Aula Teórica 10

Bom Dia

Lembre que:

- **Produto Escalar** (ou produto interno) de V^3:

$\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ (COORDENADAS NA BASE CANÔNICA)

$$\boxed{\langle \vec{u}, \vec{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 \in \mathbb{R}}$$

- Base O.N. de V^3:

$\mathcal{F} = (\vec{f_1}, \vec{f_2}, \vec{f_3})$ Base tal que

  - $\| \vec{f_i} \| = 1 \quad \forall i \in \{1, 2, 3\}$
  - $\underbrace{\langle \vec{f_i}, \vec{f_j} \rangle}_{\vec{f_i} \perp \vec{f_j}} = 0 \quad \forall i \neq j , \quad i, j \in \{1, 2, 3\}$

- Se $\mathcal{F} = (\vec{f_1}, \vec{f_2}, \vec{f_3})$ é base O.N. e

$\vec{u} = (u_1, u_2, u_3)_{\mathcal{F}}$, $\vec{v} = (v_1, v_2, v_3)_{\mathcal{F}}$

$$\Rightarrow \quad \langle \vec{u}, \vec{v} \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Propriedades: $\langle \cdot , \cdot \rangle : V^3 \times V^3 \longrightarrow \mathbb{R}$

(i) $\langle \cdot , \cdot \rangle$ é uma forma bilinear

- $\langle \vec{u}+\vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$
- $\langle \vec{u}, \vec{v}+\vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
- $\langle \lambda\vec{u}, \vec{v} \rangle = \lambda \langle \vec{u}, \vec{v} \rangle = \langle \vec{u}, \lambda\vec{v} \rangle$

(ii) $\langle \cdot , \cdot \rangle$ é simétrico:

$$\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$$

(iii) $\langle \cdot , \cdot \rangle$ é não-degenerado:

$\forall \vec{u} \neq \vec{0}$, $\exists \vec{v} \in V^3$ tal que $\langle \vec{u}, \vec{v} \rangle \neq 0$

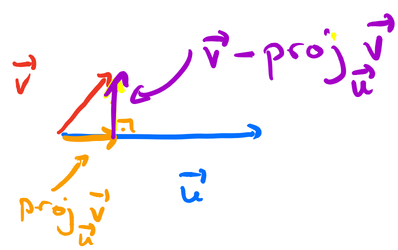
(iv) $\langle \cdot , \cdot \rangle$ é positivo definido

$\langle \vec{u}, \vec{u} \rangle \geq 0$ e $\langle \vec{u}, \vec{u} \rangle = 0 \iff \vec{u} = \vec{0}$

- Projeção ortogonal de $\vec{v}$ em $\vec{u} \neq \vec{0}$

$\rightarrow \operatorname{proj}_{\vec{u}} \vec{v} \parallel \vec{u}$

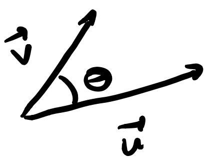
$\rightarrow \vec{v} - \operatorname{proj}_{\vec{u}} \vec{v} \perp \vec{u}$



$$\boxed{\operatorname{proj}_{\vec{u}} \vec{v} = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|^2} \vec{u}}$$

Definição: O ângulo entre dois vetores $\vec{u}$ e $\vec{v}$ é definido da seguinte forma:

Se $\vec{u} = \vec{PQ}$, $\vec{v} = \vec{PR}$ então θ é o menor ângulo entre os segmentos de reta $\overline{PQ}$ e $\overline{PR}$



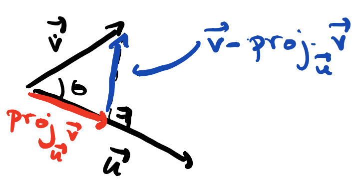
$(\theta \in [0, \pi])$

Teorema: $\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$

Dem: Caso 0: $\vec{u} = 0$ ou $\vec{v} = 0 \Rightarrow$ o.k.

Caso 1: $0 \leq \theta \leq \frac{\pi}{2}$ $\quad (\langle \vec{u}, \vec{v} \rangle \geq 0)$

$\vec{u} \neq \vec{0}$ e $\vec{v} \neq \vec{0}$

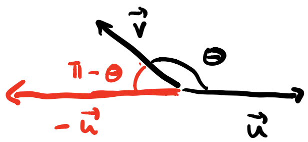


$$\cos\theta = \frac{\|\text{proj}_{\vec{u}} \vec{v}\|}{\|\vec{v}\|} = \frac{\left\| \langle \vec{u}, \vec{v} \rangle / \|\vec{u}\|^2 \ \vec{u} \right\|}{\|\vec{v}\|} = \frac{|\langle \vec{u}, \vec{v} \rangle|}{\|\vec{u}\| \, \|\vec{v}\|}$$

$$\Rightarrow \langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

Caso 2 : $\quad \pi/2 < \theta \leq \pi \qquad (\langle \vec{u}, \vec{v} \rangle < 0)$

$\vec{u} \neq 0$ e $\vec{v} \neq 0$



$$\cos(\pi - \theta) = \frac{\| \text{proj}_{-\vec{u}} \vec{v} \|}{\| \vec{v} \|} = \frac{\left\| \frac{\langle -\vec{u}, \vec{v} \rangle}{\| \vec{u} \|^2} \, -\vec{u} \right\|}{\| \vec{v} \|}$$

$$= \frac{| \langle \vec{u}, \vec{v} \rangle |}{\| \vec{u} \| \, \| \vec{v} \|} = - \frac{\langle \vec{u}, \vec{v} \rangle}{\| \vec{u} \| \| \vec{v} \|}$$

Logo, $\quad \langle \vec{u}, \vec{v} \rangle = - \| \vec{u} \| \| \vec{v} \| \cos(\pi - \theta) = \| \vec{u} \| \| \vec{v} \| \cos \theta$ ∎

Por definição, dado $\vec{u} \neq \vec{0}$ a projeção de $\vec{v}$ em $\vec{u}$ é o vetor $\text{proj}_{\vec{u}} \vec{v}$ que satisfaz as seguintes propriedades:

1. $\text{proj}_{\vec{u}} \vec{v} \parallel \vec{u}$
2. $\vec{v} - \text{proj}_{\vec{u}} \vec{v} \perp \vec{u}$

De 1 tiramos que

$$\mathrm{proj}_{\vec{u}}\vec{v} = \lambda \vec{u}$$

Queremos calcular λ.

Usando 2

$$\langle \vec{v} - \lambda\vec{u}, \vec{u} \rangle = 0$$

$$\Rightarrow \langle \vec{v}, \vec{u} \rangle + \langle -\lambda\vec{u}, \vec{u} \rangle = 0$$

$$\Rightarrow \langle \vec{v}, \vec{u} \rangle - \lambda \langle \vec{u}, \vec{u} \rangle = 0$$

$$\Rightarrow \lambda \|\vec{u}\|^2 = \langle \vec{v}, \vec{u} \rangle$$

$$\Rightarrow \lambda = \frac{\langle \vec{v}, \vec{u} \rangle}{\|\vec{u}\|^2}$$

Suponha $\vec{v}, \vec{u}$ L.D. $\quad \vec{u} \neq \vec{0}$

$$\Rightarrow \vec{v} = a\vec{u}$$

$$\Rightarrow \quad \text{proj}_{\vec{u}} \vec{v} = \frac{\langle \vec{v}, \vec{u} \rangle}{\|\vec{u}\|^2} \vec{u}$$

$$= \frac{\langle a\vec{u}, \vec{u} \rangle \vec{u}}{\|\vec{u}\|^2}$$

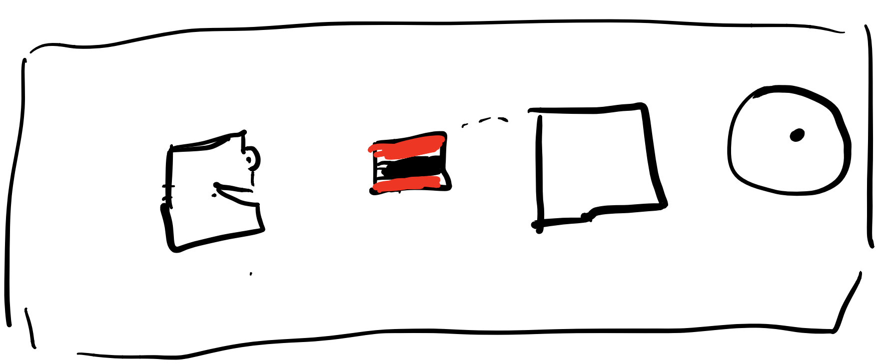
$$= a \frac{\cancel{\langle \vec{u}, \vec{u} \rangle}}{\cancel{\|\vec{u}\|^2}} \vec{u}$$

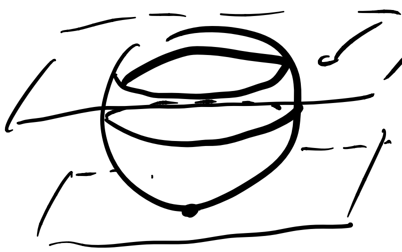
$$= a\vec{u}$$

$$= \vec{v}$$

Visualização de $\mathbb{R}^4$? Flat land

- Geometry, Relativity and the fourth dimension (R. Rucker)

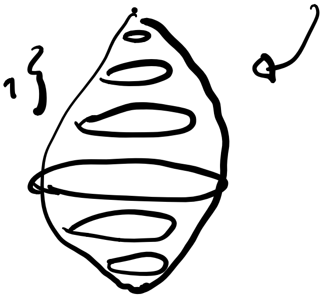




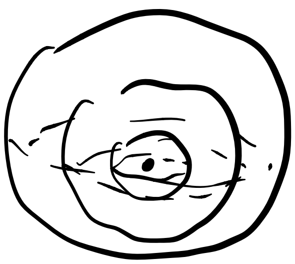
$S^2 \subseteq \mathbb{R}^3$

$\parallel$

$\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}$



$$S^3 = \{(x,y,z,w)\in\mathbb{R}^4 \mid x^2+y^2+z^2+\underbrace{w^2}_{=t}=1\} \subset \mathbb{R}^4$$



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