11.9 SOLUÇÕES

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"R" representa "raio da convergência" e "I" representa "intervalo da convergência" nessa seção.

1. $f(x) = \dfrac{x}{1-x} = x\left(\dfrac{1}{1-x}\right) = x\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}$

$= \sum_{n=1}^{\infty} x^n$

com $R = 1$ e $I = (-1, 1)$.

2. $f(x) = \dfrac{1}{4+x^2} = \dfrac{1}{4}\left(\dfrac{1}{1+x^2/4}\right)$

$= \dfrac{1}{4}\left(\dfrac{1}{1-(-x^2/4)}\right)$

$= \dfrac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{x^2}{4}\right)^n = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{4^{n+1}}$

com $\left|\dfrac{x^2}{4}\right| < 1 \Leftrightarrow x^2 < 4 \Leftrightarrow |x| < 2$, logo $R = 2$ e $I = (-2, 2)$.

3. $f(x) = \dfrac{1+x^2}{1-x^2} = 1 + \dfrac{2x^2}{1-x^2} = 1 + 2x^2\sum_{n=0}^{\infty}(x^2)^n$

$= 1 + \sum_{n=0}^{\infty} 2x^{2n+2} = 1 + \sum_{n=1}^{\infty} 2x^{2n}$

com $|x^2| < 1 \Leftrightarrow |x| < 1$, logo $R = 1$ e $I = (-1, 1)$.

4. $f(x) = \dfrac{1}{1+4x^2} = \dfrac{1}{1-(-4x^2)}$

$= \sum_{n=0}^{\infty}(-1)^n(4x^2)^n = \sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}$

com $|4x^2| < 1$, logo $x^2 < \frac{1}{4} \Leftrightarrow |x| < \frac{1}{2}$, e logo $R = \frac{1}{2}$ e $I = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

5. $f(x) = \dfrac{1}{x^4+16} = \dfrac{1}{16}\left[\dfrac{1}{1+(x/2)^4}\right]$

$= \dfrac{1}{16}\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{x}{2}\right)^{4n} = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{4n}}{2^{4n+4}}$

para $\left|\dfrac{x}{2}\right| < 1 \Leftrightarrow |x| < 2$, logo, $R = 2$ e $I = (-2, 2)$.

6. $f(x) = \dfrac{x}{x-3} = 1 + \dfrac{3}{x-3} = 1 - \dfrac{1}{1-x/3}$

$= 1 - \sum_{n=0}^{\infty}\left(\dfrac{x}{3}\right)^n = -\sum_{n=1}^{\infty}\left(\dfrac{x}{3}\right)^n$

Para convergência, $\dfrac{|x|}{3} < 1 \Leftrightarrow |x| < 3$, logo $R = 3$ e $I = (-3, 3)$.

Outro Método:

$f(x) = \dfrac{x}{x-3} = -\dfrac{x}{3(1-x/3)} = -\dfrac{x}{3}\sum_{n=0}^{\infty}\left(\dfrac{x}{3}\right)^n$

$= -\sum_{n=0}^{\infty}\dfrac{x^{n+1}}{3^{n+1}} = -\sum_{n=1}^{\infty}\dfrac{x^n}{3^n}$

7. $f(x) = \dfrac{2}{3x+4} = \dfrac{1}{2}\left(\dfrac{1}{1+3x/4}\right)$

$= \dfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{3x}{4}\right)^n = \sum_{n=0}^{\infty}\dfrac{(-1)^n 3^n x^n}{2^{2n+1}}$

$\left|\dfrac{3x}{4}\right| < 1$, logo $R = \frac{4}{3}$ e $I = \left(-\frac{4}{3}, \frac{4}{3}\right)$.

8. $\dfrac{3x-2}{2x^2-3x+1} = \dfrac{3x-2}{(2x-1)(x-1)} = \dfrac{A}{2x-1} + \dfrac{B}{x-1}$

$\Leftrightarrow A + 2B = 3$ e $-A - B = -2 \Leftrightarrow A = B = 1$, logo

$f(x) = \dfrac{3x-2}{2x^2-3x+1} = \dfrac{1}{2x-1} + \dfrac{1}{x-1}$

$= -\sum_{n=0}^{\infty}(2x)^n - \sum_{n=0}^{\infty}x^n = -\sum_{n=0}^{\infty}(2^n+1)x^n$

com $R = \frac{1}{2}$. Em $x = \pm\frac{1}{2}$, a série diverge pelo Teste para Divergência, então $I = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

9. $\dfrac{x}{x^2-3x+2} = \dfrac{x}{(x-2)(x-1)} = \dfrac{A}{x-2} + \dfrac{B}{x-1} \Leftrightarrow$

$A + B = 1$ e $-A - 2B = 0 \Leftrightarrow A = 2, B = -1$, logo

$f(x) = \dfrac{x}{(x-2)(x-1)} = \dfrac{2}{x-2} - \dfrac{1}{x-1}$

$= -\dfrac{1}{1-x/2} + \dfrac{1}{1-x} = -\sum_{n=0}^{\infty}\left(\dfrac{x}{2}\right)^n + \sum_{n=0}^{\infty}x^n$

$= \sum_{n=0}^{\infty}(1-2^{-n})x^n$

com $R = 1$. Em $x = \pm 1$, a série diverge pelo Teste para Divergência, então $I = (-1, 1)$.

10. $f(x) = \operatorname{tg}^{-1} 2x = 2\displaystyle\int\dfrac{dx}{1+4x^2}$

$= 2\displaystyle\int\sum_{n=0}^{\infty}(-1)^n(4x^2)^n\,dx$

$= 2\displaystyle\int\sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}\,dx$

$= C + 2\sum_{n=0}^{\infty}\dfrac{(-1)^n 4^n x^{2n+1}}{2n+1}$

$= \sum_{n=0}^{\infty}\dfrac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}$

para $|4x^2| < 1$, logo $|x| < \frac{1}{2}$ e $R = \frac{1}{2}$.

11. $f(x) = \ln(1+x) - \ln(1-x) = \int \frac{dx}{1+x} + \int \frac{dx}{1-x}$

$$= \int \left[\sum_{n=0}^{\infty} (-1)^n x^n + \sum_{n=0}^{\infty} x^n \right] dx$$

$$= \int \sum_{n=0}^{\infty} 2x^{2n}\, dx = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1} + C$$

Mas $f(0) = \ln 1 - \ln 1 = 0$, logo $C = 0$ e temos

$$f(x) = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1} \text{ com } R = 1.$$

12. $\int \frac{dx}{1+x^4} = \int \sum_{n=0}^{\infty} (-1)^n x^{4n}\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4n+1}$

com $R = 1$.

13. $\frac{1}{1+x^5} = \sum_{n=0}^{\infty} (-1)^n x^{5n} \quad \Rightarrow$

$$\frac{x}{1+x^5} = \sum_{n=0}^{\infty} (-1)^n x^{5n+1} \quad \Rightarrow$$

$$\int \frac{x}{1+x^5}\, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+2}}{5n+2} \text{ com } R = 1.$$

14. Pelo Exemplo 7, $\operatorname{arctg} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$, logo

$$\int \frac{\operatorname{arctg} x}{x}\, dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}\, dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2}$$

com $R = 1$.

15. Usamos a representação

$$\int \frac{dx}{1+x^4} = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{4n+1}, \text{ do}$$

Problema 12 com $C = 0$. Logo,

$$\int_0^{0,2} \frac{dx}{1+x^4} = \left[x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \cdots \right]_0^{0,2}$$

$$= 0{,}2 - \frac{0{,}2^5}{5} + \frac{0{,}2^9}{9} - \frac{0{,}2^{13}}{13} + \cdots$$

Uma vez que a série está alternando, o erro na aproximação de n-ésima ordem é menor que o primeiro termo negligenciado, pelo Teorema da Estimativa da Série Alternada. Se usarmos somente os dois primeiros termos da série, então o erro é no máximo $0{,}2^9/9 \approx 5{,}7 \times 10^{-8}$. Então, até a sexta casa decimal, $\int_0^{0,2} \frac{dx}{1+x^4} \approx 0{,}2 - \frac{0{,}2^5}{5} = 0{,}199936$.

16. Usamos a representação

$$\int \operatorname{tg}^{-1}(x^2)\, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)}$$

do Exercício 26 no texto com $C = 0$:

$$\int_0^{1/2} \operatorname{tg}^{-1}(x^2)\, dx$$

$$= \left[\frac{x^3}{3} - \frac{x^7}{21} + \frac{x^{11}}{55} - \frac{x^{15}}{105} + \frac{x^{19}}{171} - \cdots \right]_0^{1/2}$$

$$= \frac{0{,}5^3}{3} - \frac{0{,}5^7}{21} + \frac{0{,}5^{11}}{55} - \frac{0{,}5^{15}}{105} + \frac{0{,}5^{19}}{171} - \cdots$$

A série é alternanda, então usando somente os primeiros quatro termos da série, o erro é no máximo $0{,}5^{19}/171 \approx 1{,}1 \times 10^{-8}$. Então, até a sexta casa decimal, $\int_0^{1/2} \operatorname{tg}^{-1}(x^2)\, dx$

$$\approx \tfrac{1}{3}(0{,}5)^3 - \tfrac{1}{21}(0{,}5)^7 + \tfrac{1}{55}(0{,}5)^{11} - \tfrac{1}{105}(0{,}5)^{15}$$

$$\approx 0{,}041303$$