Computer Vision

Spring 2012 15-385,-685

Instructor: S. Narasimhan

Wean Hall 5409

T-R 10:30am – 11:50am

Frequency domain analysis and Fourier Transform

Lecture #4

How to Represent Signals?

- Option 1: Taylor series represents any function using polynomials.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots.$$

- Polynomials are not the best - unstable and not very physically meaningful.

- Easier to talk about “signals” in terms of its “frequencies” (how fast/often signals change, etc).

Jean Baptiste Joseph Fourier (1768-1830)

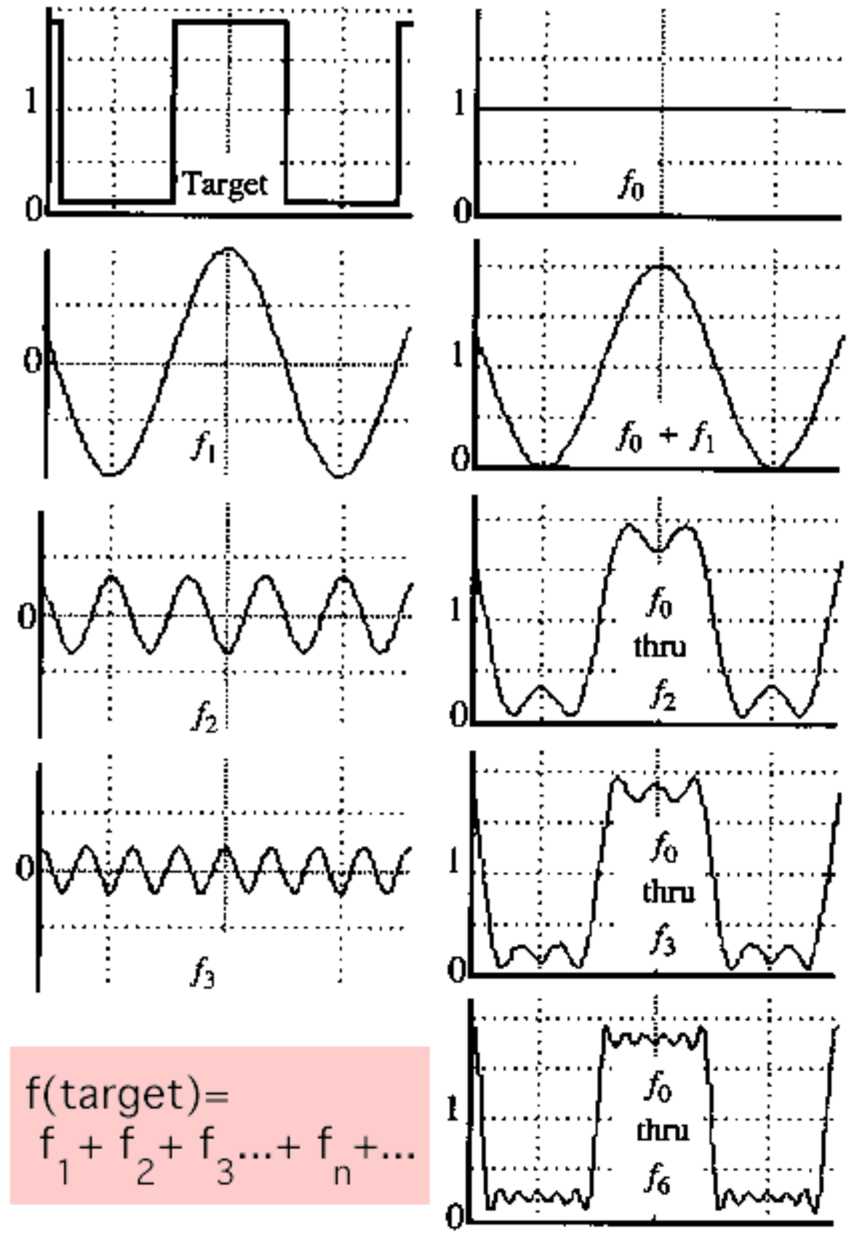
- Had crazy idea (1807):
- **Any** periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies.
- Don't believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it's true!
  - called Fourier Series
  - Possibly the greatest tool used in Engineering

A Sum of Sinusoids

- Our building block:

$$A\sin(\omega x+\phi)$$

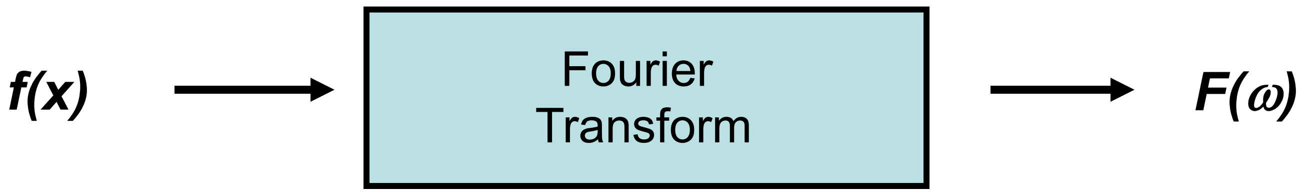
- Add enough of them to get any signal *f(x)* you want!
- How many degrees of freedom?
- What does each control?
- Which one encodes the coarse vs. fine structure of the signal?



f(target)= $f_1 + f_2 + f_3 ... + f_n + ...$

Fourier Transform

- We want to understand the frequency ω of our signal. So, let's reparametrize the signal by ω instead of *x*:



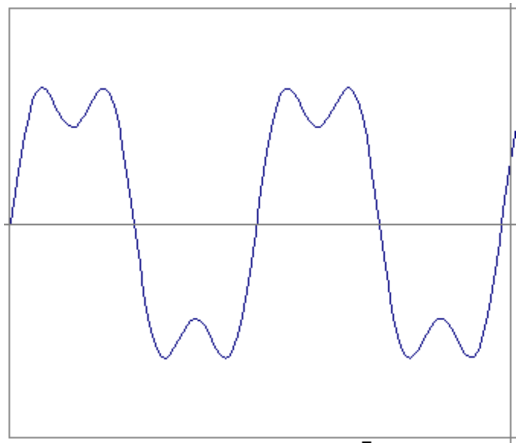
- For every ω from 0 to inf, ***F(ω)*** holds the amplitude *A* and phase ϕ of the corresponding sine $A\sin(\omega x + \phi)$
  - How can *F* hold both? Complex number trick!

$$F(\omega) = R(\omega) + iI(\omega)$$

$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2} \qquad \phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

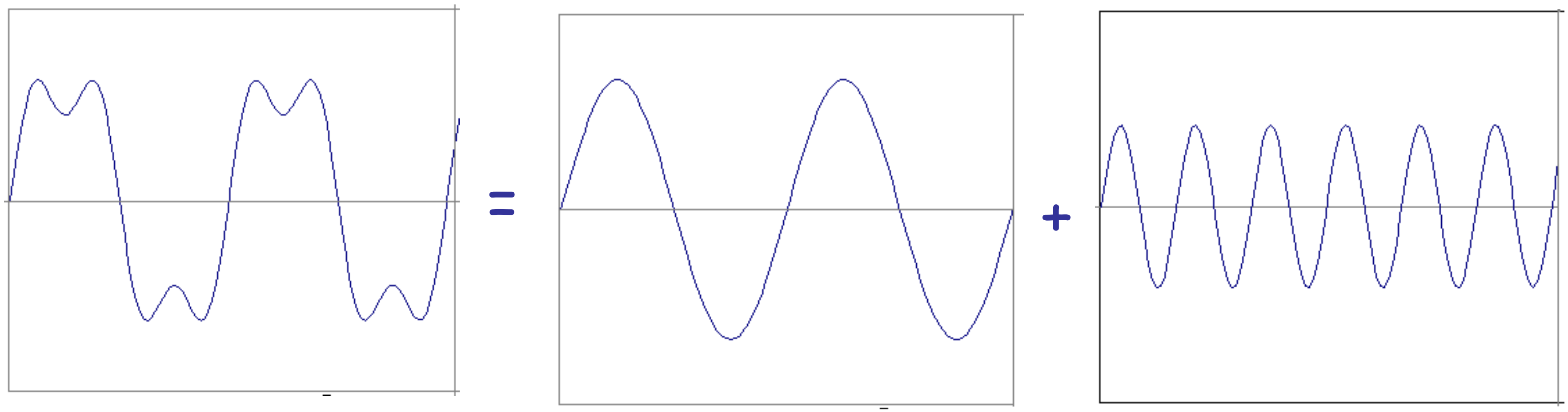
Time and Frequency

- example : $g(t) = \sin(2pi\, f\, t) + (1/3)\sin(2pi\, (3f)\, t)$



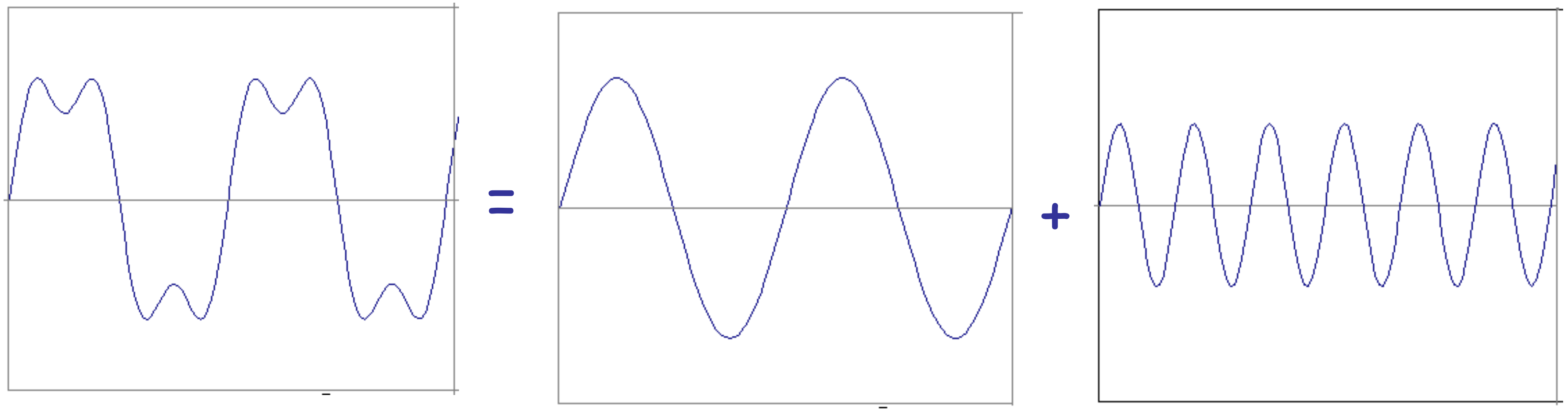
Time and Frequency

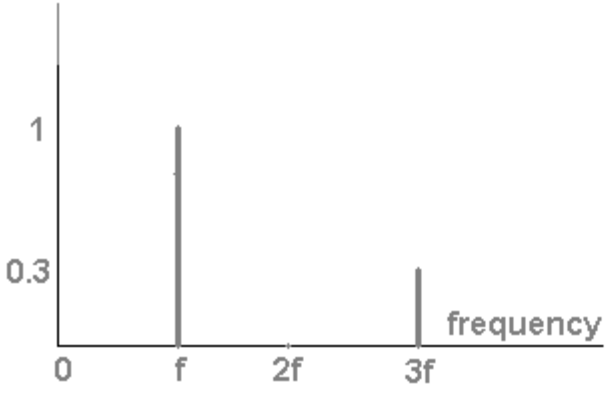
- example : $g(t) = \sin(2\pi f t) + (1/3)\sin(2\pi (3f) t)$



Frequency Spectra

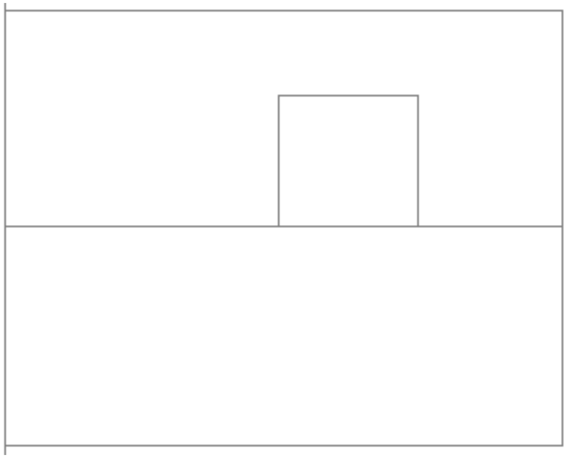
- example : $g(t) = \sin(2pi\ f\ t) + (1/3)\sin(2pi\ (3f)\ t)$



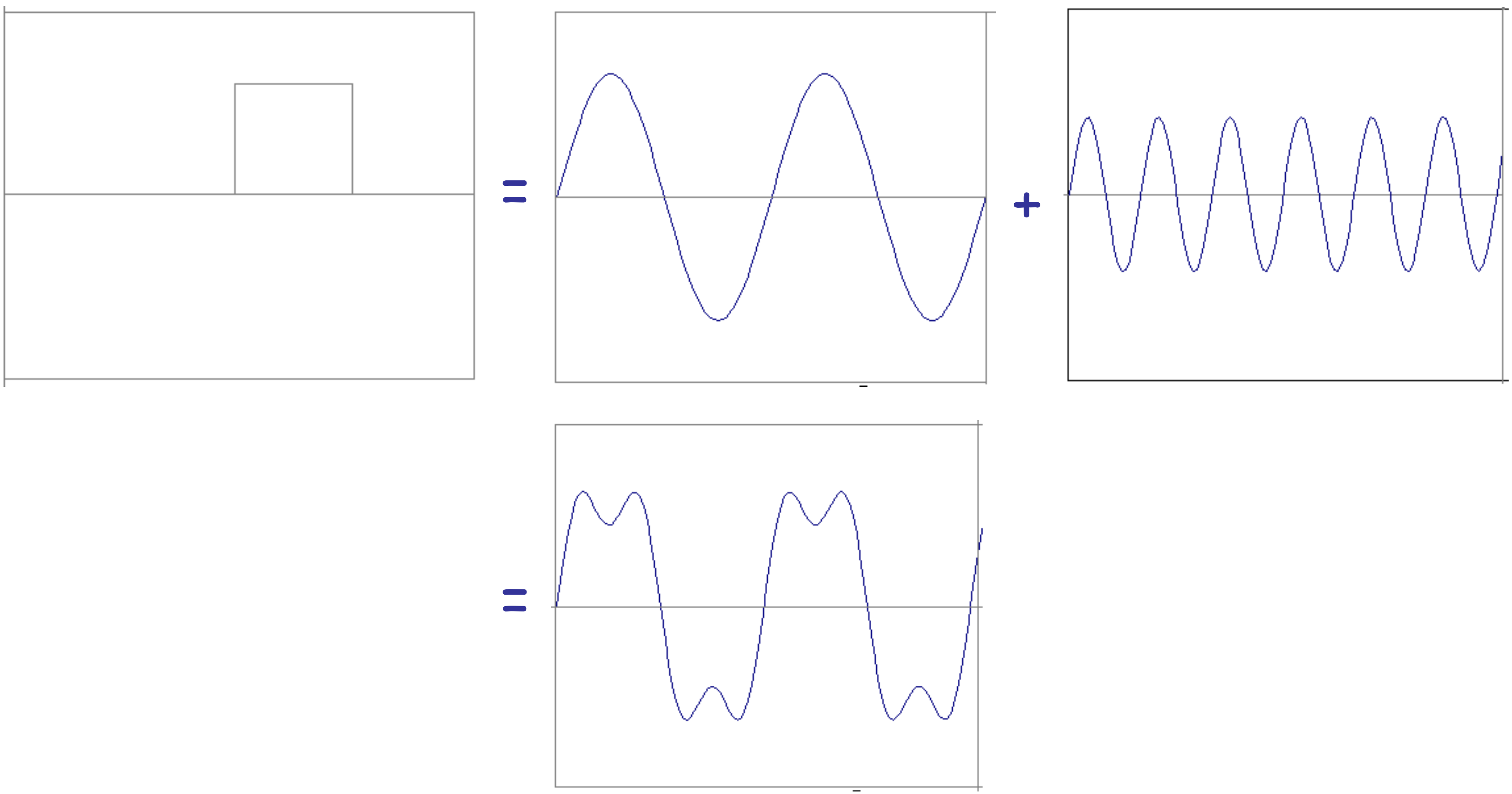


Frequency Spectra

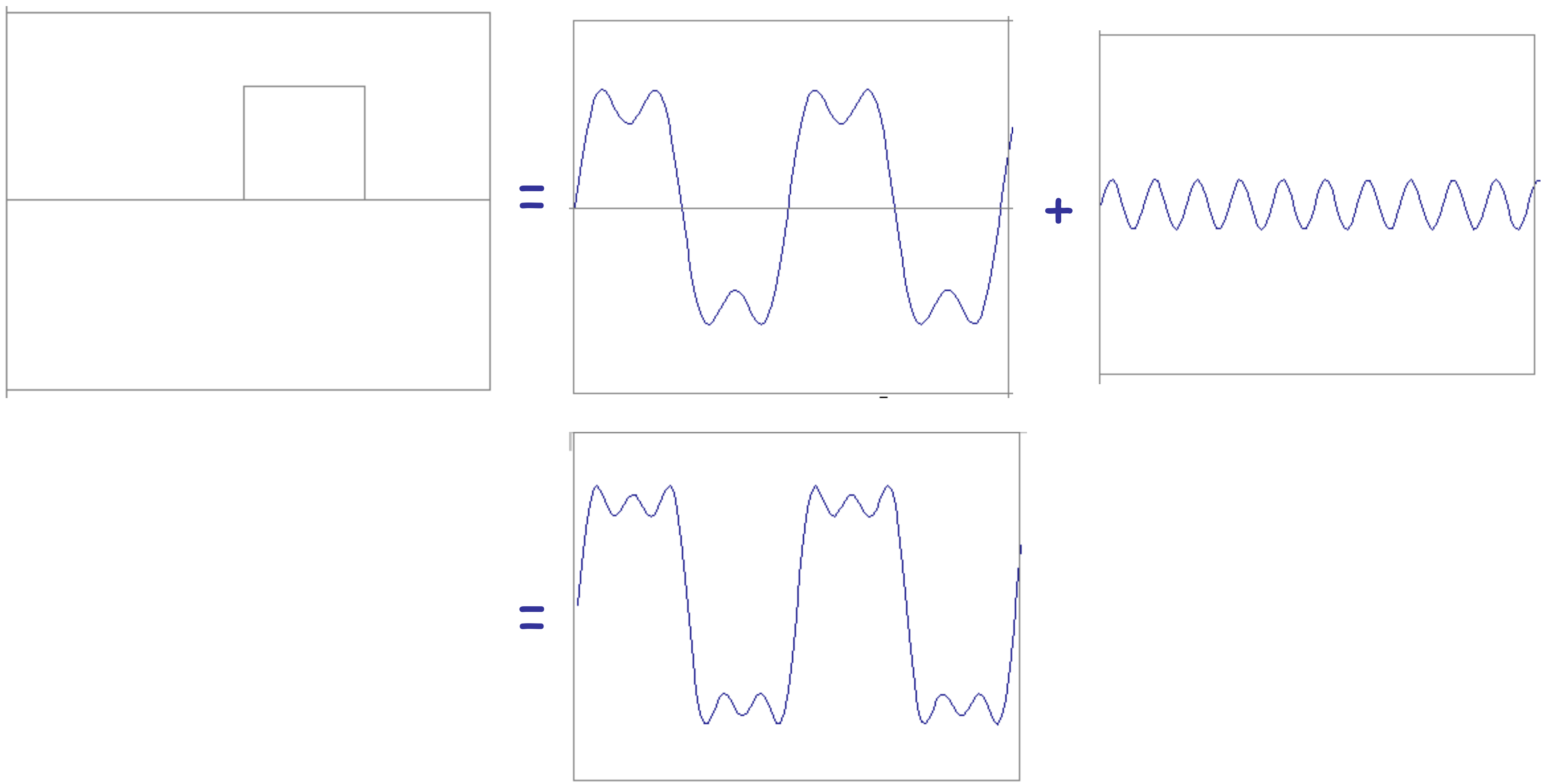
- Usually, frequency is more interesting than the phase



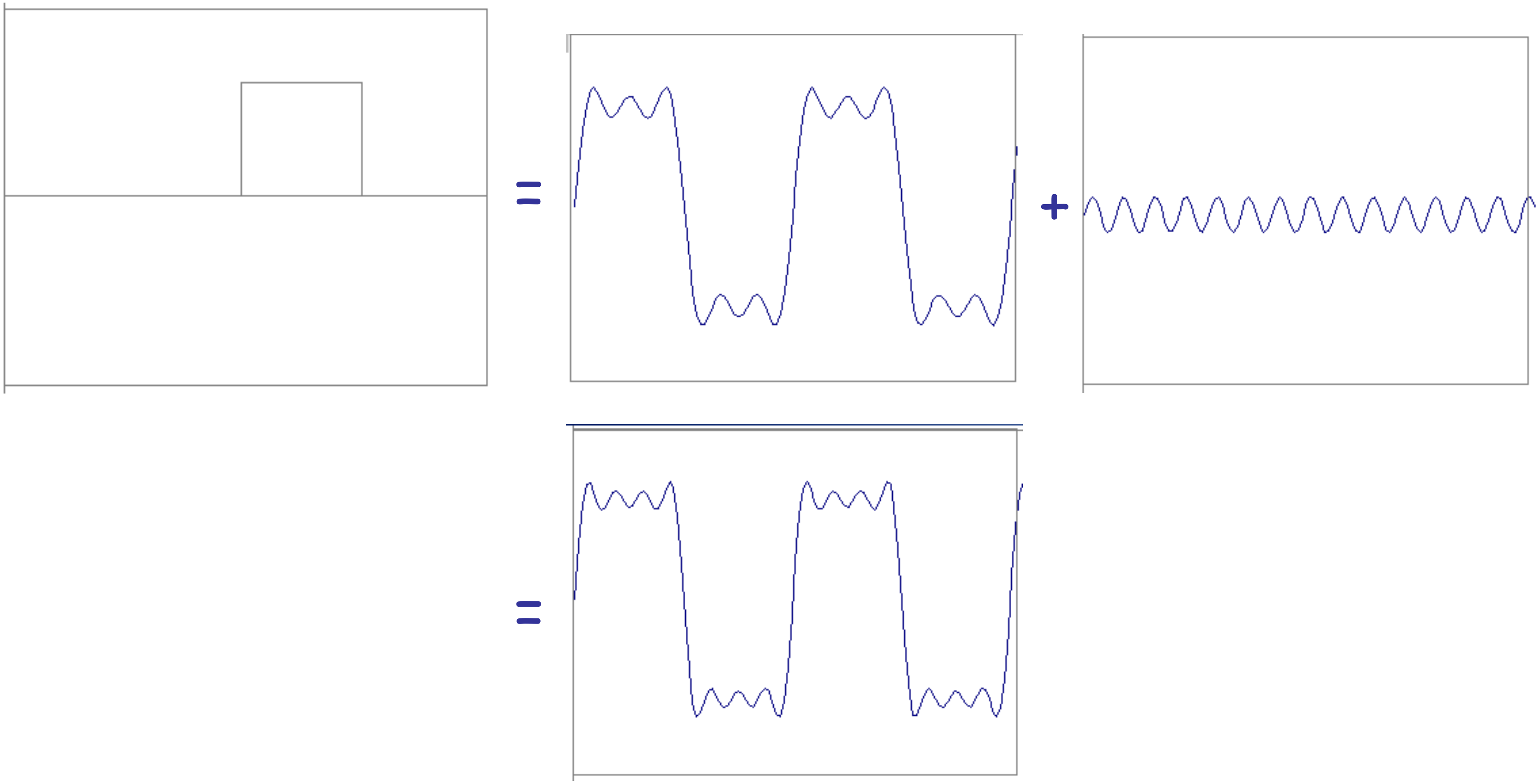
Frequency Spectra



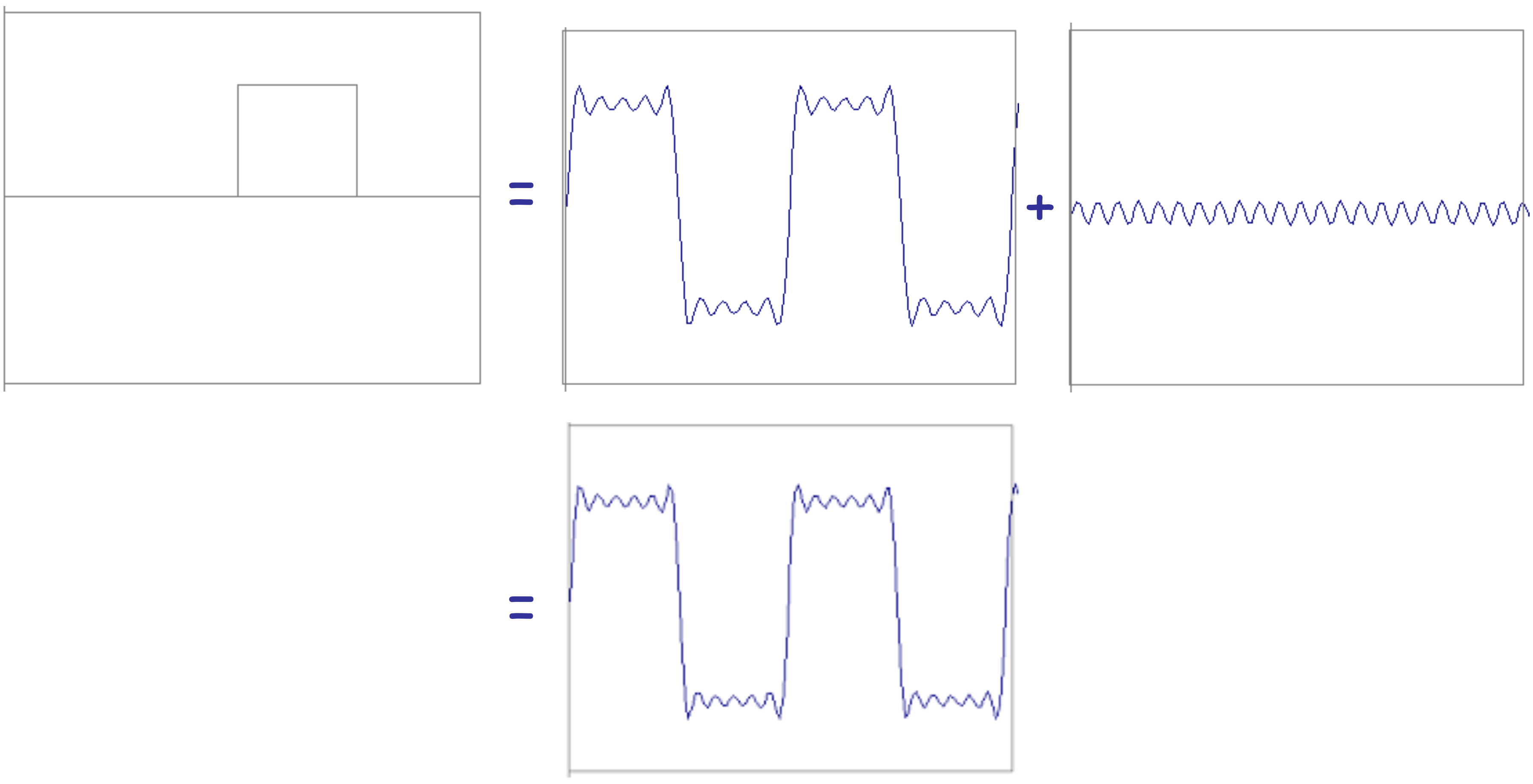
Frequency Spectra



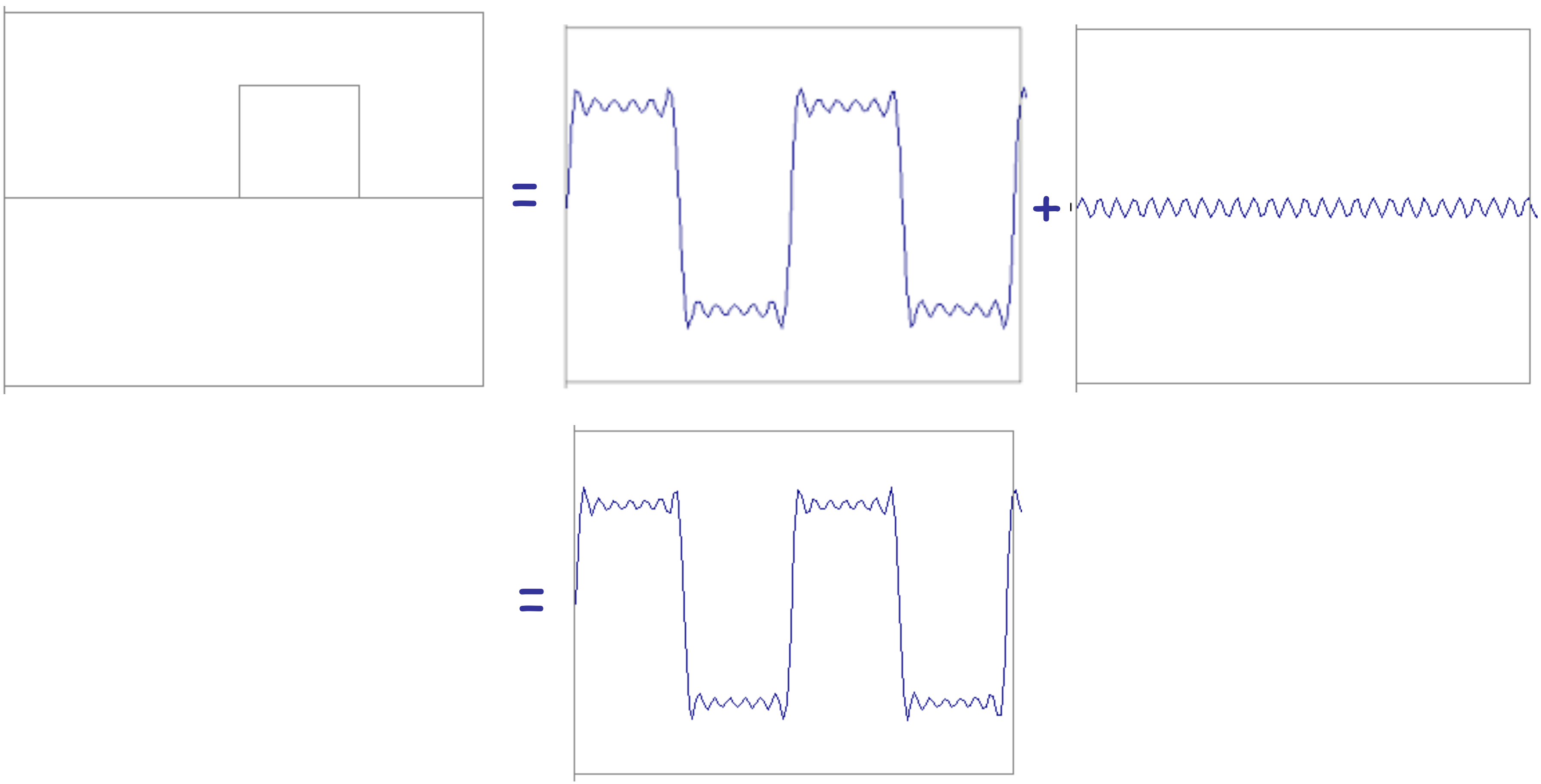
Frequency Spectra



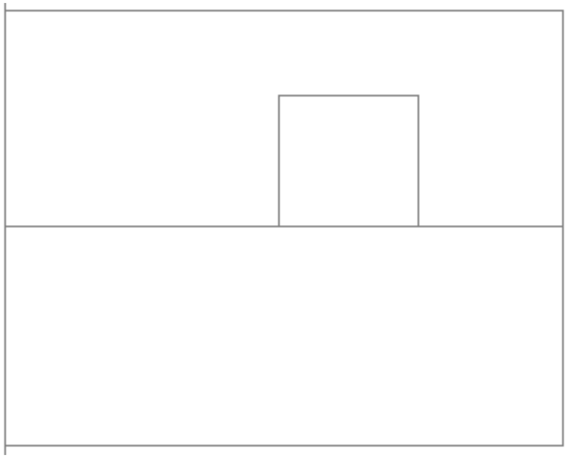
Frequency Spectra



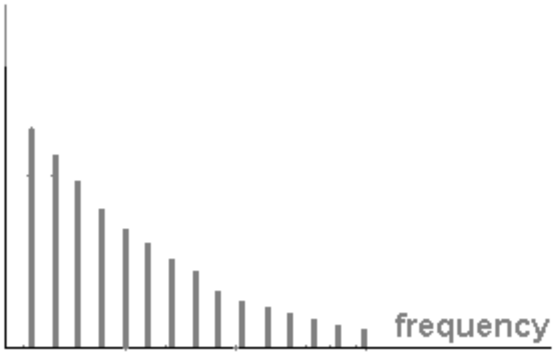
Frequency Spectra



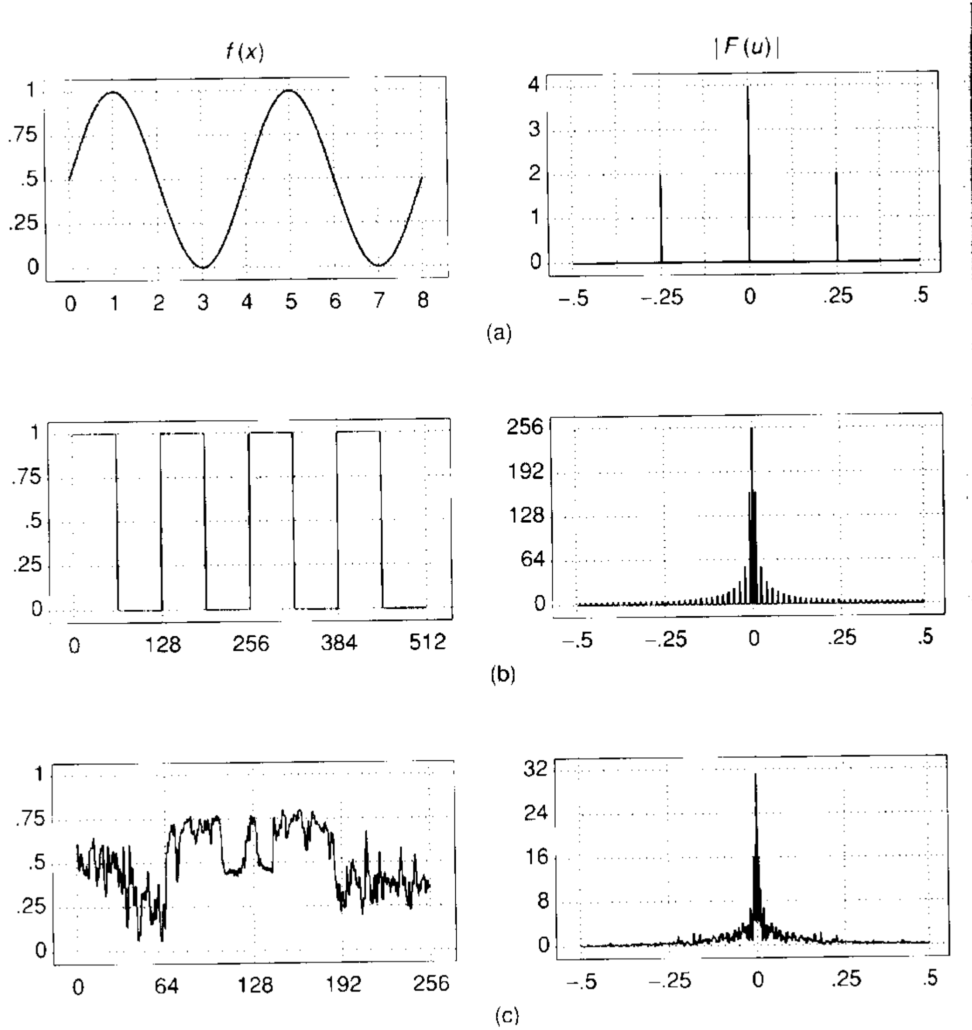
Frequency Spectra



$$= \quad A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)$$

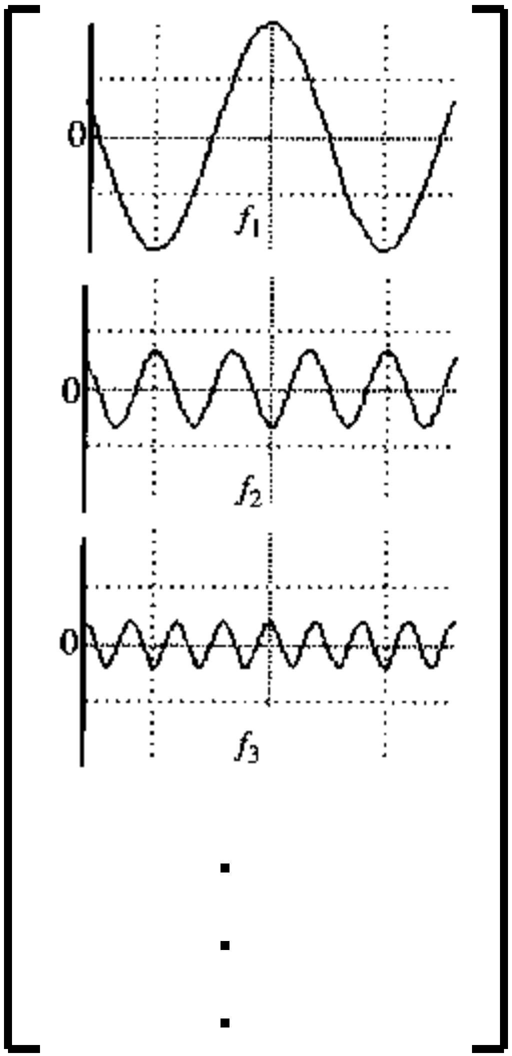


Frequency Spectra



FT: Just a change of basis

$$M * f(x) = F(\omega)$$



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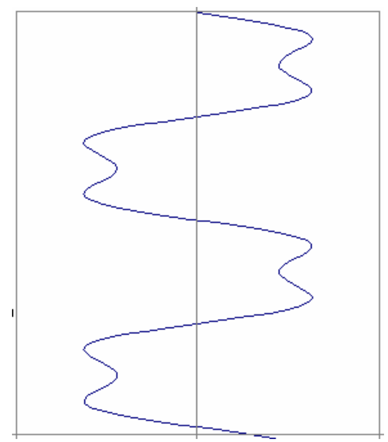

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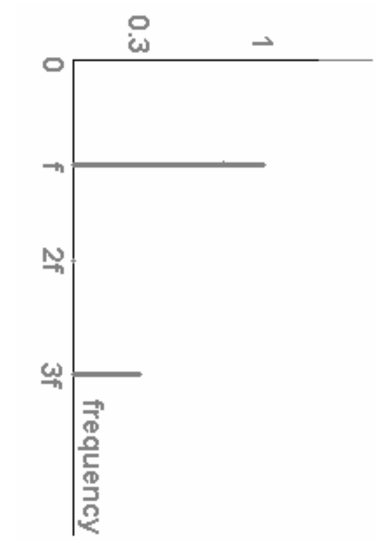
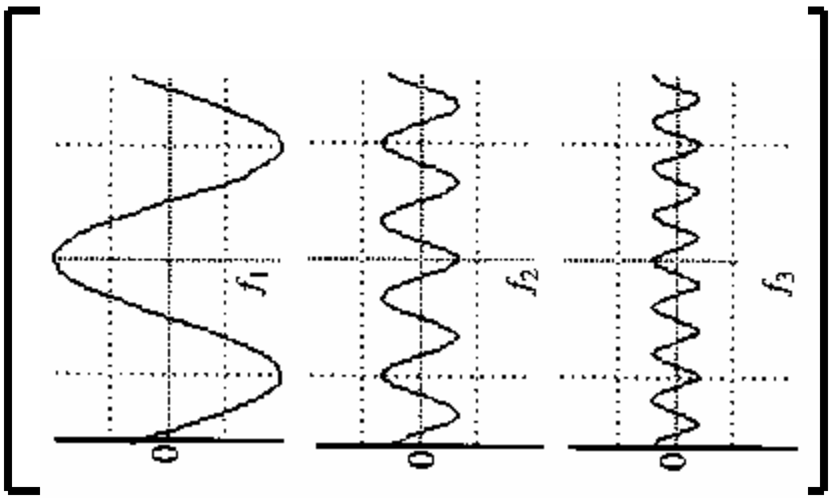


IFT: Just a change of basis

$$M^{-1} * F(\omega) = f(x)$$



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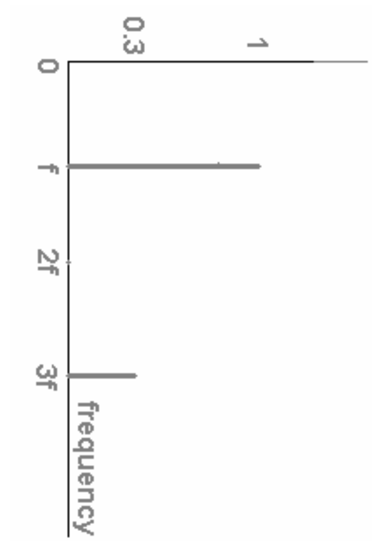

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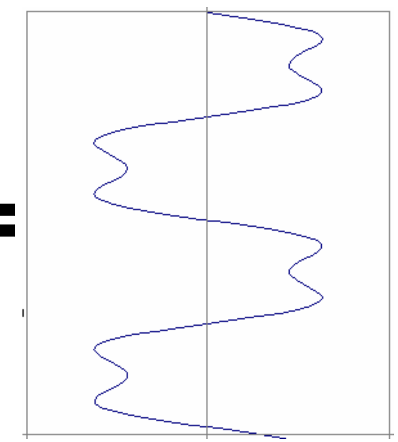

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Fourier Transform – more formally

Represent the signal as an infinite weighted sum of an infinite number of sinusoids

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i 2\pi ux} dx$$

Note: $e^{ik} = \cos k + i \sin k \qquad i = \sqrt{-1}$

Arbitrary function ⟶ Single Analytic Expression

Spatial Domain (*x*) ⟶ Frequency Domain (*u*)

(Frequency Spectrum *F(u)*)

Inverse Fourier Transform (IFT)

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{i 2\pi ux} dx$$

Fourier Transform

- Also, defined as:

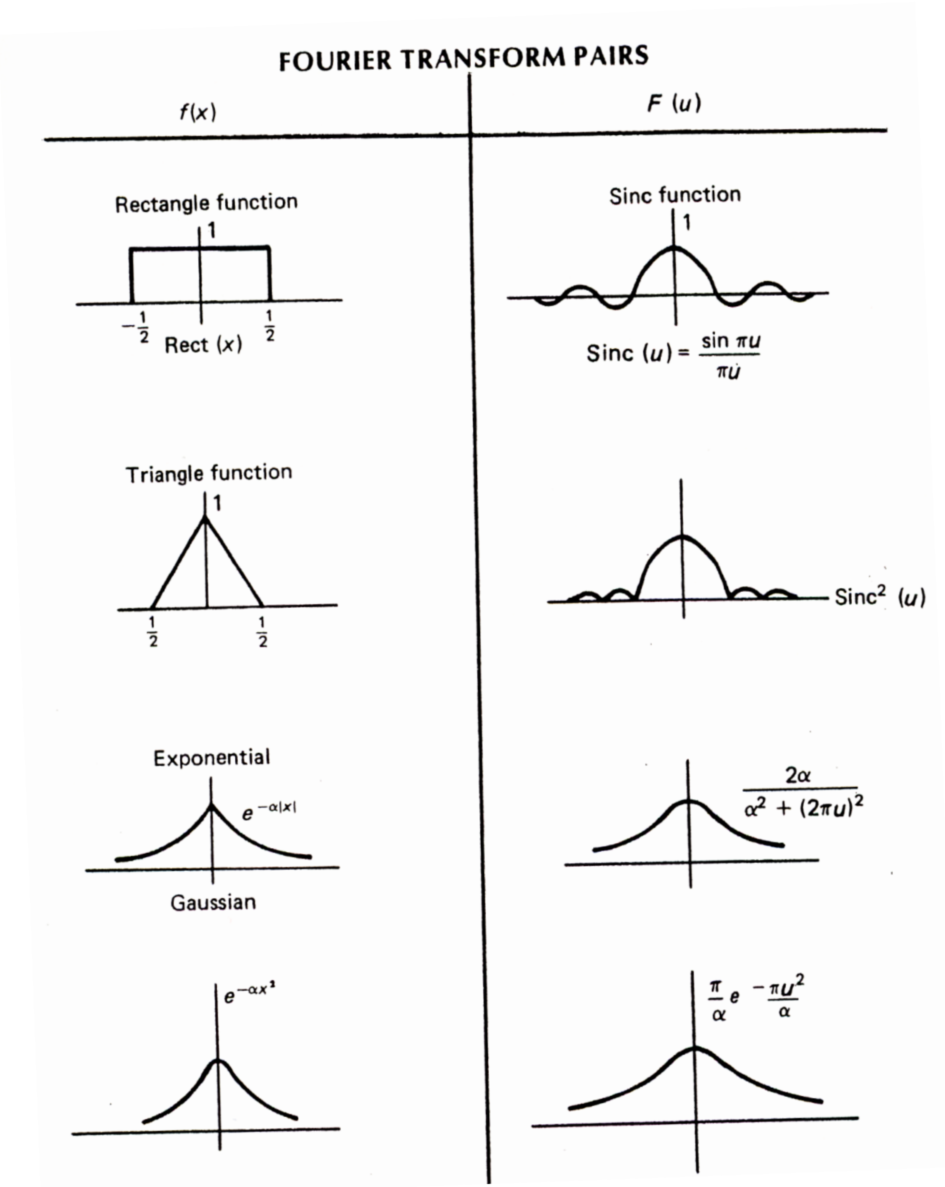
$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-iux} dx$$

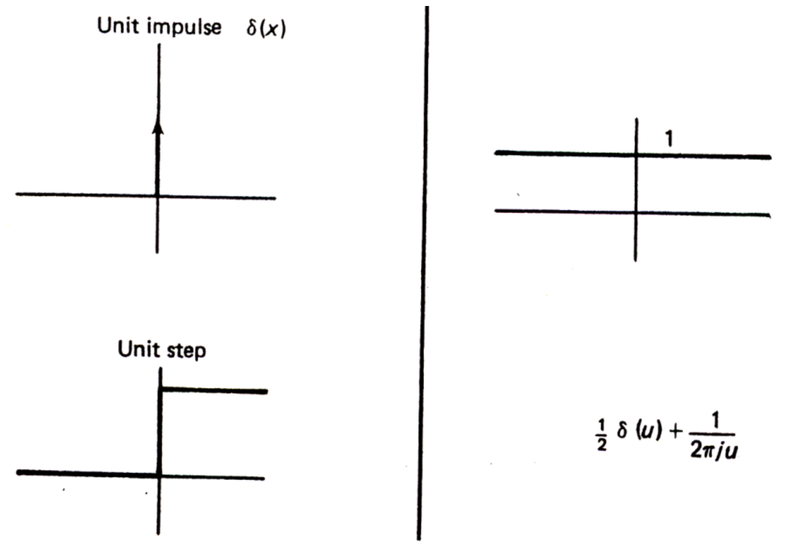
Note: $e^{ik} = \cos k + i \sin k \qquad i = \sqrt{-1}$

- Inverse Fourier Transform (IFT)

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{iux} dx$$

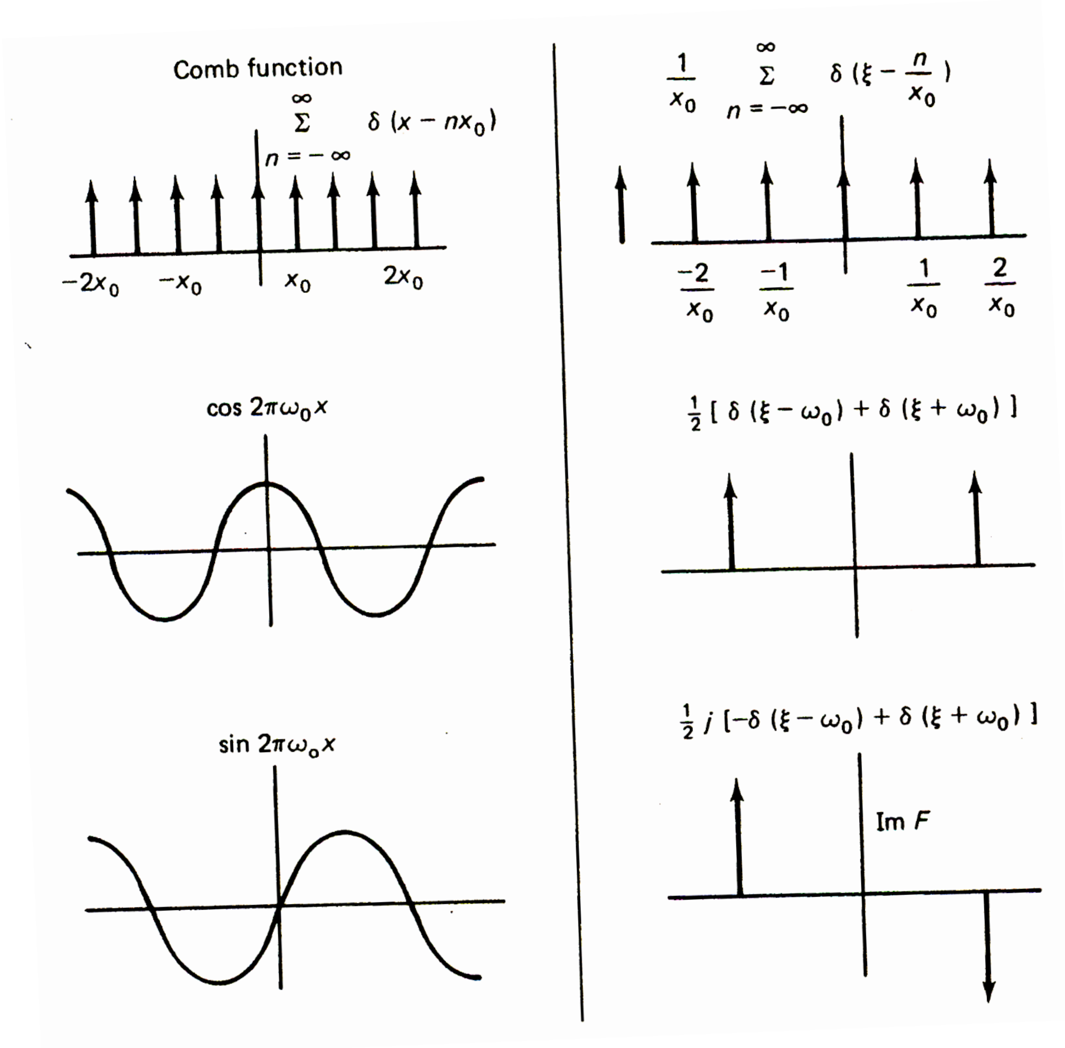
Fourier Transform Pairs (I)





Note that these are derived using angular frequency (e^{-iux})

Fourier Transform Pairs (I)



Note that these are derived using angular frequency (e^{-iux})

Fourier Transform and Convolution

Let $g = f * h$

Then $G(u) = \int_{-\infty}^{\infty} g(x) e^{-i2\pi ux} dx$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) h(x-\tau) e^{-i2\pi ux} d\tau dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \left[h(x-\tau) e^{-i2\pi u(x-\tau)} dx \right]$$

$$= \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \int_{-\infty}^{\infty} \left[h(x') e^{-i2\pi ux'} dx' \right]$$

$$= F(u) H(u)$$

Convolution in spatial domain

$\Leftrightarrow$ Multiplication in frequency domain

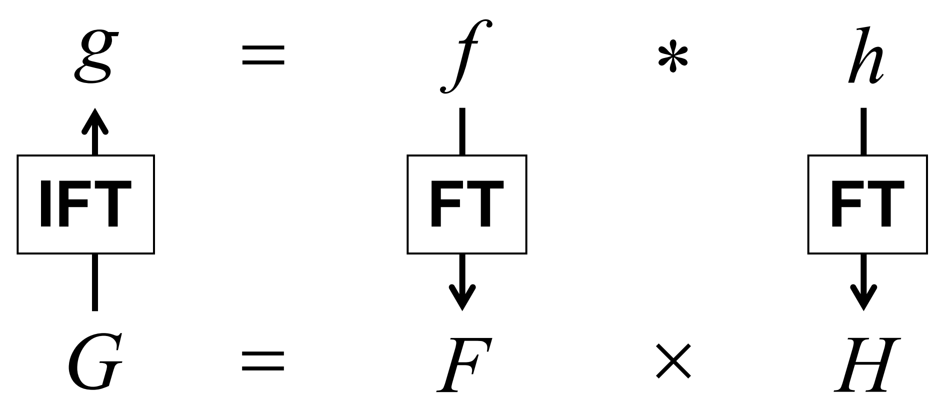
Fourier Transform and Convolution

Spatial Domain *(x)* | Frequency Domain *(u)*

$$g = f * h \quad \longleftrightarrow \quad G = FH$$

$$g = fh \quad \longleftrightarrow \quad G = F * H$$

So, we can find *g(x)* by Fourier transform



Properties of Fourier Transform

	Spatial Domain *(x)*	Frequency Domain *(u)*
Linearity	$c_1 f(x) + c_2 g(x)$	$c_1 F(u) + c_2 G(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{u}{a}\right)$
Shifting	$f(x - x_0)$	$e^{-i2\pi u x_0} F(u)$
Symmetry	$F(x)$	$f(-u)$
Conjugation	$f^*(x)$	$F^*(-u)$
Convolution	$f(x) * g(x)$	$F(u)G(u)$
Differentiation	$\frac{d^n f(x)}{dx^n}$	$(i2\pi u)^n F(u)$

Note that these are derived using frequency ($e^{-i2\pi ux}$)

Properties of Fourier Transform

Parseval's theorem:

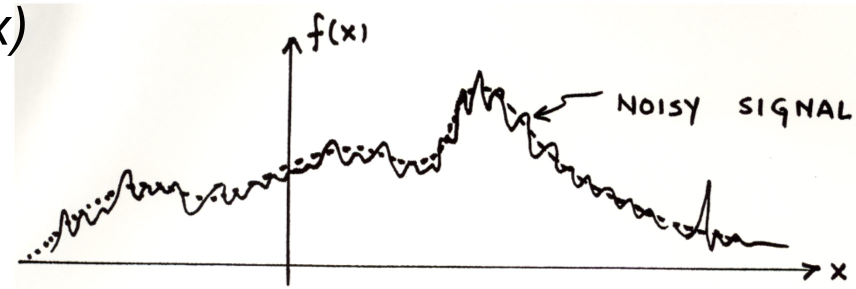
$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\xi)|^2 d\xi$$

$$\int_{-\infty}^{\infty} f(x) g^*(x)\, dx = \int_{-\infty}^{\infty} F(\xi) G^*(\xi)\, d\xi$$

$f(x)$	$F(\xi)$
Real (R)	Real part even (RE) Imaginary part odd (IO)
Imaginary (I)	RO,IE
RE,IO	R
RE,IE	I
RE	RE
RO	IO
IE	IE
IO	RO
Complex even (CE)	CE
CO	CO

Example use: Smoothing/Blurring

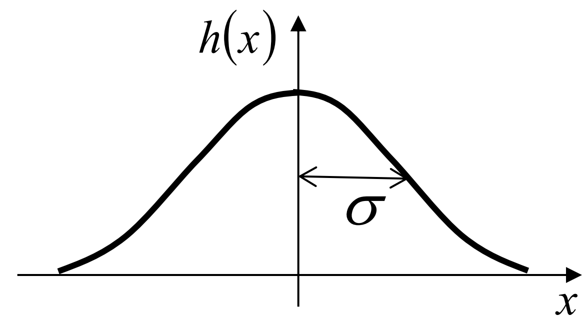
- We want a smoothed function of *f(x)*



$$g(x) = f(x) * h(x)$$

- Let us use a Gaussian kernel

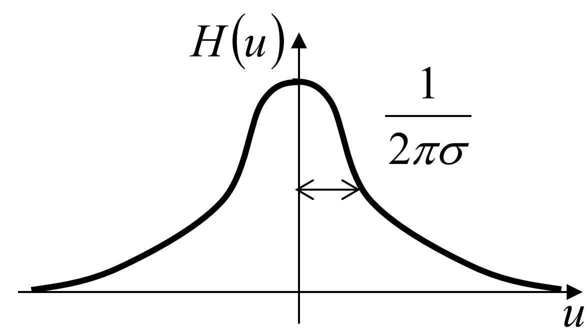
$$h(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2} \frac{x^2}{\sigma^2} \right]$$



- Then

$$H(u) = \exp\left[-\frac{1}{2} (2\pi u)^2 \sigma^2 \right]$$

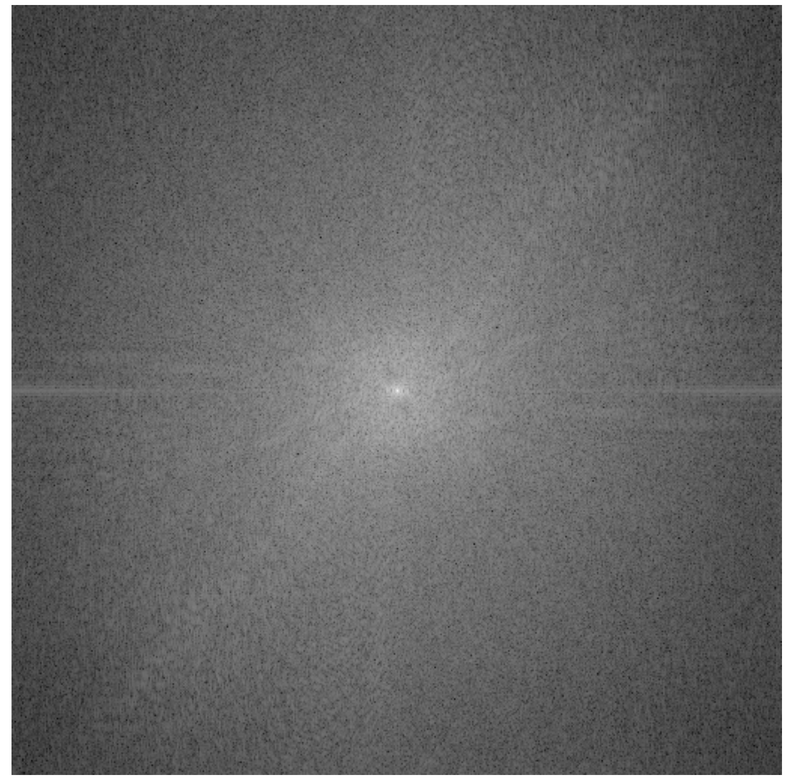
$$G(u) = F(u)H(u)$$



H(u) attenuates high frequencies in *F(u)* (Low-pass Filter)!

Image Processing in the Fourier Domain

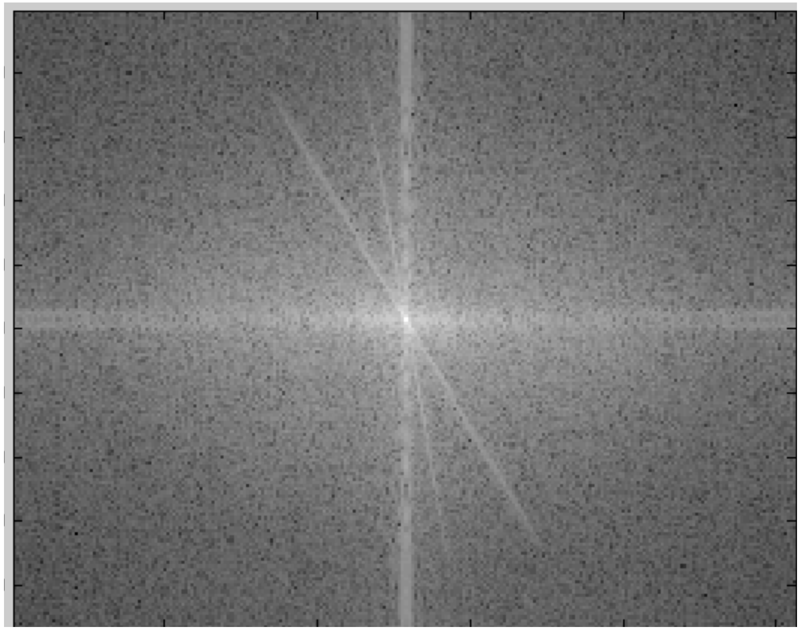
Magnitude of the FT



Does not look anything like what we have seen

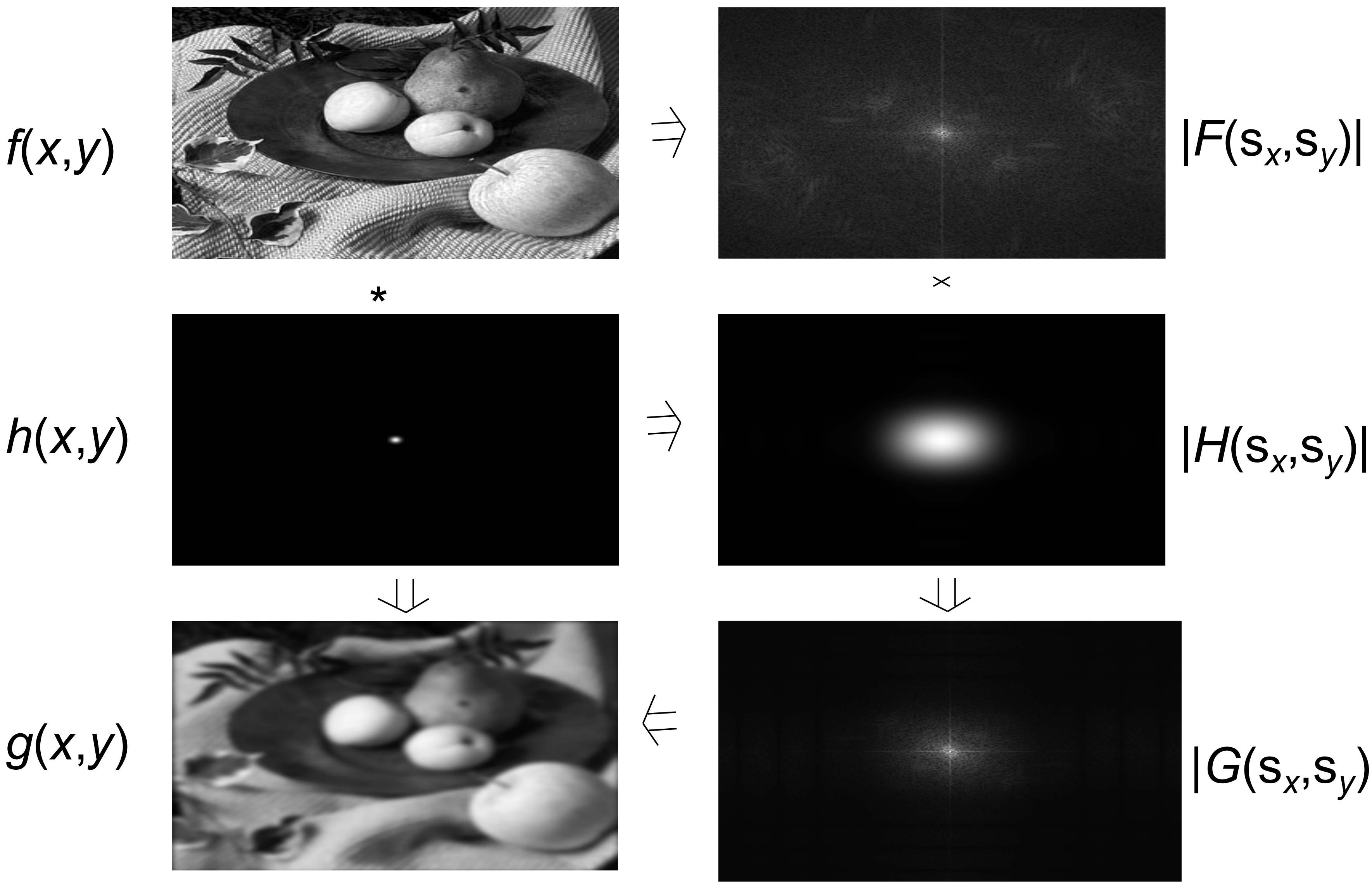
Image Processing in the Fourier Domain

Magnitude of the FT



Does not look anything like what we have seen

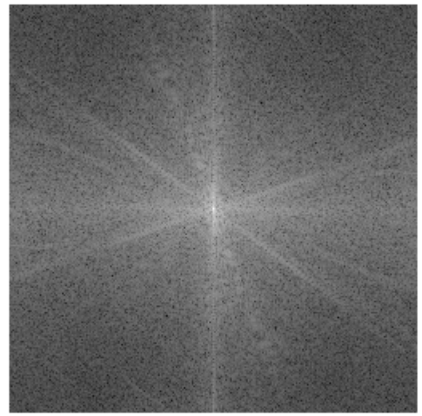
Convolution is Multiplication in Fourier Domain



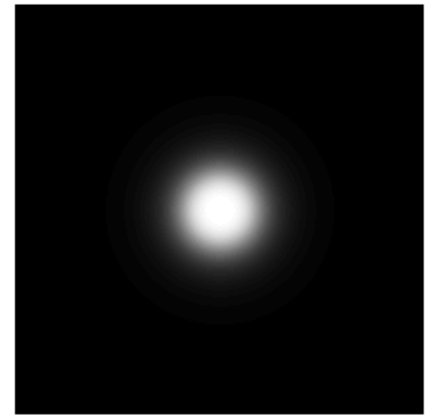
Low-pass Filtering

Original image

FFT of original image



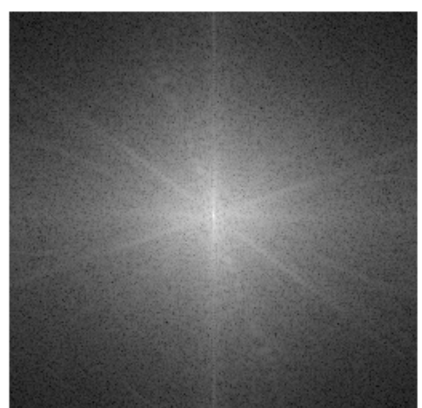
Low−pass filter



Let the low frequencies pass and eliminating the high frequencies.

Low−pass image

FFT of low−pass image

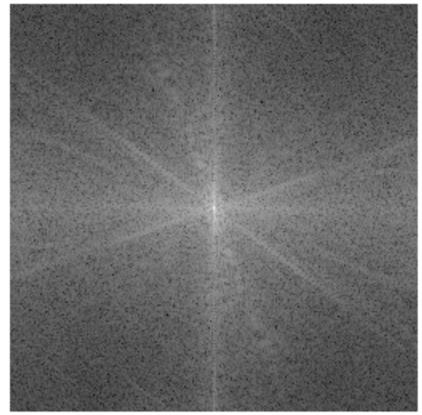


Generates image with overall shading, but not much detail

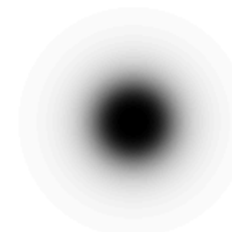
High-pass Filtering

Original image

FFT of original image



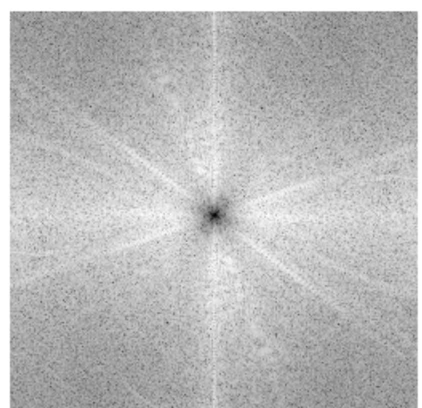
High-pass filter



Lets through the high frequencies (the detail), but eliminates the low frequencies (the overall shape). It acts like an edge enhancer.

High-pass image

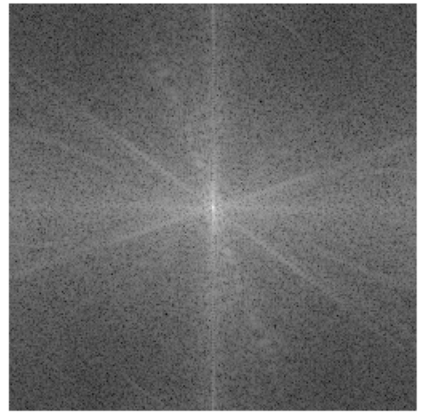
FFT of high-pass image



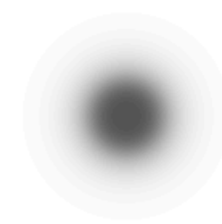
Boosting High Frequencies

Original image

FFT of original image

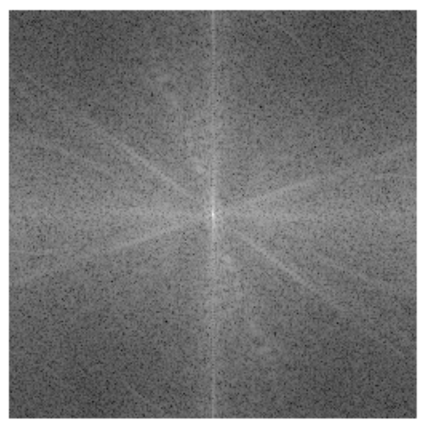


High–boost filter

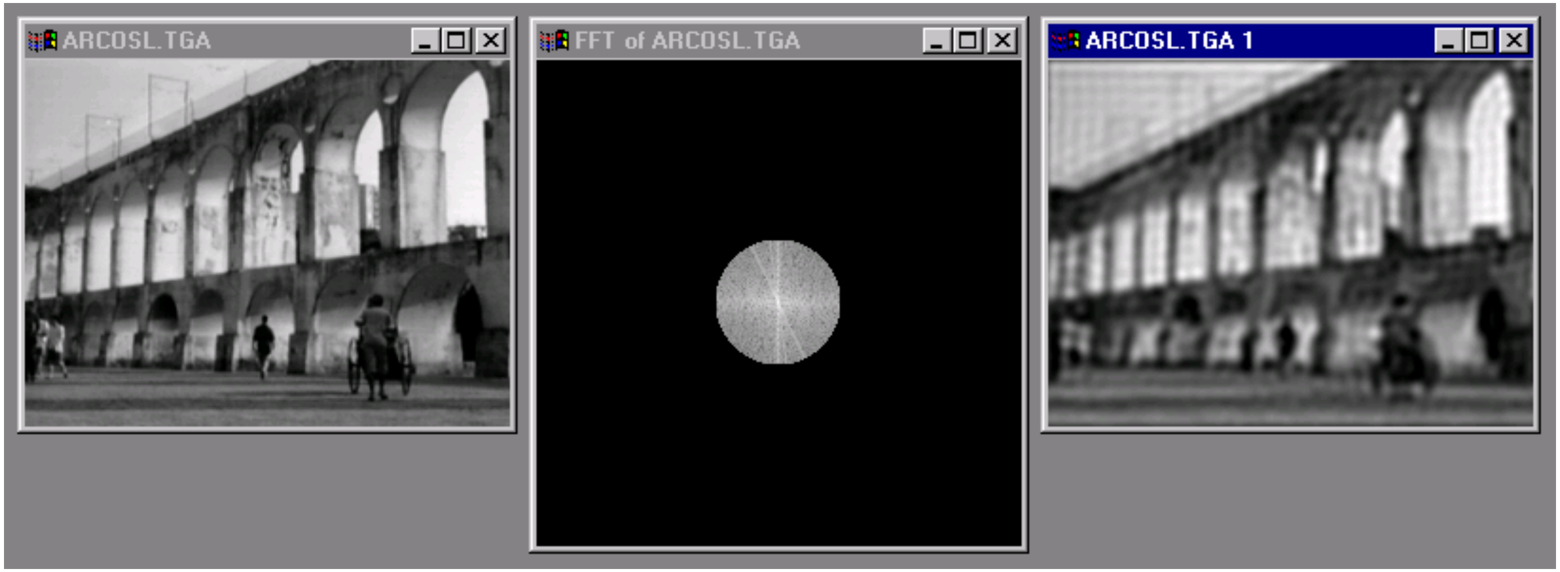


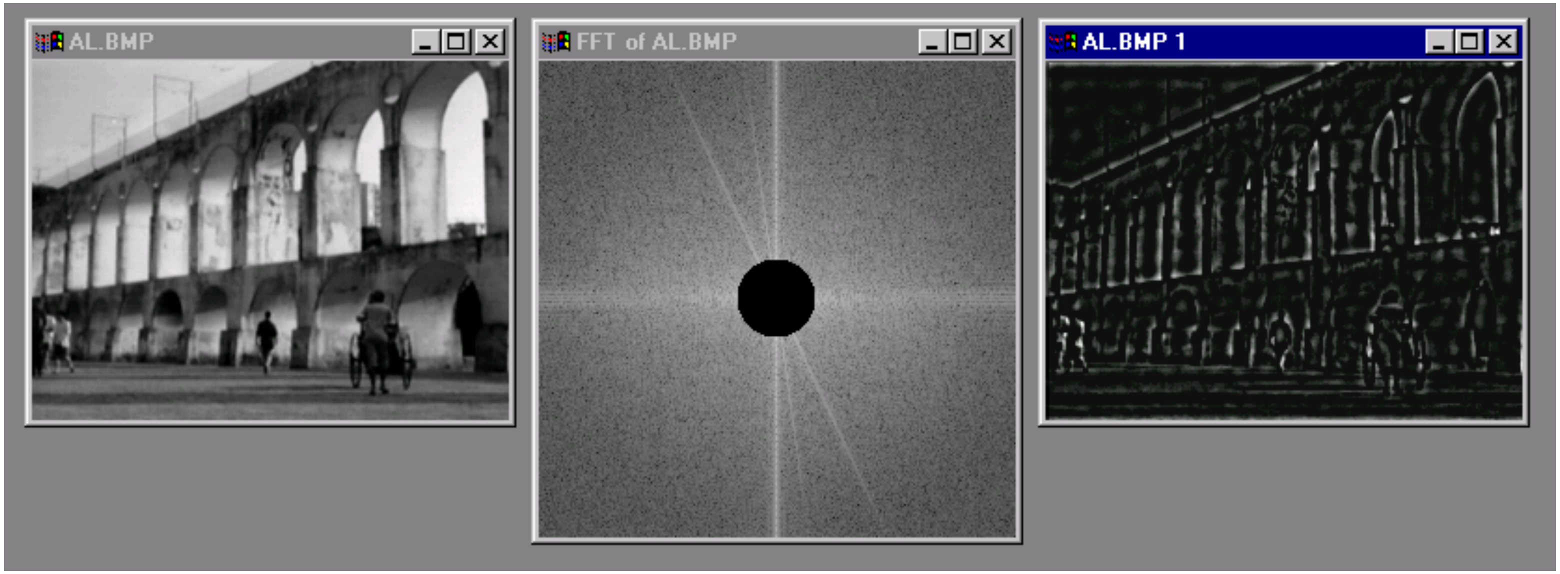
High boosted image

FFT of high boosted image



Most information at low frequencies!





Fun with Fourier Spectra

Next Class

- Image resampling and image pyramids
- Horn, Chapter 6