$ e $\frac{\partial u}{\partial y} = b$. Logo,

$\frac{\partial z}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} = a\frac{df}{d(ax + by)} = af'(ax + by)$ e

$\frac{\partial z}{\partial y} = b\frac{df}{d(ax + by)} = bf'(ax + by).$

47. $f(x, y) = x^2y + x\sqrt{y} \Rightarrow f_x = 2xy + \sqrt{y},$

$f_y = x^2 + \frac{x}{2\sqrt{y}}$. Então $f_{xx} = 2y, f_{xy} = 2x + \frac{1}{2\sqrt{y}},$

$f_{yx} = 2x + \frac{1}{2\sqrt{y}}$ e $f_{yy} = -\frac{x}{4y^{3/2}}.$

48. $f(x, y) = \text{sen}(x + y) + \cos(x - y) \Rightarrow$

$f_x = \cos(x + y) - \text{sen}(x - y),$

$f_y = \cos(x + y) + \text{sen}(x - y)$. Então,

$f_{xx} = -\text{sen}(x + y) - \cos(x - y),$

$f_{xy} = -\text{sen}(x + y) + \cos(x - y),$

$f_{yx} = -\text{sen}(x + y) + \cos(x - y)$ e

$f_{yy} = -\text{sen}(x + y) - \cos(x - y).$

49. $z = \left(x^2+y^2\right)^{3/2} \quad\Rightarrow$

$z_x = \frac{3}{2}\left(x^2+y^2\right)^{1/2}(2x) = 3x\left(x^2+y^2\right)^{1/2}$ e

$z_y = 3y\left(x^2+y^2\right)^{1/2}$. Então,

$z_{xx} = 3\left(x^2+y^2\right)^{1/2} + 3x\left(x^2+y^2\right)^{-1/2}\left(\frac{1}{2}\right)(2x)$

$= \dfrac{3\left(x^2+y^2\right)+3x^2}{\sqrt{x^2+y^2}} = \dfrac{3\left(2x^2+y^2\right)}{\sqrt{x^2+y^2}}$ e

$z_{xy} = 3x\left(\frac{1}{2}\right)\left(x^2+y^2\right)^{-1/2}(2y) = \dfrac{3xy}{\sqrt{x^2+y^2}}$. Por

simetria $z_{yx} = \dfrac{3xy}{\sqrt{x^2+y^2}}$ e $z_{yy} = \dfrac{3\left(x^2+2y^2\right)}{\sqrt{x^2+y^2}}$.

50. $z = \cos^2(5x+2y) \quad\Rightarrow$

$z_x = [2\cos(5x+2y)][-\operatorname{sen}(5x+2y)](5)$

$= -10\cos(5x+2y)\operatorname{sen}(5x+2y)$ e

$z_y = [2\cos(5x+2y)][-\operatorname{sen}(5x+2y)](2)$

$= -4\cos(5x+2y)\operatorname{sen}(5x+2y)$

Então

$z_{xx} = (10)(5)\operatorname{sen}^2(5x+2y) + (-10)(5)\cos^2(5x+2y)$

$= 50\left[\operatorname{sen}^2(5x+2y) - \cos^2(5x+2y)\right]$,

$z_{xy} = (10)(2)\operatorname{sen}^2(5x+2y) + (-10)(2)\cos^2(5x+2y)$

$= 20\left[\operatorname{sen}^2(5x+2y) - \cos^2(5x+2y)\right]$,

$z_{yx} = -(-4)(5)\operatorname{sen}^2(5x+2y) + (-4)(5)\cos^2(5x+2y)$

$= 20\left[\operatorname{sen}^2(5x+2y) - \cos^2(5x+2y)\right]$,

$z_{yy} = -(-4)(2)\operatorname{sen}^2(5x+2y) + (-4)(2)\cos^2(5x+2y)$

$= 8\left[\operatorname{sen}^2(5x+2y) - \cos^2(5x+2y)\right]$

51. $z = t\operatorname{sen}^{-1}\sqrt{x} \quad\Rightarrow$

$z_x = t\dfrac{1}{\sqrt{1-(\sqrt{x})^2}}\left(\frac{1}{2}\right)x^{-1/2} = \dfrac{t}{2\sqrt{x-x^2}}$,

$z_t = \operatorname{sen}^{-1}\sqrt{x}$. Então

$z_{xx} = \frac{1}{2}t\left(-\frac{1}{2}\right)\left(x-x^2\right)^{-3/2}(1-2x) = \dfrac{t(2x-1)}{4\left(x-x^2\right)^{3/2}}$,

$z_{xt} = \dfrac{1}{2\sqrt{x-x^2}}$,

$z_{tx} = \dfrac{1}{\sqrt{1-(\sqrt{x})^2}}\left(\frac{1}{2}x^{-1/2}\right) = \dfrac{1}{2\sqrt{x-x^2}}$, e $z_{tt} = 0$.

52. $z = x^{\ln t} \quad\Rightarrow\quad z_x = (\ln t)\,x^{(\ln t)-1}$, $\ln z = (\ln t)(\ln x)$, logo

$z_t = \left(x^{\ln t}\right)\left(\dfrac{1}{t}\right)\ln x = x^{\ln t}\dfrac{\ln x}{t}$. Então

$z_{xx} = (\ln t)[(\ln t)-1]x^{(\ln t)-2}$,

$\ln z_x = \ln(\ln t) + [(\ln t)-1]\ln x$, logo

$z_{xt} = z_x\left[\dfrac{1}{\ln t}\left(\dfrac{1}{t}\right) + \dfrac{1}{t}\ln x\right]$

$= (\ln t)\,x^{(\ln t)-1}\dfrac{1+(\ln t)(\ln x)}{t\ln t}$

$= x^{(\ln t)-1}\dfrac{1+\ln t\ln x}{t}$

Além disso,

$z_{tx} = \dfrac{(\ln t)\,x^{(\ln t)-1}\ln x + (1/x)\,x^{\ln t}}{t}$

$= x^{(\ln t)-1}\dfrac{1+\ln t\ln x}{t}$ e

$z_{tt} = \left[\dfrac{\partial}{\partial t}\left(x^{\ln t}\right)\right]\left(\dfrac{\ln t}{t}\right) + x^{\ln t}\left[-(\ln x)\,t^{-2}\right]$

$= x^{\ln t}\ln x\dfrac{(\ln x)-1}{t^2}$

53. $u = x^5y^4 - 3x^2y^3 + 2x^2 \quad\Rightarrow\quad u_x = 5x^4y^4 - 6xy^3 + 4x$,

$u_{xy} = 20x^4y^3 - 18xy^2$ e $u_y = 4x^5y^3 - 9x^2y^2$,

$u_{yx} = 20x^4y^3 - 18xy^2$. Logo, $u_{xy} = u_{yx}$.

54. $u = \operatorname{sen}^2 x\cos y \quad\Rightarrow\quad u_x = 2\operatorname{sen} x\cos x\cos y$,

$u_{xy} = -2\operatorname{sen} x\cos x\operatorname{sen} y$ e $u_y = -\operatorname{sen}^2 x\operatorname{sen} y$,

$u_{yx} = -2\operatorname{sen} x\cos x\operatorname{sen} y$. Logo, $u_{xy} = u_{yx}$.

55. $u = \operatorname{sen}^{-1}\left(xy^2\right) \quad\Rightarrow$

$u_x = \dfrac{1}{\sqrt{1-(xy^2)^2}}\left(y^2\right) = y^2\sqrt{1-x^2y^4}$,

$u_{xy} = 2y\left(1-x^2y^4\right)^{-1/2} + y^2\left(-\frac{1}{2}\right)\left(1-x^2y^4\right)^{-3/2}\left(-4x^2y^3\right)$

$= \dfrac{2y\left(1-x^2y^4\right)+2x^2y^5}{\left(1-x^2y^4\right)^{3/2}} = \dfrac{2y}{\left(1-x^2y^4\right)^{3/2}}$ e

$u_y = \dfrac{1}{\sqrt{1-(xy^2)^2}}(2xy) = \dfrac{2xy}{\sqrt{1-x^2y^4}}$,

$u_{yx} = \dfrac{2y\sqrt{1-x^2y^4} - 2xy\left(\frac{1}{2}\right)\left(1-x^2y^4\right)^{-1/2}\left(-2xy^4\right)}{1-x^2y^4}$

$= \dfrac{2y-2x^2y^5+2x^2y^5}{\left(1-x^2y^4\right)^{3/2}} = \dfrac{2y}{\left(1-x^2y^4\right)^{3/2}}$

Logo, $u_{xy} = u_{yx}$.

56. $u = x^2y^3z^4 \quad\Rightarrow\quad u_x = 2xy^3z^4$, $u_{xy} = 6xy^2z^4$,

$u_{xz} = 8xy^3z^3$; $u_y = 3x^2y^2z^4$, $u_{yx} = 6xy^2z^4$,

$u_{yz} = 12x^2y^2z^3$; $u_z = 4x^2y^3z^3$, $u_{zx} = 8xy^3z^3$,

$u_{zy} = 12x^2y^2z^3$. Então $u_{xy} = u_{yx}$, $u_{xz} = u_{zy}$, e

$u_{yz} = u_{zy}$.

57. $f(x, y) = x^2y^3 - 2x^4y \quad\Rightarrow\quad f_x = 2xy^3 - 8x^3y$,

$f_{xx} = 2y^3 - 24x^2y$, $f_{xxx} = -48xy$

58. $f(x, y) = e^{xy^2} \quad\Rightarrow\quad f_x = y^2e^{xy^2}$, $f_{xx} = y^4e^{xy^2}$,

$f_{xxy} = 4y^3e^{xy^2} + 2xy^5e^{xy^2} = 2y^3e^{xy^2}\left(2+xy^2\right)$

59. $f(x, y, z) = x^5 + x^4y^4z^3 + yz^2 \quad\Rightarrow$

$f_x = 5x^4 + 4x^3y^4z^3$, $f_{xy} = 16x^3y^3z^3$, e

$f_{xyz} = 48x^3y^3z^2$

60. $f(x, y, z) = e^{xyz} \quad \Rightarrow \quad f_y = xze^{xyz}$,

$f_{yz} = xe^{xyz} + xz\,(xy)\,e^{xyz} = xe^{xyz}\,(1 + yxz)$, e

$$f_{yzy} = x\,(xz)\,e^{xyz}\,(1 + xyz) + xe^{xyz}\,(xz) = x^2 z\,(2 + xyz)\,e^{xyz}$$

61. $z = x \operatorname{sen} y \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = \operatorname{sen} y,\ \dfrac{\partial^2 z}{\partial y\,\partial x} = \cos y$, e

$$\frac{\partial^3 z}{\partial y^2\,\partial x} = -\operatorname{sen} y.$$

62.

$z = \ln \operatorname{sen}(x - y) \quad \Rightarrow$

$$\frac{\partial z}{\partial x} = \frac{1}{\operatorname{sen}(x - y)} \cos(x - y) = \operatorname{cotg}(x - y),$$

$\dfrac{\partial^2 z}{\partial x^2} = -\operatorname{cossec}^2(x - y)$ e

$$\frac{\partial^3 z}{\partial y\,\partial x^2} = -2 \operatorname{cossec}(x - y)\,[-\operatorname{cossec}(x - y) \operatorname{cotg}(x - y)\,(-1)] = -2 \operatorname{cossec}^2(x - y) \operatorname{cotg}(x - y)$$

63. $u = \ln\left(x + 2y^2 + 3z^3\right) \quad \Rightarrow$

$$\frac{\partial u}{\partial z} = \frac{1}{x + 2y^2 + 3z^3}\left(9z^2\right) = \frac{9z^2}{x + 2y^2 + 3z^3},$$

$$\frac{\partial^2 u}{\partial y\,\partial z} = -9z^2\left(x + 2y^2 + 3z^3\right)^{-2}(4y) = -\frac{36yz^2}{(x + 2y^2 + 3z^3)^2},$$

e $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \dfrac{72yz^2}{(x + 2y^2 + 3z^3)^3}$.

64. Seja $w = x + y$. Então $\dfrac{\partial w}{\partial x} = 1 = \dfrac{\partial w}{\partial y}$, e, pela Regra da Cadeia,

$$u_x = f(w) + x\frac{df}{dw}\frac{\partial w}{\partial x} + y\frac{dg}{dw}\frac{\partial w}{\partial x} = f(w) + xf'(w) + yg'(w),$$

$$u_{xx} = \frac{df}{dw}\frac{\partial w}{\partial x} + f'(w) + x\frac{d[f'(w)]}{dw}\frac{\partial w}{\partial x} + y\frac{d[g'(w)]}{dw}\frac{\partial w}{\partial x} = 2f'(w) + xf''(w) + yg''(w),$$

e

$$u_{xy} = \frac{df}{dw}\frac{\partial w}{\partial y} + x\frac{d[f'(w)]}{dw}\frac{\partial w}{\partial y} + g'(w) + y\frac{d[g'(w)]}{dw}\frac{\partial w}{\partial y} = f'(w) + xf''(w) + g'(w) + yg''(w)$$

Analogamente, $u_y = xf'(w) + g(w) + yg'(w)$ e $u_{yy} = xf''(w) + 2g'(w) + yg''(w)$. Logo,

$$u_{xx} - 2u_{xy} + u_{yy} = 2f'(w) + xf''(w) + yg''(w) - 2f'(w) - 2xf''(w) - 2g'(w) - 2yg''(w) + xf''(w) + 2g'(w) + yg''(w) = 0$$

65. $f(x_1, \ldots, x_n) = \left(x_1^2 + \cdots + x_n^2\right)^{(2-n)/2} \quad \Rightarrow$

$$\frac{\partial f}{\partial x_i} = \left(1 - \frac{n}{2}\right) 2x_i\left(x_1^2 + \cdots + x_n^2\right)^{-n/2},\ 1 \le i \le n \quad \Rightarrow$$

$$\frac{\partial^2 f}{\partial x_i^2} = 2\left(1 - \frac{n}{2}\right)\left(x_1^2 + \cdots + x_n^2\right)^{-n/2} - (2n)\left(1 - \frac{n}{2}\right)\left(x_i^2\right)\left(x_1^2 + \cdots + x_n^2\right)^{-(2+n)/2},$$

$1 \le i \le n$. Portanto,

$$\begin{aligned}
\frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}
&= \sum_{i=1}^{n}\Big[(2 - n)\left(x_1^2 + \cdots + x_n^2\right)^{-n/2} - n(2 - n)\left(x_i^2\right)\left(x_1^2 + \cdots + x_n^2\right)^{-(2+n)/2}\Big] \\
&= n(2 - n)\left(x_1^2 + \cdots + x_n^2\right)^{-n/2} - n(2 - n)\left(x_1^2 + \cdots + x_n^2\right)\left(x_1^2 + \cdots + x_n^2\right)^{-(2+n)/2} \\
&= n(2 - n)\left(x_1^2 + \cdots + x_n^2\right)^{-n/2} - n(2 - n)\left(x_1^2 + \cdots + x_n^2\right)^{-n/2} \\
&= 0
\end{aligned}$$