*This article is about Newton's method for finding roots. For Newton's method for finding minima, see Newton's method in optimization.*

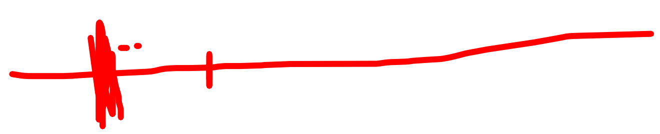
In numerical analysis, **Newton's method**, also known as the **Newton–Raphson method**, named after Isaac Newton and Joseph Raphson, is a root-finding algorithm which produces successively better approximations to the roots (or zeroes) of a real-valued function. The most basic version starts with a single-variable function $f$ defined for a real variable $x$, the function's derivative $f'$, and an initial guess $x_0$ for a root of $f$. If the function satisfies sufficient assumptions and the initial guess is close, then

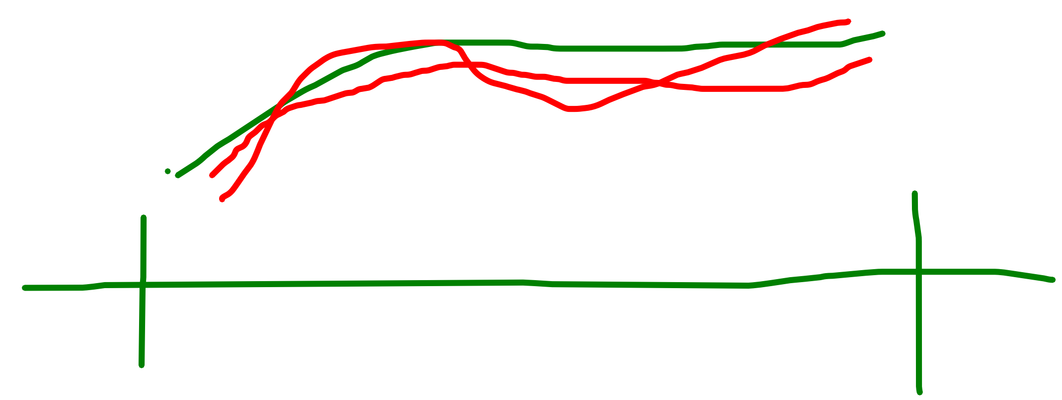
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

is a better approximation of the root than $x_0$. Geometrically, $(x_1, 0)$ is the intersection of the $x$-axis and the tangent of the graph of $f$ at $(x_0, f(x_0))$: that is, the improved guess is the unique root of the linear approximation at the initial point. The process is repeated as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until a sufficiently precise value is reached. This algorithm is first in the class of Householder's methods, succeeded by Halley's method. The method can also be extended to complex functions and to systems of equations.





## Square root [ edit ]

Consider the problem of finding the square root of a number *a*, that is to say the positive number $x$ such that $x^2 = a$. Newton's method is one of many methods of computing square roots. We can rephrase that as finding the zero of $f(x) = x^2 - a$. We have $f'(x) = 2x$.

For example, for finding the square root of 612 with an initial guess $x_0 = 10$, the sequence given by Newton's method is:

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} &= 10 - \frac{10^2 - 612}{2 \times 10} &= 35.6 \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} &= 35.6 - \frac{35.6^2 - 612}{2 \times 35.6} &= \underline{2}6.395\,505\,617\,978\ldots \\
x_3 &= \vdots &= \vdots &= \underline{24.7}90\,635\,492\,455\ldots \\
x_4 &= \vdots &= \vdots &= \underline{24.738\,6}88\,294\,075\ldots \\
x_5 &= \vdots &= \vdots &= \underline{24.738\,633\,753\,7}67\ldots
\end{aligned}$$

where the correct digits are underlined. With only a few iterations one can obtain a solution accurate to many decimal places.

Rearranging the formula as follows yields the Babylonian method of finding square roots:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left(2x_n - \left(x_n - \frac{a}{x_n}\right)\right) = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

i.e. the arithmetic mean of the guess, $x_n$ and $\frac{a}{x_n}$.

## Proof of quadratic convergence for Newton's iterative method [ edit ]

According to Taylor's theorem, any function $f(x)$ which has a continuous second derivative can be represented by an expansion about a point that is close to a root of $f(x)$. Suppose this root is $\alpha$. Then the expansion of $f(\alpha)$ about $x_n$ is:

$$f(\alpha) = f(x_n) + f'(x_n)(\alpha - x_n) + R_1 \qquad (1)$$

where the Lagrange form of the Taylor series expansion remainder is

$$R_1 = \frac{1}{2!} f''(\xi_n)(\alpha - x_n)^2 ,$$

where $\xi_n$ is in between $x_n$ and $\alpha$.

Since $\alpha$ is the root, (1) becomes:

$$0 = f(\alpha) = f(x_n) + f'(x_n)(\alpha - x_n) + \tfrac{1}{2} f''(\xi_n)(\alpha - x_n)^2 \qquad (2)$$

Dividing equation (2) by $f'(x_n)$ and rearranging gives

$$\frac{f(x_n)}{f'(x_n)} + (\alpha - x_n) = \frac{-f''(\xi_n)}{2f'(x_n)}(\alpha - x_n)^2 \qquad (3)$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$$\underbrace{\alpha - x_{n+1}}_{\varepsilon_{n+1}} = \frac{f(x_n)}{f'(x_n)} + (\alpha - x_n) = -\frac{f''(\xi_n)}{2f'(x_n)}(\underbrace{\alpha - x_n}_{\varepsilon_n})^2$$

That is,

$$\varepsilon_{n+1} = \frac{-f''(\xi_n)}{2f'(x_n)} \cdot {\varepsilon_n}^2 \,.$$

Taking the absolute value of both sides gives

$$\left|\varepsilon_{n+1}\right| = \frac{\left|f''(\xi_n)\right|}{2\left|f'(x_n)\right|} \cdot {\varepsilon_n}^2 \,.$$

Equation (6) shows that the rate of convergence is at least quadratic if the following conditions are satisfied:

1. $f'(x) \neq 0$; for all $x \in I$, where $I$ is the interval $[\alpha - r, \alpha + r]$ for some $r \geq |\alpha - x_0|$;
2. $f''(x)$ is continuous, for all $x \in I$;
3. $x_0$ is *sufficiently* close to the root $\alpha$.

The term *sufficiently* close in this context means the following:

a. Taylor approximation is accurate enough such that we can ignore higher order terms;

b. $\frac{1}{2}\left|\frac{f''(x_n)}{f'(x_n)}\right| < C\left|\frac{f''(\alpha)}{f'(\alpha)}\right|$, for some $C < \infty$;

c. $C\left|\frac{f''(\alpha)}{f'(\alpha)}\right|\varepsilon_n < 1$, for $n \in \mathbb{Z}$, $n \geq 0$ and $C$ satisfying condition b.

Finally, (6) can be expressed in the following way:

$$|\varepsilon_{n+1}| \leq M{\varepsilon_n}^2$$

where $M$ is the supremum of the variable coefficient of ${\varepsilon_n}^2$ on the interval $I$ defined in condition 1, that is:

$$M = \sup_{x \in I} \frac{1}{2} \left| \frac{f''(x)}{f'(x)} \right| .$$

The initial point $x_0$ has to be chosen such that conditions 1 to 3 are satisfied, where the third condition requires that $M|\varepsilon_0| < 1$.

**Questão 1.** *(1,0) A função* $f(x) = \frac{x^3+3x^2-x-6}{4x+8}$ *se* $x > -2$ *e* $f(x) = \frac{3}{8}x^2 + \frac{3}{4}x - \frac{1}{4}$ *se* $x \leq -2$ *é contínua em* $-2$*? É derivável em* $-2$*?*

$$\lim_{x \to -2^-} \frac{f(x) - f(-2)}{x+2}$$

**Questão 2.** *(3,0) Calcule três dos seguintes limites:*

*a)* $\lim_{x\longrightarrow+\infty} \sqrt[3]{2x^3 - x} - \sqrt[3]{2x^3 + 3x}$;

*b)* $\lim_{x\longrightarrow-1^-} \frac{x^{11}+2x^7+x^2+1}{x+1}$;

*c)* $\lim_{x\longrightarrow-\infty} \frac{11x^8+7x^5+3x^2+1}{3x^7+x^6-2}$;

*d)* $\lim_{x\longrightarrow 2} \frac{\sqrt{x^2+1}-\sqrt{5}}{x^2+2x-8}$.

*e)* $\lim_{x\longrightarrow 3} \frac{\sin(2x^2-x-15)}{x^2-x-6}$.

*f)* $\lim_{x\to 0} cos(1 + e^{\frac{1}{x}})[\frac{\sin(x^3-2x^2)}{x}]$

$$\frac{-1}{x+1} = +\infty \qquad x+1 \to 0^-$$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

$$x < -1 \Rightarrow x+1 < 0$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$\approx \frac{(x^2-2x)}{x^2-2x}$$

**Questão 3.** *(1,5) Calcule* $f'(x)$ *onde* $f(x) = \cos(x^3 - 5).e^{x^2+1} + \ln(\arcsin(2x)) + \frac{x^5+7x}{1+\arctan(x^2)}$.

$$x \to 1 \quad \frac{x^7(11x^{\ldots})}{x^7(1+\frac{1}{x})}$$

$$\frac{x^8}{x^8}$$

$$\sin(2x^2-x-15) \approx 2x^2-x-15$$

$$[f(g(h(x)))]' = f'(g(h(x)))\,(g(h(x)))'$$

$$\left(\frac{f}{g}\right)' \neq \frac{f'g - f g'}{g^2}$$

$$u \to 0 \quad v \to 0 \qquad \frac{\operatorname{sen} u}{v} = 1 \qquad \frac{\operatorname{sen} u}{u}$$

$$(f \cdot g)' = f g' + f' g = f' g + f g'$$

$$(g(h(x)))' \neq g'(h(x))$$

9.6 $\lim y = e^{\frac{x-1}{x^2}}$ $\quad R-\{0\}$.

$$y' = e^{\frac{x-1}{x^2}} \cdot \left[ \frac{1 \cdot x^2 - (x-1)2x}{x^4} \right]$$

$$\lim_{x \to 0} y(x) = \lim_{x \to 0} e^{\frac{x-1}{x^2}} \to 0^+ = 0$$

$\frac{x-1}{x^2}$: $\to -1$, $\to 0^+$, $\to -\infty$

$$x^2 - (x-1)(2x) = x^2 - 2x^2 + 2x = 0$$

$$\Leftrightarrow -x^2 + 2x = 0$$

$$\Leftrightarrow x = 0 \text{ ou } x = 2$$

| | | 0 | | 2 | |
|---|---|---|---|---|---|
| $-x^2+2x$ | $-$ | 0 | $+$ | 0 | $-$ |
| $x^4$ | $+$ | 0 | $+$ | $+$ | $+$ |
| $f'$ | $-$ | | $+$ | 0 | $-$ |

↗ máximo ↘

$$\lim_{x \to -\infty} y(x) = \lim_{x \to -\infty} e^{\frac{x-1}{x^2}} = 1$$

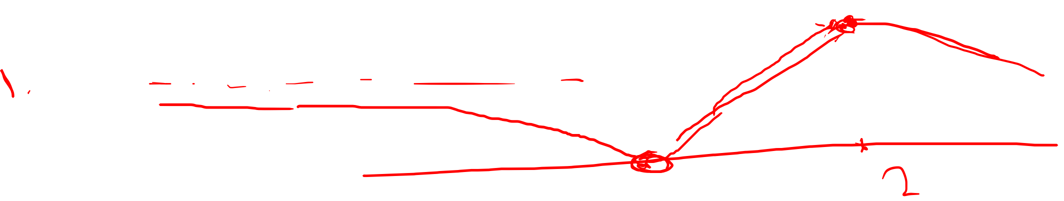
$$\lim_{x \to -\infty} \frac{x-1}{x^2} = \lim \frac{\cancel{x}(1-\frac{1}{x})}{\cancel{x} \cdot x} = 0$$

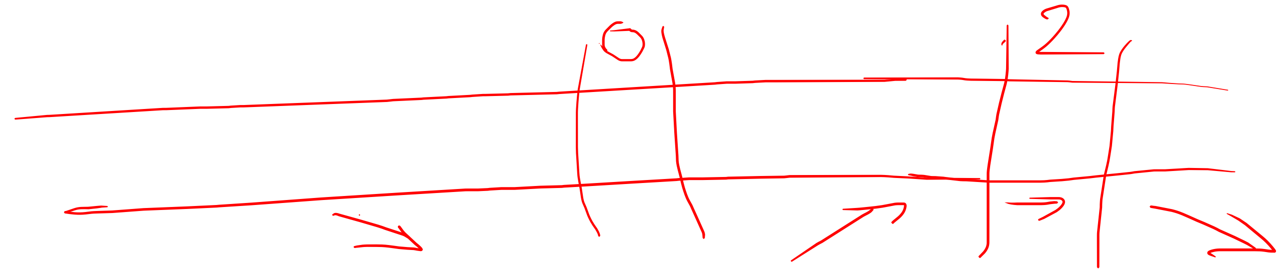
$$f(2) = e^{\frac{2-1}{4}} = e^{\frac{1}{4}}$$

$$0 < \frac{1}{4} \Rightarrow e^{0} < e^{\frac{1}{4}}$$

$$e^{\frac{1}{4}} > 1 = e^{0}$$

$f'$






$$\lim_{x \to 0} f(x) = 0$$

$$\lim_{x \to -\infty} f(x) = 1 < f(2)$$

2 é máximo global

Quarta (7/07)

Envio do gabarito

ñ haverá aula online

(aproveitem para reunir os grupos para fazer a lista).

Sexta (9/07)

Feriado

ñ haverá aula.