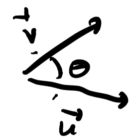
Aula Teórica 11 BOM DiA

Lembre que:

- Produto escalar (interno) de V^3

$$\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$



- Base O.N. de V^3:

  $\mathcal{F} = (\vec{f}_1, \vec{f}_2, \vec{f}_3)$ Base tal que

  - $\|\vec{f}_i\| = 1 \quad \forall i \in \{1,2,3\}$
  - $\underbrace{\langle \vec{f}_i, \vec{f}_j \rangle}_{\vec{f}_i \perp \vec{f}_j} = 0 \quad \forall i \neq j, \quad i,j \in \{1,2,3\}$

- Se $\mathcal{F} = (\vec{f}_1, \vec{f}_2, \vec{f}_3)$ é base O.N. e
  $\vec{u} = (u_1, u_2, u_3)_{\mathcal{F}}$, $\vec{v} = (v_1, v_2, v_3)_{\mathcal{F}}$

$\Rightarrow \quad \langle \vec{u}, \vec{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$

<u>Propriedades</u>: $\langle \cdot , \cdot \rangle : V^3 \times V^3 \longrightarrow \mathbb{R}$

(i) $\langle \cdot , \cdot \rangle$ é uma forma bilinear

- $\langle \vec{u} + \vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$
- $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
- $\langle \lambda \vec{u}, \vec{v} \rangle = \lambda \langle \vec{u}, \vec{v} \rangle = \langle \vec{u}, \lambda \vec{v} \rangle$

(ii) $\langle \cdot, \cdot \rangle$ é simétrico:

$$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$$

(iii) $\langle \cdot, \cdot \rangle$ é não-degenerado:

$$\forall \vec{u} \neq \vec{0} \;,\; \exists \vec{v} \in V^3 \text{ tal que } \langle \vec{u}, \vec{v} \rangle \neq 0$$

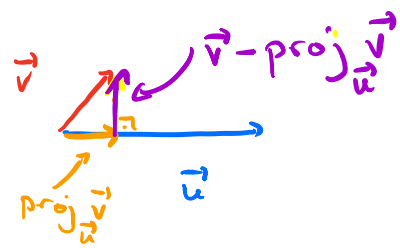
(iv) $\langle \cdot, \cdot \rangle$ é positivo definido

$$\langle \vec{u}, \vec{u} \rangle \geq 0 \quad \text{e} \quad \langle \vec{u}, \vec{u} \rangle = 0 \iff \vec{u} = \vec{0}$$

- Projeção ortogonal de $\vec{v}$ em $\vec{u} \neq \vec{0}$

→ $\text{proj}_{\vec{u}} \vec{v} \parallel \vec{u}$

→ $\vec{v} - \text{proj}_{\vec{u}} \vec{v} \perp \vec{u}$



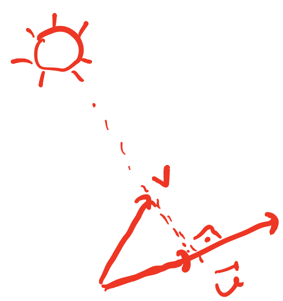
$$\boxed{\text{proj}_{\vec{u}} \vec{v} = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|^2} \vec{u}}$$

$$\|\text{proj}_{\vec{u}} \vec{v}\|^2 = \left\langle \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|^2} \vec{u}, \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|^2} \vec{u} \right\rangle$$

$$= \frac{\langle \vec{u}, v \rangle^2}{\|\vec{u}\|^4} \underbrace{\langle \vec{u}, \vec{u} \rangle}_{\|\vec{u}\|^2}$$

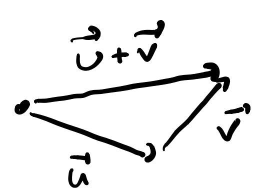
$$= \frac{\langle \vec{u}, \vec{v} \rangle^2}{\|\vec{u}\|^2}$$

$$\Rightarrow \|proj_{\vec{u}} \vec{v}\| = \frac{|\langle \vec{u}, \vec{v} \rangle|}{\|\vec{u}\|}$$



- Desigualdade Triangular

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$



Para demonstrar a desigualdade triangular eu vou utilizar a desigualde de Cauchy - Schwartz

<u>Desigualde de Cauchy - Schwartz</u>

$$\langle \vec{u}, \vec{v} \rangle^2 \leq \|\vec{u}\|^2 \|v\|^2$$

$$\left(|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \|\vec{v}\| \right)$$

Uma forma de provar a desigualdade de C-S é a seguinte: (No case específico do produto interno Euclideano de V^3)

$$\langle \vec{u}, \vec{v} \rangle^2 = \left(\|\vec{u}\| \|\vec{v}\| \cos\theta \right)^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 \cos^2\theta$$

$$\leq \|\vec{u}\|^2 \|\vec{v}\|^2$$

///

Mas quero apresentar uma demonstração que vale em casos muito mais gerais:

<u>Queremos</u>:

$$\langle \vec{u}, \vec{v} \rangle^2 \leq \|\vec{u}\|^2 \|\vec{v}\|^2$$

<u>Dem</u>: Considere o polinômio

$$\begin{aligned} p(t) &= \langle \vec{u} + t\vec{v}, \vec{u} + t\vec{v} \rangle \\ &= \langle \vec{u}, \vec{u} \rangle + t \langle \vec{v}, \vec{u} \rangle + t \langle \vec{u}, \vec{v} \rangle + t^2 \langle \vec{v}, \vec{v} \rangle \\ &= \|\vec{u}\|^2 + 2t \langle \vec{u}, \vec{v} \rangle + t^2 \|\vec{v}\|^2 \end{aligned}$$

$at^2 + bt + c$

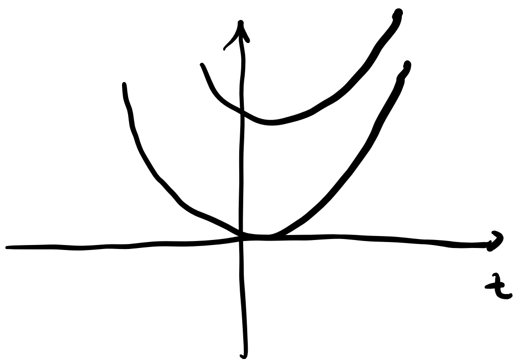
$a = \|\vec{v}\|^2$

$b = 2\langle \vec{u}, \vec{v} \rangle$

$c = \|\vec{u}\|^2$

Sabemos que $p(t) = \|\vec{u} + t\vec{v}\|^2 \geq 0$

$\forall t$



$p(t)$ tem no máximo uma raiz

logo $\Delta = b^2 - 4ac \leq 0$

pois raízes de $p(t)$ são

$$\frac{-b \pm \sqrt{\Delta}}{2a} \qquad \left(\begin{array}{l}\Delta > 0 \\ \Rightarrow 2 \text{ raízes}\end{array}\right)$$

Mas

$$\Delta = 4\langle \vec{u}, v\rangle^2 - 4\|\vec{u}\|^2\|\vec{v}\|^2 \leq 0$$

$$\Rightarrow \quad \langle \vec{u}, \vec{v}\rangle^2 \leq \|\vec{u}\|^2 \|\vec{v}\|^2 \qquad \blacksquare$$

Dem da desigualdade triangular

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

$$\|\vec{u}+\vec{v}\|^2 = \langle \vec{u}+\vec{v}, \vec{u}+\vec{v} \rangle = \|\vec{u}\|^2 + 2\langle \vec{u},\vec{v}\rangle + \|\vec{v}\|^2$$

$$\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 = \left(\|\vec{u}\| + \|\vec{v}\|\right)^2$$

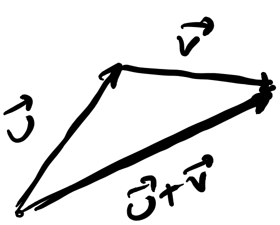
C-S

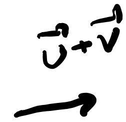
$$\Rightarrow \|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| \qquad \blacksquare$$

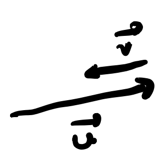
<u>OBS</u>:

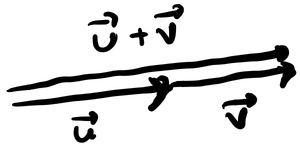
$$\|\vec{u}+\vec{v}\| = \|\vec{u}\| + \|\vec{v}\| \iff \vec{v} = \lambda\vec{u} \text{ ou } \vec{u} = \lambda\vec{v}$$

com $\lambda > 0$









Alguma Aplicações da Projeção Ortogonal

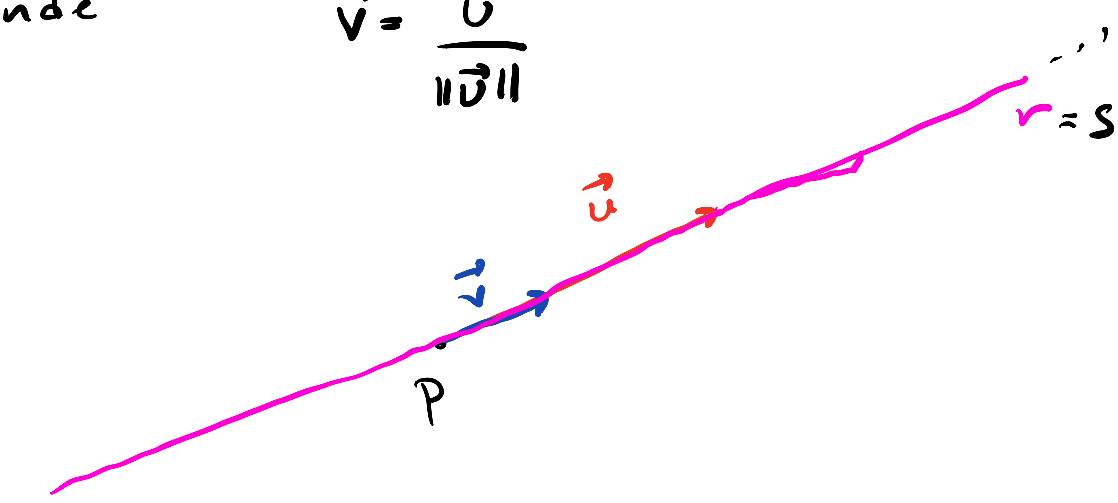
① Dado uma reta

$$r: \quad X = P + \lambda \vec{u}, \qquad \begin{array}{l}\lambda \in \mathbb{R} \\ \vec{u} \neq 0\end{array}$$

Podemos descrever a mesma reta usando um vetor diretor de norma 1

$$s: \quad X = P + \mu \vec{v}, \qquad \mu \in \mathbb{R}$$

onde $\quad \vec{v} = \dfrac{\vec{u}}{\|\vec{u}\|}$



Vamos ver que $r = s$

$$X \in s \iff X = P + \mu \vec{v} \quad \text{para algum } \mu \in \mathbb{R}$$

$$\iff X = P + \mu \frac{\vec{u}}{\|\vec{u}\|} \quad \text{para algum } \mu \in \mathbb{R}$$

$$\iff X = P + \left(\frac{\mu}{\|\vec{v}\|}\right) \vec{u} \quad \text{para algum } \mu \in \mathbb{R}$$

$$\iff X = P + \lambda \vec{u} \quad \text{para algum } \lambda \in \mathbb{R}.$$

$\lambda = \frac{\mu}{\|\vec{v}\|}$

$$\iff X \in r$$

2 Seja

$$\pi : X = P + \lambda \vec{u} + \mu \vec{v}, \quad \lambda, \mu \in \mathbb{R}$$

$\vec{u}, \vec{v}$ L.I.

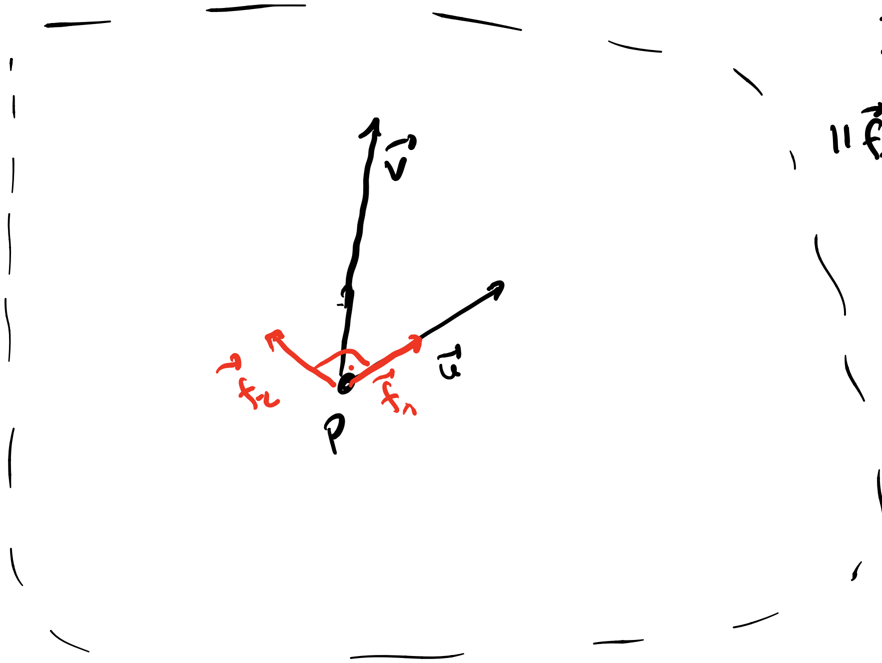
(π é um plano)

Podemos descrever π como

$$\pi': X = P + \alpha \vec{f_1} + \beta \vec{f_2}, \quad \alpha, \beta \in \mathbb{R}$$

$\vec{f_1} \perp \vec{f_2}$

$\|\vec{f_1}\| = \|\vec{f_2}\| = 1$



$$\vec{f_1} = \frac{\vec{u}}{\|\vec{u}\|}$$

$$\vec{f_2} = \frac{\vec{v} - \text{proj}_{\vec{f_1}} \vec{v}}{\|\vec{v} - \text{proj}_{\vec{f_1}} \vec{v}\|}$$

Note que

(i) $\|\vec{f}_1\| = \|\vec{f}_2\| = 1$

(ii) $\vec{f}_1 \perp \vec{f}_2$

Vamos fazer a conta: $\left(\text{Notação } \lambda = \dfrac{1}{\|\vec{v} - \text{proj}_{\vec{f}_1}\vec{v}\|}\right)$

$$\langle \vec{f}_1, \vec{f}_2 \rangle = \langle \vec{f}_1, \lambda(\vec{v} - \text{proj}_{\vec{f}_1}\vec{v}) \rangle$$

$$= \lambda \langle \vec{f}_1, \vec{v} - \text{proj}_{\vec{f}_1}\vec{v} \rangle =$$

$$= \lambda \left(\langle \vec{f}_1, \vec{v} \rangle - \langle \vec{f}_1, \text{proj}_{\vec{f}_1}\vec{v} \rangle \right)$$

$$= \lambda \left(\langle \vec{f}_1, \vec{v} \rangle - \left\langle \vec{f}_1, \frac{\langle \vec{f}_1, \vec{v} \rangle}{\|\vec{f}_1\|^2} \vec{f}_1 \right\rangle \right)$$

$$= \lambda \left(\langle \vec{f}_1, \vec{v} \rangle - \frac{\langle \vec{f}_1, \vec{v} \rangle}{\cancel{\|\vec{f}_1\|^2}} \cancel{\langle \vec{f}_1, \vec{f}_1 \rangle} \right)$$

$$= \lambda \left(\langle \vec{f}_1, \vec{v} \rangle - \langle \vec{f}_1, \vec{v} \rangle \right) = \lambda \cdot 0 = 0 \quad ///$$

Precisamos mostrar que

$$\pi = \pi'$$

(Exercício que farei no começo da aula de sexta que vem)

$$\left\{ \begin{array}{c} P + \lambda \vec{u} + \mu \vec{v} \\ \lambda, \mu \in \mathbb{R} \end{array} \right\} \overset{?}{=} \left\{ \begin{array}{c} P + \alpha \vec{f_1} + \beta \vec{f_2} \\ \alpha, \beta \in \mathbb{R} \end{array} \right\}$$