

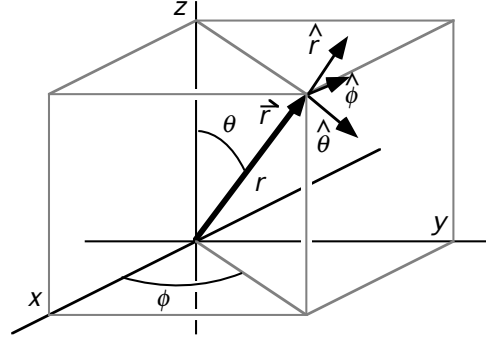
# Spherical Coordinates

## Transforms

The forward and reverse coordinate transformations are

$$\begin{aligned}
 r &= \sqrt{x^2 + y^2 + z^2} & x &= r \sin \theta \cos \phi \\
 \theta &= \arctan\left(\sqrt{x^2 + y^2}, z\right) & y &= r \sin \theta \sin \phi \\
 \phi &= \arctan(y, x) & z &= r \cos \theta
 \end{aligned}$$

where we *formally* take advantage of the *two argument* arctan function to eliminate quadrant confusion.



## Unit Vectors

The unit vectors in the spherical coordinate system are functions of position. It is convenient to express them in terms of the *spherical* coordinates and the unit vectors of the *rectangular* coordinate system which are *not* themselves functions of position.

$$\begin{aligned}
 \hat{r} &= \frac{\vec{r}}{r} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{r} = \hat{x} \sin \theta \cos \phi + \hat{y} \sin \theta \sin \phi + \hat{z} \cos \theta \\
 \hat{\phi} &= \frac{\hat{z} \times \hat{r}}{\sin \theta} = -\hat{x} \sin \phi + \hat{y} \cos \phi \\
 \hat{\theta} &= \hat{\phi} \times \hat{r} = \hat{x} \cos \theta \cos \phi + \hat{y} \cos \theta \sin \phi - \hat{z} \sin \theta
 \end{aligned}$$

## Variations of unit vectors with the coordinates

Using the expressions obtained above it is easy to derive the following handy relationships:

$$\begin{aligned}
 \frac{\partial \hat{r}}{\partial r} &= 0 \\
 \frac{\partial \hat{r}}{\partial \theta} &= \hat{x} \cos \theta \cos \phi + \hat{y} \cos \theta \sin \phi - \hat{z} \sin \theta = \hat{\theta} \\
 \frac{\partial \hat{r}}{\partial \phi} &= -\hat{x} \sin \theta \sin \phi + \hat{y} \sin \theta \cos \phi = (-\hat{x} \sin \phi + \hat{y} \cos \phi) \sin \theta = \hat{\phi} \sin \theta \\
 \frac{\partial \hat{\phi}}{\partial r} &= 0 \\
 \frac{\partial \hat{\phi}}{\partial \theta} &= 0 \\
 \frac{\partial \hat{\phi}}{\partial \phi} &= -\hat{x} \cos \phi - \hat{y} \sin \phi = -(\hat{r} \sin \theta + \hat{\theta} \cos \theta) \\
 \frac{\partial \hat{\theta}}{\partial r} &= 0 \\
 \frac{\partial \hat{\theta}}{\partial \theta} &= -\hat{x} \sin \theta \cos \phi - \hat{y} \sin \theta \sin \phi - \hat{z} \cos \theta = -\hat{r} \\
 \frac{\partial \hat{\theta}}{\partial \phi} &= -\hat{x} \cos \theta \sin \phi + \hat{y} \cos \theta \cos \phi = \hat{\phi} \cos \theta
 \end{aligned}$$

## Path increment

We will have many uses for the path increment  $d\vec{r}$  expressed in spherical coordinates:

$$\begin{aligned} d\vec{r} &= d(r\hat{r}) = \hat{r}dr + r d\hat{r} = \hat{r}dr + r\left(\frac{\partial\hat{r}}{\partial r}dr + \frac{\partial\hat{r}}{\partial\theta}d\theta + \frac{\partial\hat{r}}{\partial\phi}d\phi\right) \\ &= \hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi \end{aligned}$$

## Time derivatives of the unit vectors

We will also have many uses for the time derivatives of the unit vectors expressed in spherical coordinates:

$$\begin{aligned} \dot{\hat{r}} &= \frac{\partial\hat{r}}{\partial r}\dot{r} + \frac{\partial\hat{r}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{r}}{\partial\phi}\dot{\phi} = \hat{\theta}\dot{\theta} + \hat{\phi}\dot{\phi}\sin\theta \\ \dot{\hat{\theta}} &= \frac{\partial\hat{\theta}}{\partial r}\dot{r} + \frac{\partial\hat{\theta}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\theta}}{\partial\phi}\dot{\phi} = -\hat{r}\dot{\theta} + \hat{\phi}\dot{\phi}\cos\theta \\ \dot{\hat{\phi}} &= \frac{\partial\hat{\phi}}{\partial r}\dot{r} + \frac{\partial\hat{\phi}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\phi}}{\partial\phi}\dot{\phi} = -(\hat{r}\sin\theta + \hat{\theta}\cos\theta)\dot{\phi} \end{aligned}$$

## Velocity and Acceleration

The velocity and acceleration of a particle may be expressed in spherical coordinates by taking into account the associated rates of change in the unit vectors:

$$\vec{v} = \dot{\vec{r}} = \dot{\hat{r}}r + \hat{r}\dot{r}$$

$$\boxed{\vec{v} = \hat{r}\dot{r} + \hat{\theta}r\dot{\theta} + \hat{\phi}r\dot{\phi}\sin\theta}$$

$$\begin{aligned} \vec{a} = \dot{\vec{v}} &= \dot{\hat{r}}\dot{r} + \hat{r}\ddot{r} + \dot{\hat{\theta}}r\dot{\theta} + \hat{\theta}r\ddot{\theta} + \dot{\hat{\theta}}\dot{\theta} + \hat{\theta}r\ddot{\theta} + \dot{\hat{\phi}}r\dot{\phi}\sin\theta + \hat{\phi}r\dot{\phi}\dot{\theta}\sin\theta + \hat{\phi}r\ddot{\phi}\sin\theta + \hat{\phi}r\dot{\phi}\dot{\theta}\cos\theta \\ &= (\hat{\theta}\dot{\theta} + \hat{\phi}\dot{\phi}\sin\theta)\dot{r} + \hat{r}\ddot{r} + (-\hat{r}\dot{\theta} + \hat{\phi}\dot{\phi}\cos\theta)r\dot{\theta} + \hat{\theta}r\ddot{\theta} + \hat{\theta}r\dot{\theta} \\ &\quad + [-(\hat{r}\sin\theta + \hat{\theta}\cos\theta)\dot{\phi}]r\dot{\phi}\sin\theta + \hat{\phi}r\dot{\phi}\sin\theta + \hat{\phi}r\ddot{\phi}\sin\theta + \hat{\phi}r\dot{\phi}\dot{\theta}\cos\theta \end{aligned}$$

$$\boxed{\vec{a} = \hat{r}(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta) + \hat{\theta}(r\ddot{\theta} + 2r\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta) + \hat{\phi}(r\ddot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + 2r\dot{\phi}\dot{\theta}\sin\theta)}$$

## The del operator from the definition of the gradient

Any (static) scalar field  $u$  may be considered to be a function of the spherical coordinates  $r$ ,  $\theta$ , and  $\phi$ . The value of  $u$  changes by an infinitesimal amount  $du$  when the point of observation is changed by  $d\vec{r}$ . That change may be determined from the partial derivatives as

$$du = \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial\theta}d\theta + \frac{\partial u}{\partial\phi}d\phi.$$

But we also define the gradient in such a way as to obtain the result

$$du = \vec{\nabla}u \cdot d\vec{r}$$

Therefore,

$$\frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial\theta}d\theta + \frac{\partial u}{\partial\phi}d\phi = \vec{\nabla}u \cdot d\vec{r}$$

or, in spherical coordinates,

$$\frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial\theta}d\theta + \frac{\partial u}{\partial\phi}d\phi = (\vec{\nabla}u)_r dr + (\vec{\nabla}u)_\theta r d\theta + (\vec{\nabla}u)_\phi r \sin\theta d\phi$$

and we demand that this hold for any choice of  $dr$ ,  $d\theta$ , and  $d\phi$ . Thus,

$$(\bar{\nabla}u)_r = \frac{\partial u}{\partial r}, \quad (\bar{\nabla}u)_\theta = \frac{1}{r} \frac{\partial u}{\partial \theta}, \quad (\bar{\nabla}u)_\phi = \frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi},$$

from which we find

$$\bar{\nabla} = \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial}{\partial \phi}$$

## Divergence

The divergence  $\bar{\nabla} \cdot \bar{A}$  is carried out taking into account, once again, that the unit vectors themselves are functions of the coordinates. Thus, we have

$$\bar{\nabla} \cdot \bar{A} = \left( \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \cdot (A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi})$$

where the derivatives must be taken *before* the dot product so that

$$\begin{aligned} \bar{\nabla} \cdot \bar{A} &= \left( \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \cdot \bar{A} \\ &= \hat{r} \cdot \frac{\partial \bar{A}}{\partial r} + \frac{\hat{\theta}}{r} \cdot \frac{\partial \bar{A}}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \cdot \frac{\partial \bar{A}}{\partial \phi} \\ &= \hat{r} \cdot \left( \frac{\partial A_r}{\partial r} \hat{r} + \frac{\partial A_\theta}{\partial r} \hat{\theta} + \frac{\partial A_\phi}{\partial r} \hat{\phi} + A_r \frac{\partial \hat{r}}{\partial r} + A_\theta \frac{\partial \hat{\theta}}{\partial r} + A_\phi \frac{\partial \hat{\phi}}{\partial r} \right) \\ &\quad + \frac{\hat{\theta}}{r} \cdot \left( \frac{\partial A_r}{\partial \theta} \hat{r} + \frac{\partial A_\theta}{\partial \theta} \hat{\theta} + \frac{\partial A_\phi}{\partial \theta} \hat{\phi} + A_r \frac{\partial \hat{r}}{\partial \theta} + A_\theta \frac{\partial \hat{\theta}}{\partial \theta} + A_\phi \frac{\partial \hat{\phi}}{\partial \theta} \right) \\ &\quad + \frac{\hat{\phi}}{r \sin \theta} \cdot \left( \frac{\partial A_r}{\partial \phi} \hat{r} + \frac{\partial A_\theta}{\partial \phi} \hat{\theta} + \frac{\partial A_\phi}{\partial \phi} \hat{\phi} + A_r \frac{\partial \hat{r}}{\partial \phi} + A_\theta \frac{\partial \hat{\theta}}{\partial \phi} + A_\phi \frac{\partial \hat{\phi}}{\partial \phi} \right) \end{aligned}$$

With the help of the partial derivatives previously obtained, we find

$$\begin{aligned} \bar{\nabla} \cdot \bar{A} &= \hat{r} \cdot \left( \frac{\partial A_r}{\partial r} \hat{r} + \frac{\partial A_\theta}{\partial r} \hat{\theta} + \frac{\partial A_\phi}{\partial r} \hat{\phi} + 0 + 0 + 0 \right) \\ &\quad + \frac{\hat{\theta}}{r} \cdot \left( \frac{\partial A_r}{\partial \theta} \hat{r} + \frac{\partial A_\theta}{\partial \theta} \hat{\theta} + \frac{\partial A_\phi}{\partial \theta} \hat{\phi} + A_r \hat{\theta} + A_\theta (-\hat{r}) + 0 \right) \\ &\quad + \frac{\hat{\phi}}{r \sin \theta} \cdot \left( \frac{\partial A_r}{\partial \phi} \hat{r} + \frac{\partial A_\theta}{\partial \phi} \hat{\theta} + \frac{\partial A_\phi}{\partial \phi} \hat{\phi} + A_r \sin \theta \hat{\phi} + A_\theta \cos \theta \hat{\phi} + A_\phi \left[ -(\hat{r} \sin \theta + \hat{\theta} \cos \theta) \right] \right) \\ &= \left( \frac{\partial A_r}{\partial r} \right) + \left( \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{A_r}{r} \right) + \left( \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} + \frac{A_r}{r} + \frac{A_\theta \cos \theta}{r \sin \theta} \right) \\ &= \left( \frac{\partial A_r}{\partial r} + \frac{2A_r}{r} \right) + \left( \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{A_\theta \cos \theta}{r \sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \end{aligned}$$

$$\bar{\nabla} \cdot \bar{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

## Curl

The curl  $\bar{\nabla} \times \bar{A}$  is also carried out taking into account that the unit vectors themselves are functions of the coordinates. Thus, we have

$$\bar{\nabla} \times \bar{A} = \left( \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \times (A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi})$$

where the derivatives must be taken *before* the cross product so that

$$\begin{aligned} \bar{\nabla} \times \bar{A} &= \left( \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \times \bar{A} \\ &= \hat{r} \times \frac{\partial \bar{A}}{\partial r} + \frac{\hat{\theta}}{r} \times \frac{\partial \bar{A}}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \times \frac{\partial \bar{A}}{\partial \phi} \\ &= \hat{r} \times \left( \frac{\partial A_r}{\partial r} \hat{r} + \frac{\partial A_\theta}{\partial r} \hat{\theta} + \frac{\partial A_\phi}{\partial r} \hat{\phi} + A_r \frac{\partial \hat{r}}{\partial r} + A_\theta \frac{\partial \hat{\theta}}{\partial r} + A_\phi \frac{\partial \hat{\phi}}{\partial r} \right) \\ &\quad + \frac{\hat{\theta}}{r} \times \left( \frac{\partial A_r}{\partial \theta} \hat{r} + \frac{\partial A_\theta}{\partial \theta} \hat{\theta} + \frac{\partial A_\phi}{\partial \theta} \hat{\phi} + A_r \frac{\partial \hat{r}}{\partial \theta} + A_\theta \frac{\partial \hat{\theta}}{\partial \theta} + A_\phi \frac{\partial \hat{\phi}}{\partial \theta} \right) \\ &\quad + \frac{\hat{\phi}}{r \sin \theta} \times \left( \frac{\partial A_r}{\partial \phi} \hat{r} + \frac{\partial A_\theta}{\partial \phi} \hat{\theta} + \frac{\partial A_\phi}{\partial \phi} \hat{\phi} + A_r \frac{\partial \hat{r}}{\partial \phi} + A_\theta \frac{\partial \hat{\theta}}{\partial \phi} + A_\phi \frac{\partial \hat{\phi}}{\partial \phi} \right) \end{aligned}$$

With the help of the partial derivatives previously obtained, we find

$$\begin{aligned} \bar{\nabla} \times \bar{A} &= \hat{r} \times \left( \frac{\partial A_r}{\partial r} \hat{r} + \frac{\partial A_\theta}{\partial r} \hat{\theta} + \frac{\partial A_\phi}{\partial r} \hat{\phi} + 0 + 0 + 0 \right) \\ &\quad + \frac{\hat{\theta}}{r} \times \left( \frac{\partial A_r}{\partial \theta} \hat{r} + \frac{\partial A_\theta}{\partial \theta} \hat{\theta} + \frac{\partial A_\phi}{\partial \theta} \hat{\phi} + A_r \hat{\theta} + A_\theta (-\hat{r}) + 0 \right) \\ &\quad + \frac{\hat{\phi}}{r \sin \theta} \times \left( \frac{\partial A_r}{\partial \phi} \hat{r} + \frac{\partial A_\theta}{\partial \phi} \hat{\theta} + \frac{\partial A_\phi}{\partial \phi} \hat{\phi} + A_r \sin \theta \hat{\phi} + A_\theta \cos \theta \hat{\phi} + A_\phi \left[ -(\hat{r} \sin \theta + \hat{\theta} \cos \theta) \right] \right) \\ &= \left( \frac{\partial A_\theta}{\partial r} \hat{\phi} - \frac{\partial A_\phi}{\partial r} \hat{\theta} \right) + \left( -\frac{1}{r} \frac{\partial A_r}{\partial \theta} \hat{\phi} + \frac{1}{r} \frac{\partial A_\phi}{\partial \theta} \hat{r} + \frac{A_\theta}{r} \hat{\phi} \right) \\ &\quad + \left( \frac{1}{r \sin \theta} \frac{\partial A_r}{\partial \phi} \hat{\theta} - \frac{1}{r \sin \theta} \frac{\partial A_\theta}{\partial \phi} \hat{r} - \frac{A_\phi}{r} \hat{\theta} + \frac{A_\phi \cos \theta}{r \sin \theta} \hat{r} \right) \\ &= \hat{r} \left( \frac{1}{r} \frac{\partial A_\phi}{\partial \theta} - \frac{1}{r \sin \theta} \frac{\partial A_\theta}{\partial \phi} + \frac{A_\phi \cos \theta}{r \sin \theta} \right) \\ &\quad + \hat{\theta} \left( -\frac{\partial A_\phi}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{A_\phi}{r} \right) \\ &\quad + \hat{\phi} \left( \frac{\partial A_\theta}{\partial r} - \frac{1}{r} \frac{\partial A_r}{\partial \theta} + \frac{A_\theta}{r} \right) \end{aligned}$$

$$\boxed{\bar{\nabla} \times \bar{A} = \frac{\hat{r}}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right] + \frac{\hat{\theta}}{r \sin \theta} \left[ \frac{\partial A_r}{\partial \phi} - \sin \theta \frac{\partial}{\partial r} (r A_\phi) \right] + \frac{\hat{\phi}}{r} \left[ \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right]}$$

## Laplacian

The Laplacian is a scalar operator that can be determined from its definition as

$$\begin{aligned}\nabla^2 u &= \vec{\nabla} \cdot (\vec{\nabla} u) = \left( \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \cdot \left( \hat{r} \frac{\partial u}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial u}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial u}{\partial \phi} \right) \\ &= \hat{r} \cdot \frac{\partial}{\partial r} \left( \hat{r} \frac{\partial u}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial u}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial u}{\partial \phi} \right) \\ &\quad + \frac{\hat{\theta}}{r} \cdot \frac{\partial}{\partial \theta} \left( \hat{r} \frac{\partial u}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial u}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial u}{\partial \phi} \right) \\ &\quad + \frac{\hat{\phi}}{r \sin \theta} \cdot \frac{\partial}{\partial \phi} \left( \hat{r} \frac{\partial u}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial u}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial u}{\partial \phi} \right)\end{aligned}$$

With the help of the partial derivatives previously obtained, we find

$$\begin{aligned}\nabla^2 u &= \hat{r} \cdot \left( \hat{r} \frac{\partial^2 u}{\partial r^2} - \frac{\hat{\theta}}{r^2} \frac{\partial u}{\partial \theta} + \frac{\hat{\theta}}{r} \frac{\partial^2 u}{\partial \theta \partial r} - \frac{\hat{\phi}}{r^2 \sin \theta} \frac{\partial u}{\partial \phi} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial^2 u}{\partial \phi \partial r} \right) \\ &\quad + \frac{\hat{\theta}}{r} \cdot \left( \hat{\theta} \frac{\partial u}{\partial r} + \hat{r} \frac{\partial^2 u}{\partial r \partial \theta} - \frac{\hat{r}}{r} \frac{\partial u}{\partial \theta} + \frac{\hat{\theta}}{r} \frac{\partial^2 u}{\partial \theta^2} - \frac{\hat{\phi} \cos \theta}{r \sin^2 \theta} \frac{\partial u}{\partial \phi} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial^2 u}{\partial \phi \partial \theta} \right) \\ &\quad + \frac{\hat{\phi}}{r \sin \theta} \cdot \left( \hat{\phi} \sin \theta \frac{\partial u}{\partial r} + \hat{r} \frac{\partial^2 u}{\partial r \partial \phi} + \frac{\hat{\phi} \cos \theta}{r} \frac{\partial u}{\partial \theta} + \frac{\hat{\theta}}{r} \frac{\partial^2 u}{\partial \theta \partial \phi} - \frac{\hat{r} \sin \theta + \hat{\theta} \cos \theta}{r \sin \theta} \frac{\partial u}{\partial \phi} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial^2 u}{\partial \phi^2} \right) \\ &= \left( \frac{\partial^2 u}{\partial r^2} \right) + \left( \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right) + \left( \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} \right) \\ &= \left( \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} \right) + \left( \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial u}{\partial \theta} \right) + \left( \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} \right) \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2}\end{aligned}$$

Thus, the Laplacian operator can be written as

$$\boxed{\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}}$$