PMR-5215 – Otimização Aplicada ao Projeto de Sistemas Mecânicos

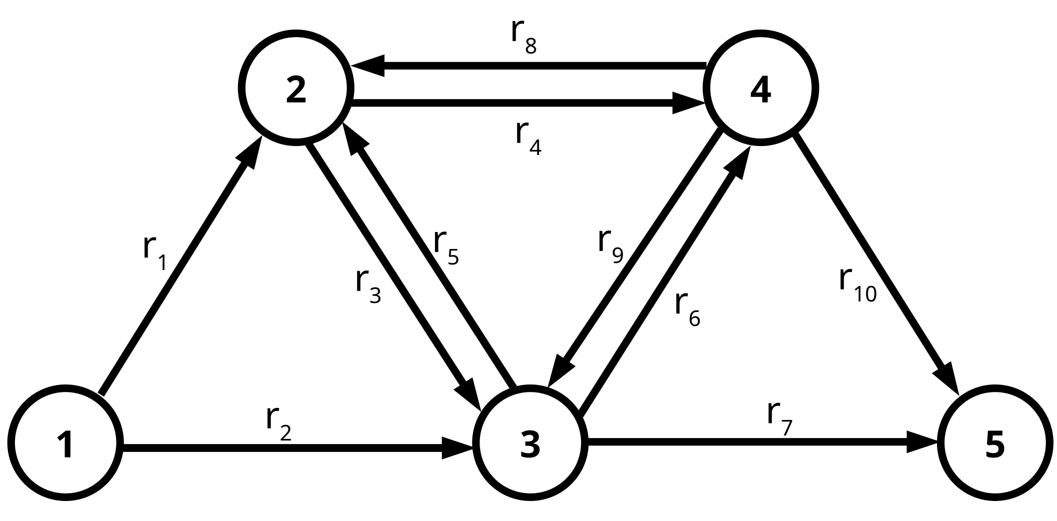
4th Assigment
(groups of at most 2 students)
Submit date: **24/04/23**

1- Consider the following optimization problems:

$$\begin{array}{ll} \text{Max} & 2x_1+3x_2 \\ \text{such that} & 4x1+x2\le 4 \\ & x_2\le 2 \\ & x_1\ge 0, x_2\ge 0 \end{array} \qquad \begin{array}{ll} \text{Min} & x_1+6x_2 \\ \text{such that} & 4x1+x2\ge 2 \\ & x_1+6x_2\ge 2 \\ & x1-2x_2\le 0 \\ & x_1\ge 0, x_2\ge 0 \end{array} \qquad \begin{array}{ll} \text{Max} & x_1+x_2 \\ \text{such that} & x_1+3x_2\le 2 \\ & 3x_1-x_2\ge 7 \\ & x_1\ge 0, x_2\ge 0 \end{array}$$

For each one draw the viable domain and the objective function's level curves.Para cada um deles desenhe o domínio viável e as curvas de nível da função objetivo. Point the problem's solutions.

2- Consider the following graph:



Let the problem of finding the path between vertex 1 and 5 with the minimal cost. Each edge's cost is given by the table:

r	1	2	3	4	5	6	7	8	9	10
Cost	3	1	1	1	1	3	4	1	2	1

This problem can be solved by linear programming. Indeed, consider that for each route can be defined a vector ***r***={r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9} where the *i*-th component contains the number of times that the edge i is crossed. For instance, to the route 1→ 3→ 5 corresponds the vector {0, 1, 0, 0, 0, 0, 1, 0, 0, 0}, while to the route 1→ 2→ 4→ 5 corresponds the vector {1, 0, 0, 1, 0, 0, 0, 0, 0, 1} (verify!).

a) Write the a route's total cost from the values r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9. Use the edge's cost from previous table.

b) Notice that for every non-extremal vertex (vertexes 2, 3 and 4), the number of times that one enters it must be greater or equal to the number of times one leaves it. For the destination vertex, one must enters it at least once (notice there are no exit edges) and for the origin vertex (1) one must leave it at least once (similarly, there are no entry edges). For each vertex, write the inequalities using variables r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9 corresponding to those constraints. Note: Those constraints could be of strict *equality* but as the problem attemtps to minimize the total cost, inequalities are enough here.

c) Add to the inequalities from item (b) the non-negativity constraints $r_i \geq 0$ and build the linear problem of minimizing the total route cost. Solve it using a numerical software (that's a problem with 10 variables and 15 constraints!). Reproduce the graph and point the obtained optimal route. What is the route's total cost?

3. Dijkstra's algorithm

Dijkstra's algorithm is a method for optimizing routes in graphs. It consists of atributing to each vertex the **minimum** cost of reaching it from the origin vertex. As such, the destiny vertex should receive the total desired route cost. Traditionally, those costs are obtained from dynamic programming.

They may be obtained by linar programming though.

Indeed, consider , the vector of vertex assigned values $\boldsymbol{v} = \{v_1, v_2, v_3, v_4, v_5\}$. Notice that if there is an edge from vertex i to j with cost r, then it follows that $v_j - v_i \leq r$. Indeed, if v_i is the *minimal* cost from origin to vertex v_i and v_j is the *minimal* cost from origin to vertex v_j then $v_j \leq v_i + r$ as one can always travels from the origin to i and then to j with aditional cost r.

a) Write for each edge the corresponding constraints imposed to the values v_i of its vertexes. Use the values from the table from question 2.

b) The values from Dijkstra's algorithm are those that *maximize* the difference between the values assigned to the origin and destiny vertexes $v_5 - v_1$, subject to the constraints from item (a). Build the liner programming problem with that objective function, constraints from item (a) and non-negativity constraints $v_i \geq 0$. Solve the problem with a numerical software. Reproduce the graph with each vertex value. What is the difference $v_5 - v_1$?

c) Show that the problem from item (b) is the *dual* problem from item 2c.