PMR5215 – Otimização Aplicada ao Projeto de Sistemas Mecânicos
6th Assignment
Numerical Constrained Optimization: Indirect Methods

Due date: 15/05/2023

1 Introduction

Consider the following constrained optimization problem:

$$\begin{array}{cc} \underset{x,y}{\text{Min}} & F(x,y) = x^2 + y^2 - 8x \\ \text{such that} & G(x,y) = y - x + 2 \geq 0 \end{array} \tag{1}$$

a) Write the corresponding Problem's Lagrangian, as a function of the (x, y) variables and the λ multiplier associated to the constraint G.

b) Show that the point $x = 3, y = 1, \lambda = 2$ is a problem minimum.

2 Extern Penalization

A constrained optimization problem such as:

$$\begin{array}{cc} \underset{X}{\text{Min}} & F(X) \\ \text{such that} & G_i(X) \geq 0 \end{array}$$

can be approached by an *unconstrained* optimization problem shaped given by:

$$\underset{X}{\text{Min}}\, \Phi(X, r) = F(X) + r \sum_i \langle -G_i(X) \rangle^2$$

Where

$$\langle x \rangle = \begin{cases} x & \text{if}\, x \geq 0 \\ 0 & \text{if}\, x < 0 \end{cases}$$

and r is the *penalization weigh.*

a) Write the penalized function $\Phi(x, y, r)$ corresponding to the problem in (1).

b) By applying the *Fastest Descent* method (see 5th assignment), solve the problem of minimizing over (x, y) the function $\Phi(x, y, r)$ with $r = 16$. Use as starting point $x = 0, y = 0$. Your stopping criteria for the unconstrained optimization is the condition:

$$|\Phi(x_i, y_i, r) - \Phi(x_{i-1}, y_{i-1}, r)| < \epsilon$$

with $\epsilon = 0.005$. Fill a table with results from each algorithm step until the stopping criteria is satisfied. Include at each point the objective function value, its gradient and the linear search value α. Use the table 1 as model.

Suggestion: Of course, one may use any appropriate line search method. Notice thought that for this function in particular, if α is a stationary point of $\Phi(x + \alpha s_x, y + \alpha s_y, r)$ then let

$$\hat{\alpha} = \frac{s_x(r(-x + y + 2) - x + 4) + s_y(r(x - 2) - (r + 1)y)}{-2rs_xs_y + (r + 1)s_x^2 + (r + 1)s_y^2}$$

and

$$\tilde{\alpha} = -\frac{(x - 4)s_x + ys_y}{s_x^2 + s_y^2}$$

Then,

$$\alpha = \begin{cases} \hat{\alpha} & \text{if } G(x + \hat{\alpha}s_x, y + \hat{\alpha}s_y) < 0 \\ \tilde{\alpha} & \text{if } G(x + \hat{\alpha}s_x, y + \hat{\alpha}s_y) \geq 0 \end{cases} \tag{2}$$

c) External Penalization produces notoriously ill-conditioned functions for numerical penalization. This problem is mitigated if values for r are slowly increased. Repeat the task for item b) with successively increasing values of $r = \{1/4, 1, 4, 16, 64, \ldots\}$, multiplying by 4 the value of r after each iteration.

The starting point at each step must be the *final* point of the preceding step. The starting point for the first step must be $x = 0, y = 0$. You must fill multiple tables with intermediary optimization results. Write on each table the value for r. You must keep optimizing for larger values of r until the produced (x, y) value satisfies the condition:

$$\left| \frac{F(x, y) - \Phi(x, y, r)}{F(x, y)} \right| < 0,2\% \tag{3}$$

Attention: This is the *outer* stopping condition. For each inner optimization step you should keep using the stopping condition from item 2b.

3 Augmented Lagrangian

The augmented Lagrangian method combine the Lagrangian function with external penalization.

A constrained optimization problem:

$$
\begin{array}{ll}
\underset{X}{\text{Min}} & F(X) \\
\text{such that} & G_i(X) \geq 0
\end{array}
$$

may be approached by an *unconstrained* optimization problem:

$$\underset{X}{\text{Min}}\, \Phi(X, \lambda, r) = F(X) + r \sum_i \left\langle \frac{\lambda_i}{2r} - G_i(X) \right\rangle^2 \tag{4}$$

where r is the penalization weight and λ_i is an estimate of the multiplier related to constraint G_i.

The estimates λ_i are improved at each optimization result X^* for problem (4).

$$\lambda_i{}^* = \max(\lambda_i - 2rG_i(X^*), 0) \tag{5}$$

Where $\lambda_i{}^*$ is the improved estimate for multiplier λ_i.

a) Repeat item 2c now with the Augmented Lagrangian Method. Again, use increasing values of $r = \{1/4, 1, 4, 16, 64, \ldots\}$. Keep augmenting r until the same stopping criteria in (3) is satisfied, using there the external penalization function Φ (use for this stopping criteria $\lambda = 0$). Again, the initial point must be $x = 0, y = 0$. Your first estimate for the multiplier associated with G must be $\lambda = 0$. At the end of each optimization by *Steepest Descent*, your estimate for λ must be improved according to (5). Fill tables with each optimization intermediary results. Write at each table the values for r and λ.

   Suggestion: Again, you may use any suitable line search method. Regardless, if α is minumum of $\Phi(\{x + \alpha s_x, y + \alpha s_y\}, \lambda, r)$ and

$$\hat{\alpha} = \frac{s_y(\lambda + 2r(x - y - 2) - 2y) - s_x(\lambda + 2r(x - y - 2) + 2x - 8)}{-4rs_xs_y + 2(r+1)s_x^2 + 2(r+1)s_y^2}$$

$$\tilde{\alpha} = -\frac{(x-4)s_x + ys_y}{s_x^2 + s_y^2}$$

   then,

$$\alpha = \begin{cases} \hat{\alpha} & \text{if}\, G(x + \hat{\alpha}s_x, y + \hat{\alpha}s_y) < \frac{\lambda}{2r} \\ \tilde{\alpha} & \text{if}\, G(x + \hat{\alpha}s_x, y + \hat{\alpha}s_y) \geq \frac{\lambda}{2r} \end{cases} \tag{6}$$

b) Compare the results of items 1b, 2b, 2c and 3a in *total* number of iterations, final values for x, y, objective function values and constraint values.

Table 1: Model for “*Steepest Descent*” intermediary results

Iter.	x	y	$F(x,y)$	$G(x,y)$	$\Phi(x,y)$	$\mathrm{d}\Phi/\mathrm{d}x$	$\mathrm{d}\Phi/\mathrm{d}y$	α
0	0	0						
1								
2								
3								
5								
6								
7								
...								