MAP 2110 – Modelagem e Matemática

1º Semestre - 2023

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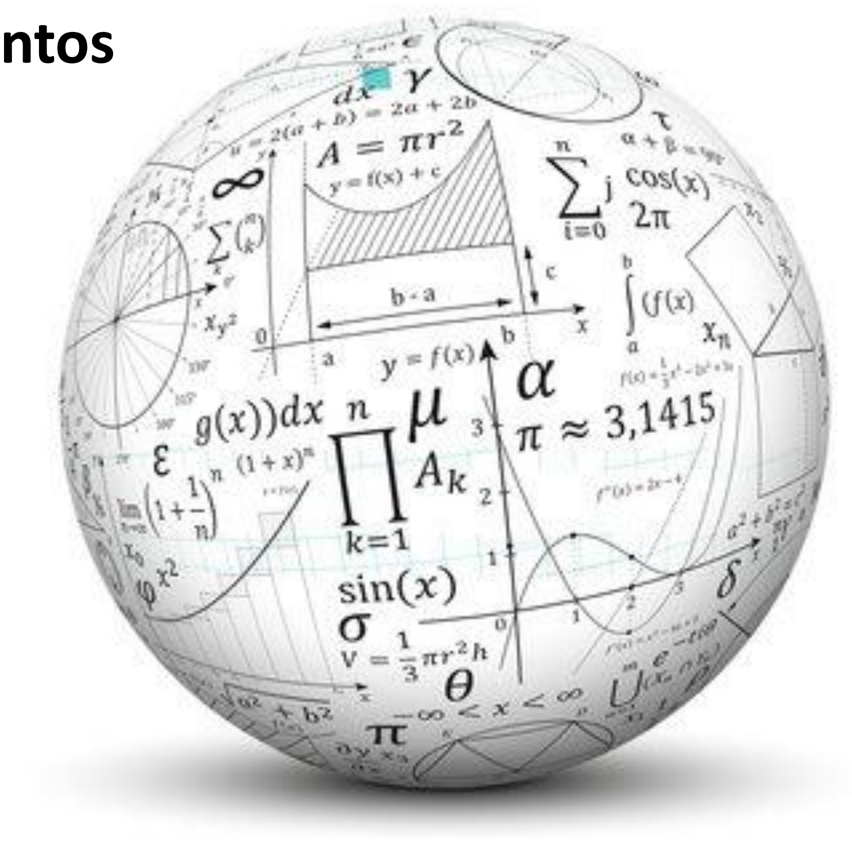


4. Suppose the spotted owls' primary food source is a single prey: mice. An ecologist wishes to predict the population levels of spotted owls and mice in a wildlife sanctuary. Letting M_n represent the mouse population after n years and O_n the predator owl population, the ecologist has suggested the model.

$$M_{n+1} = 1.2M_n - 0.001O_n M_n$$
$$O_{n+1} = 0.7O_n + 0.002O_n M_n$$

The ecologist wants to know whether the two species can coexist in the habitat and whether the outcome is sensitive to the starting populations. Find the equilibrium values of the dynamical system for this predator-prey model.

a. Compare the signs of the coefficients of the preceding model with the signs of the coefficients of the owls–hawks model in Example 3. Explain the sign of each of the four coefficients $1.2, -0.001, 0.7$, and 0.002 in terms of the predator–prey relationship being modeled.

b. Test the initial populations in the following table and predict the long-term outcome:

	Owls	Mice
Case A	150	200
Case B	150	300
Case C	100	200
Case D	10	20

c. Now experiment with different values for the coefficients using the starting values given. Then try different starting values. What is the long-term behavior? Do your experimental results indicate that the model is sensitive to the coefficients? Is it sensitive to the starting values?

$$M_{n+1} = k_1 M_n - k_3 O_n M_n$$

$$O_{n+1} = k_2 O_n + k_4 O_n M_n$$

No exemplo do livro os coeficientes tem os valores: k_1=1.2; $k_2 = 0.7$; k_3=0.001; $k_4 = 0.002$

O equilíbrio é alcançado quando : $M_{n+1} = M_n = M^*$;$O_{n+1} = O_n = O^*$

Substituindo no sistema:

$$M^* = k_1 M^* - k_3 O^* M^*$$

$$O^* = k_2 O^* + k_4 O^* M^*$$

$\Rightarrow$

$$0 = [(k_1 - 1) - k_3 O^*]M^*$$

$$0 = [(k_2 - 1) + k_4 M]O^*$$

$\Rightarrow$

Pontos de Equilíbrio:

(M^*, O^*) = (0,0)

(M^*, O^*) = ($\frac{(k_2-1)}{-k_4}$, $\frac{(k_1-1)}{k_3}$)

Dessa forma o equilíbrio não-nulo é dado por:

(M^*, O^*) = ($\frac{(0.7-1)}{-0.002}$, $\frac{(1.2-1)}{0.001}$) = ($\frac{-0.3}{-0.002}$, $\frac{0.2}{0.001}$)= (150,200)

$\Rightarrow$ (M^*, O^*) = (150,200)

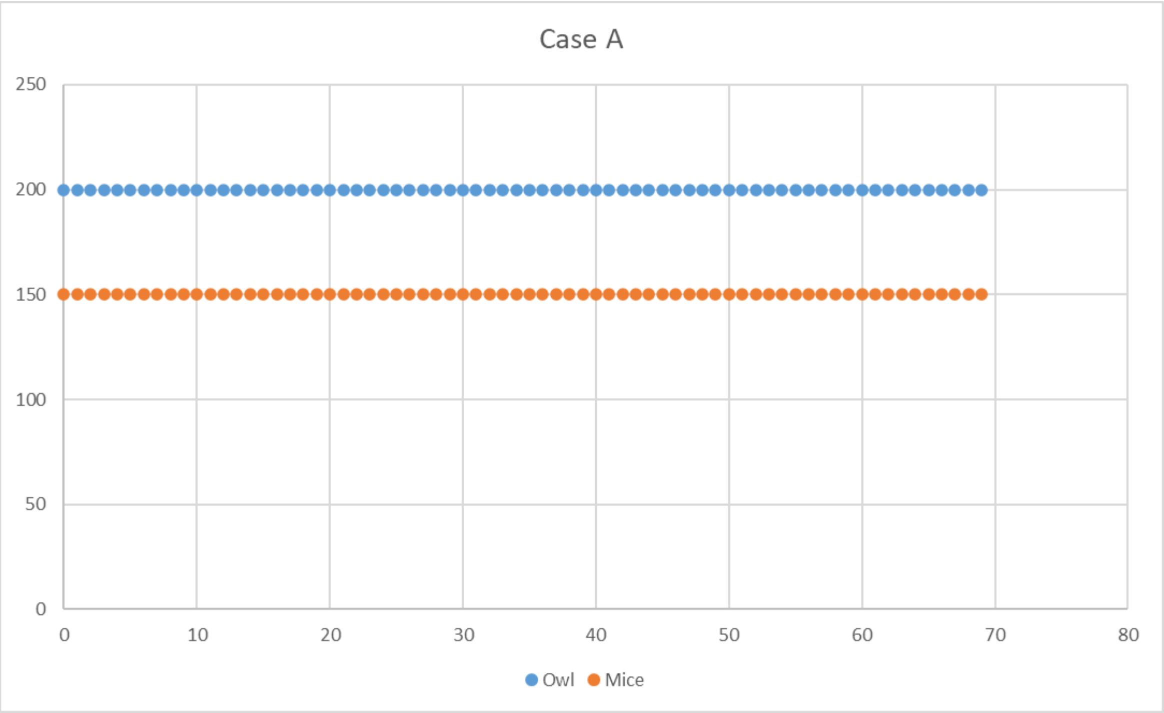
b. Test the initial populations in the following table and predict the long-term outcome:

	Owls	Mice
Case A	150	200
Case B	150	300
Case C	100	200
Case D	10	20

(M^*, O^*) = (150,200)

ERRO DE DIGITAÇÃO DA REFERÊNCIA !

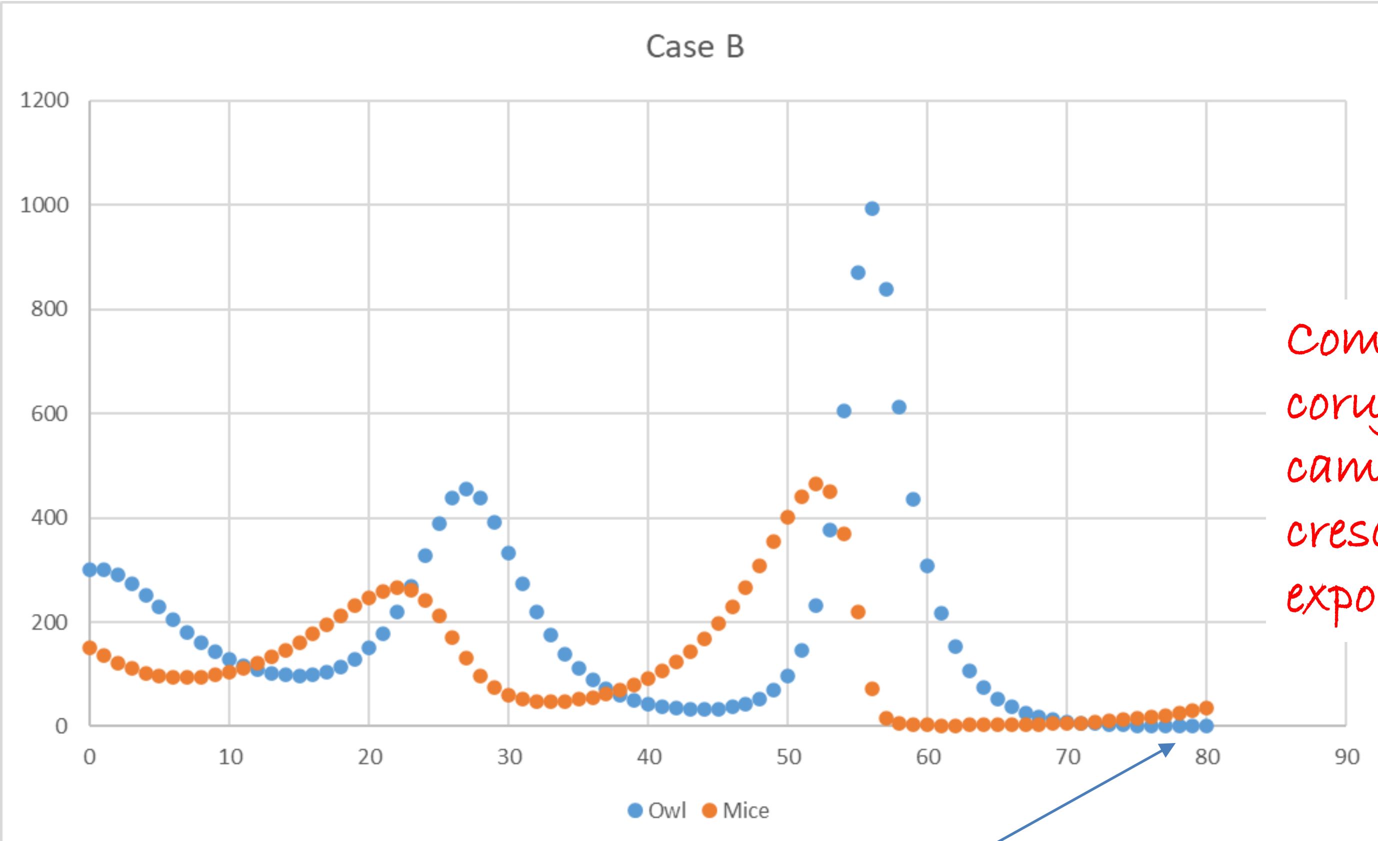
Case	A	
n	Owl	Mice
0	200	150



O ponto de equilíbrio é confirmado pelo experimento numérico

Invertendo os valores iniciais para corrigir o erro no enunciado:

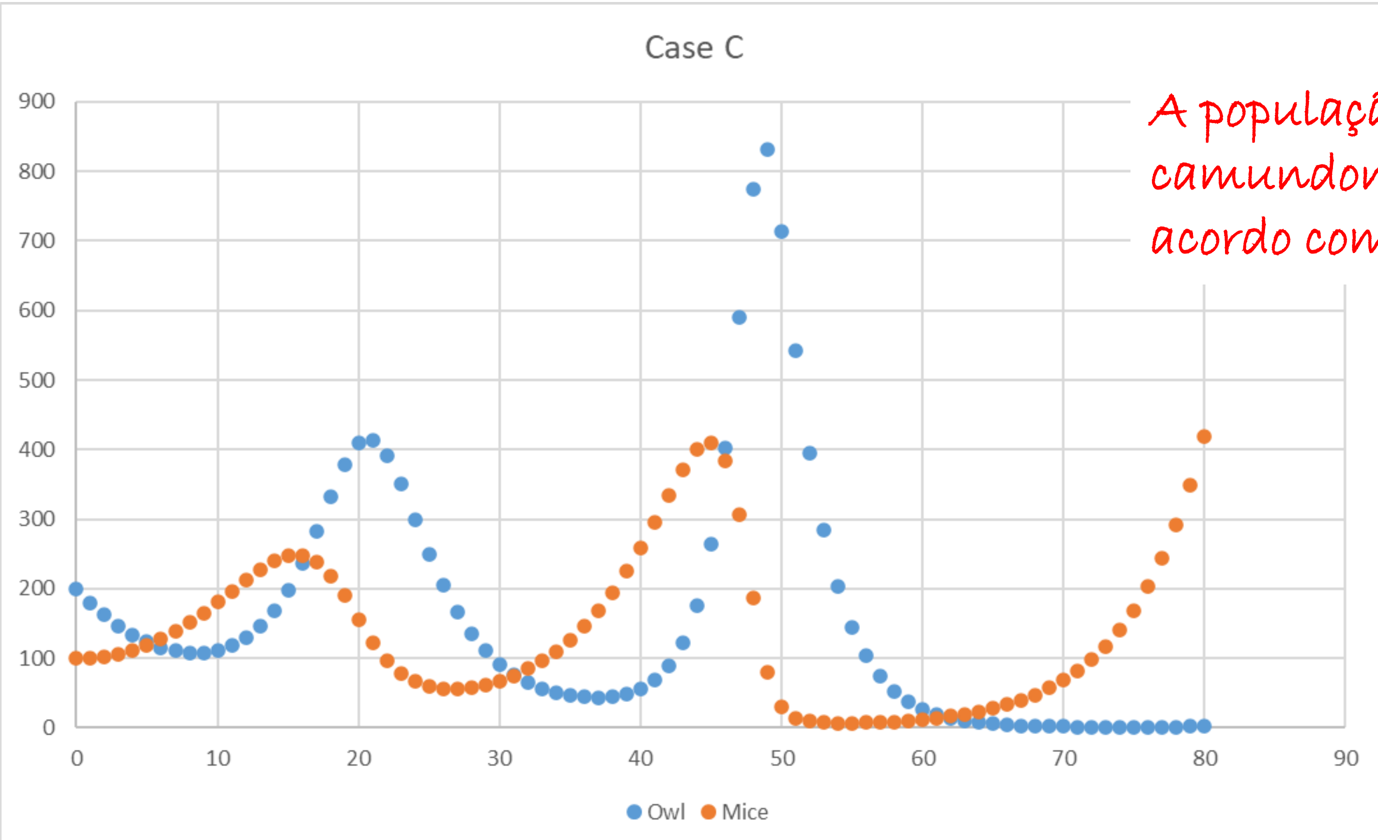
Case	B	
n	Owl	Mice
0	300	150



Como não há mais corujas a população de camundongos crescerá exponencialmente

No passo 78 a população de corujas ficou menor que 1 e foi forçada manualmente para 0 -> representando a extinção das corujas

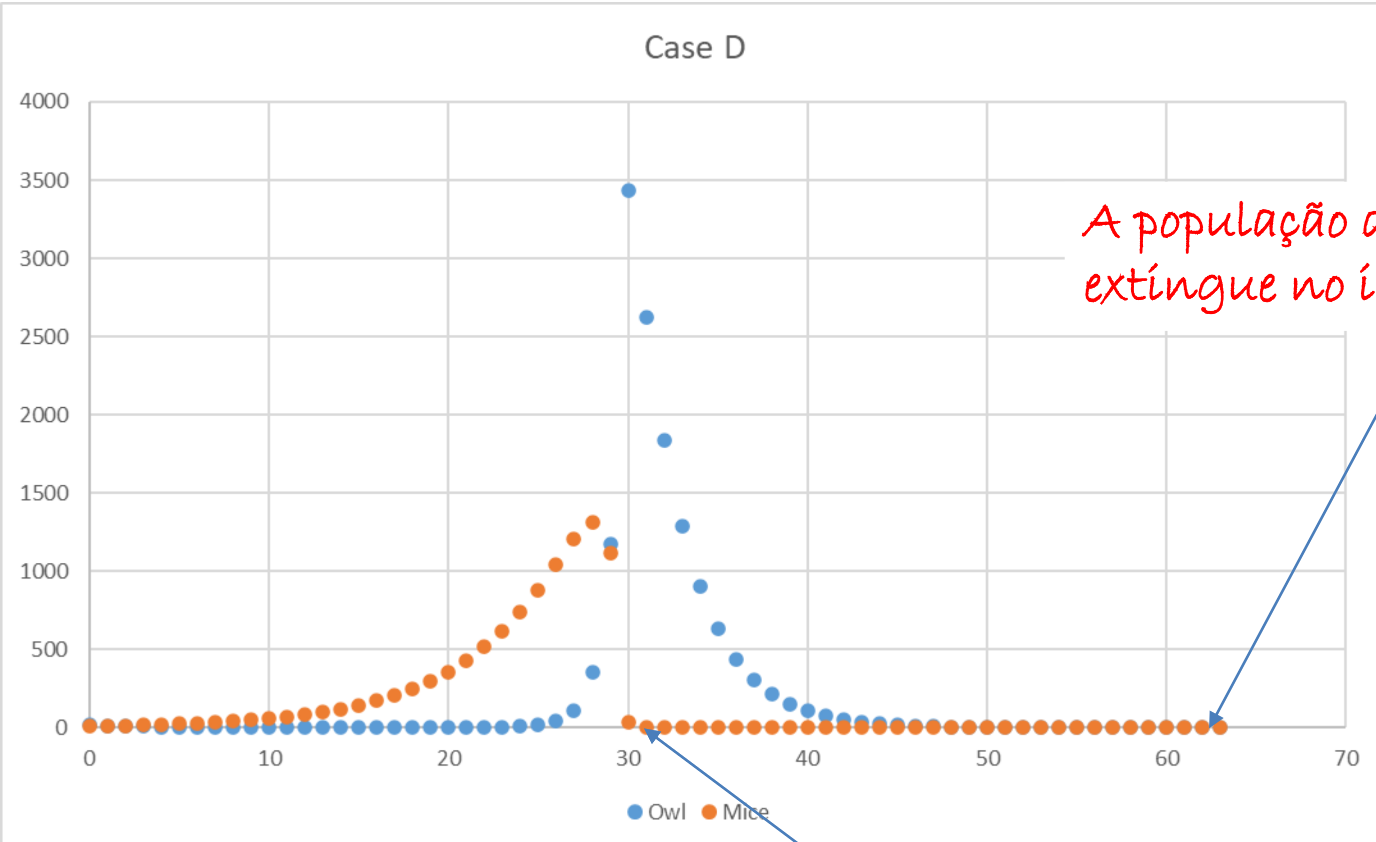
Case	C	
n	Owl	Mice
0	200	100



A população de camundongos cresce de acordo com a taxa 1.2

A população de corujas caiu a apenas 1 indivíduo no instante 74 mas depois começa subir.

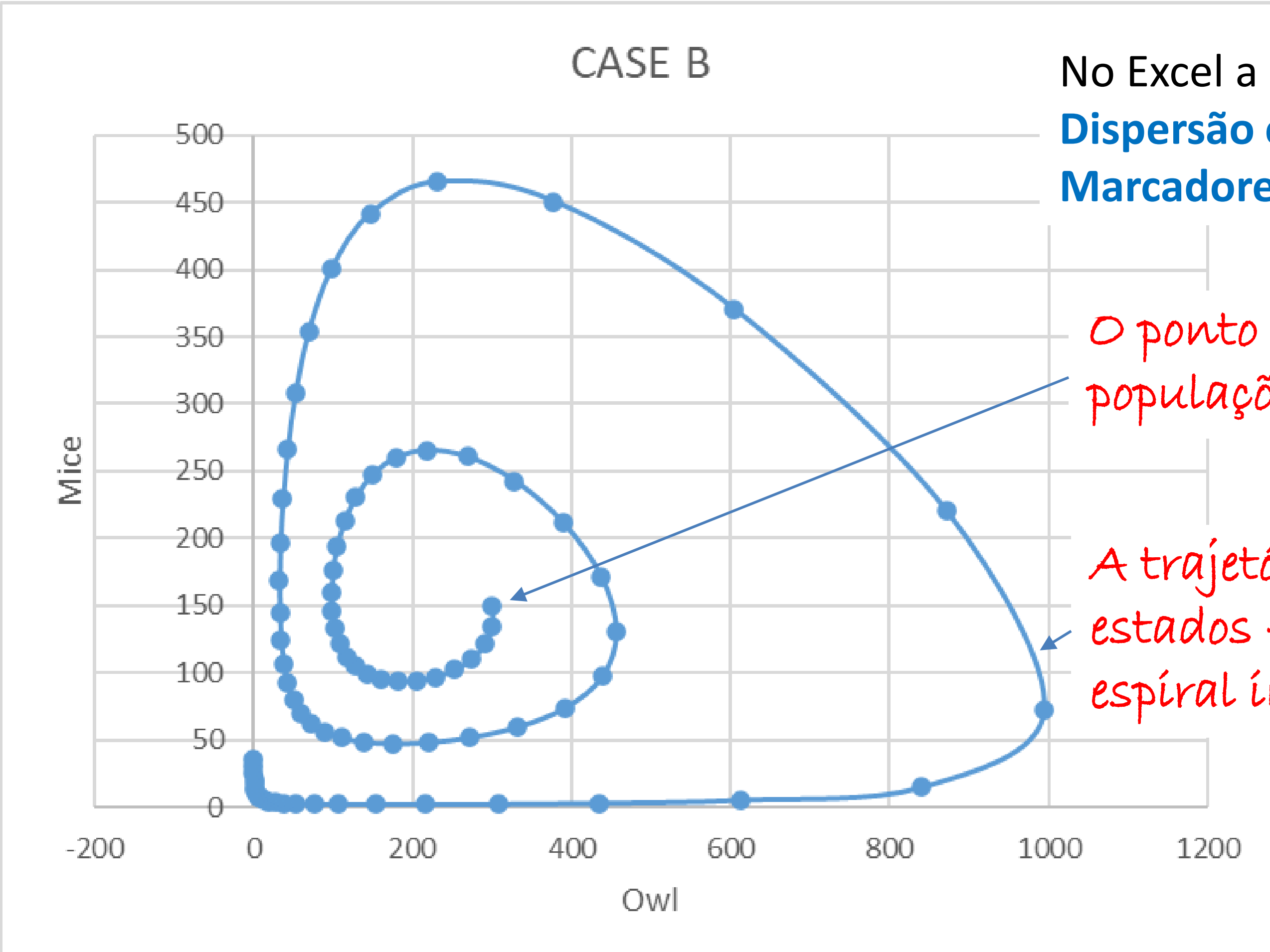
Case	D	
n	Owl	Mice
0	20	10



A população de corujas se extingue no instante 62.

No passo 31 a população de camundongos ficou negativa e foi forçada manualmente para 0 -> representando a extinção das camundongos.

No caso de sistemas bidimensionais, como o do exemplo, um recurso de visualização útil é plotar os dois estados no plano. Esse gráfico tem o nome de retrato de fase.

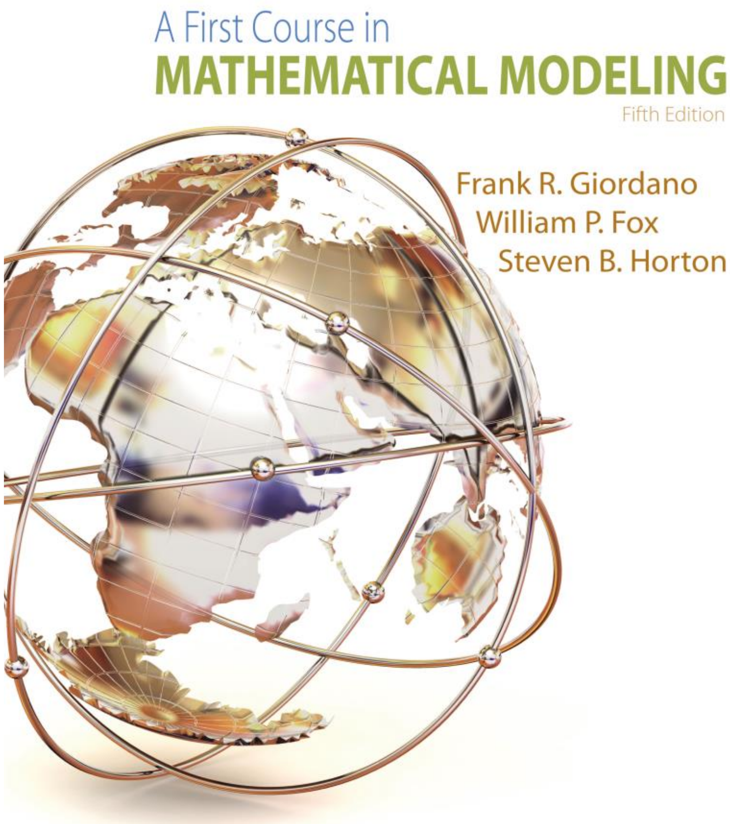


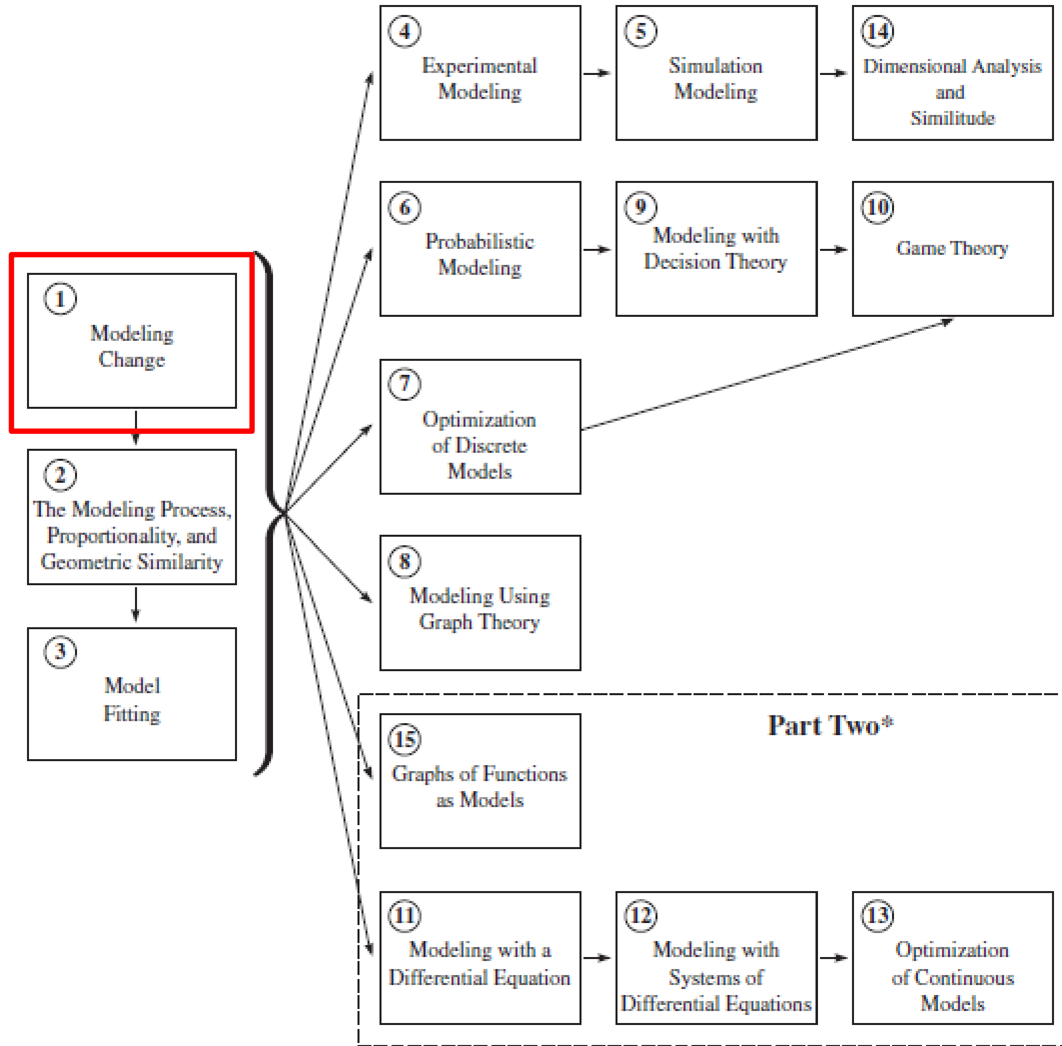
No Excel a opção é gráfico **Dispersão com Linhas Suaves e Marcadores**

O ponto de partida são as populações iniciais

A trajetória é a sequência de estados – nesse caso é uma espiral instável

A extinção das corujas irá causar a explosão da população de camundongos





*Part Two requires single-variable calculus as a corequisite.

EXAMPLE 4 *Travelers' Tendencies at a Regional Airport*

Consider a regional airport that is supported by three major airlines: American Airlines, United Airlines, and US Airways, each flying out to respective hubs. We survey the weekly local business travelers and find 75% of those who traveled on US Airways traveled again on US Airways, 5% switched to fly United, and 20% switched to fly American. Of those who traveled on United, 60% traveled again on United, but 20% switched to US Airways, and 20% switched to American. Of those who traveled on American, only 40% remained with American, 40% switched to US Airways, and 20% switched to United. We assume these tendencies continue week to week and that no additional local business travelers enter or leave the system. These tendencies are depicted in Figure 1.29.

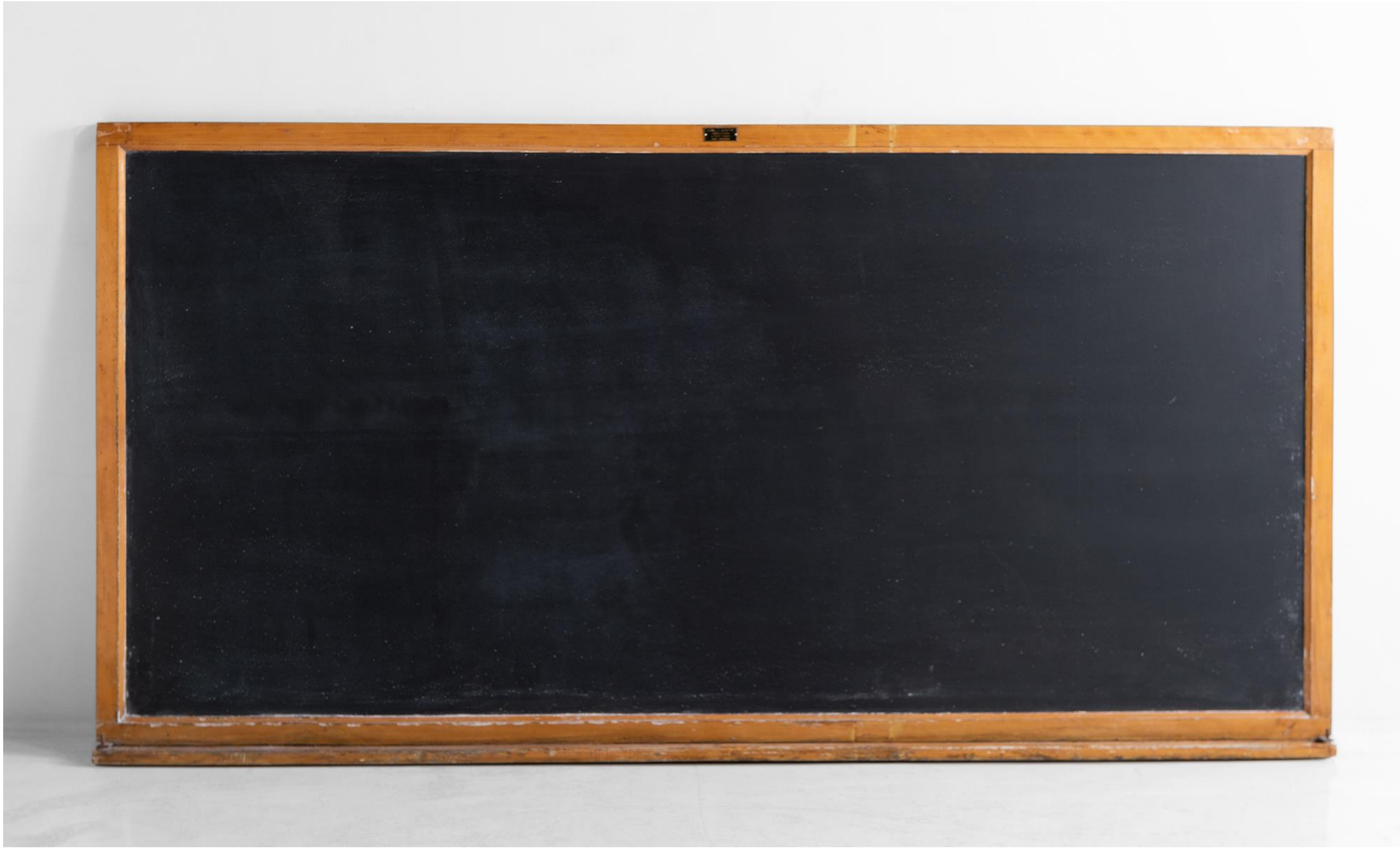
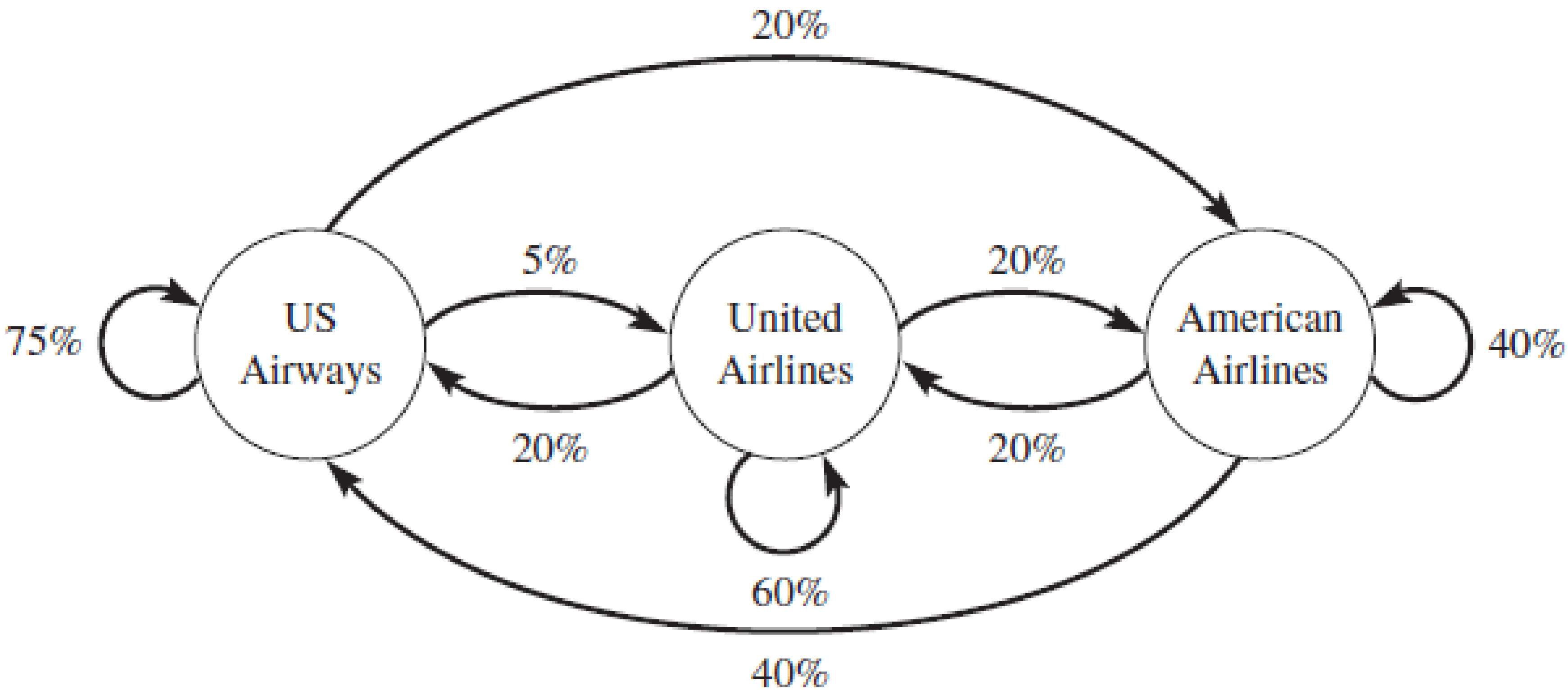
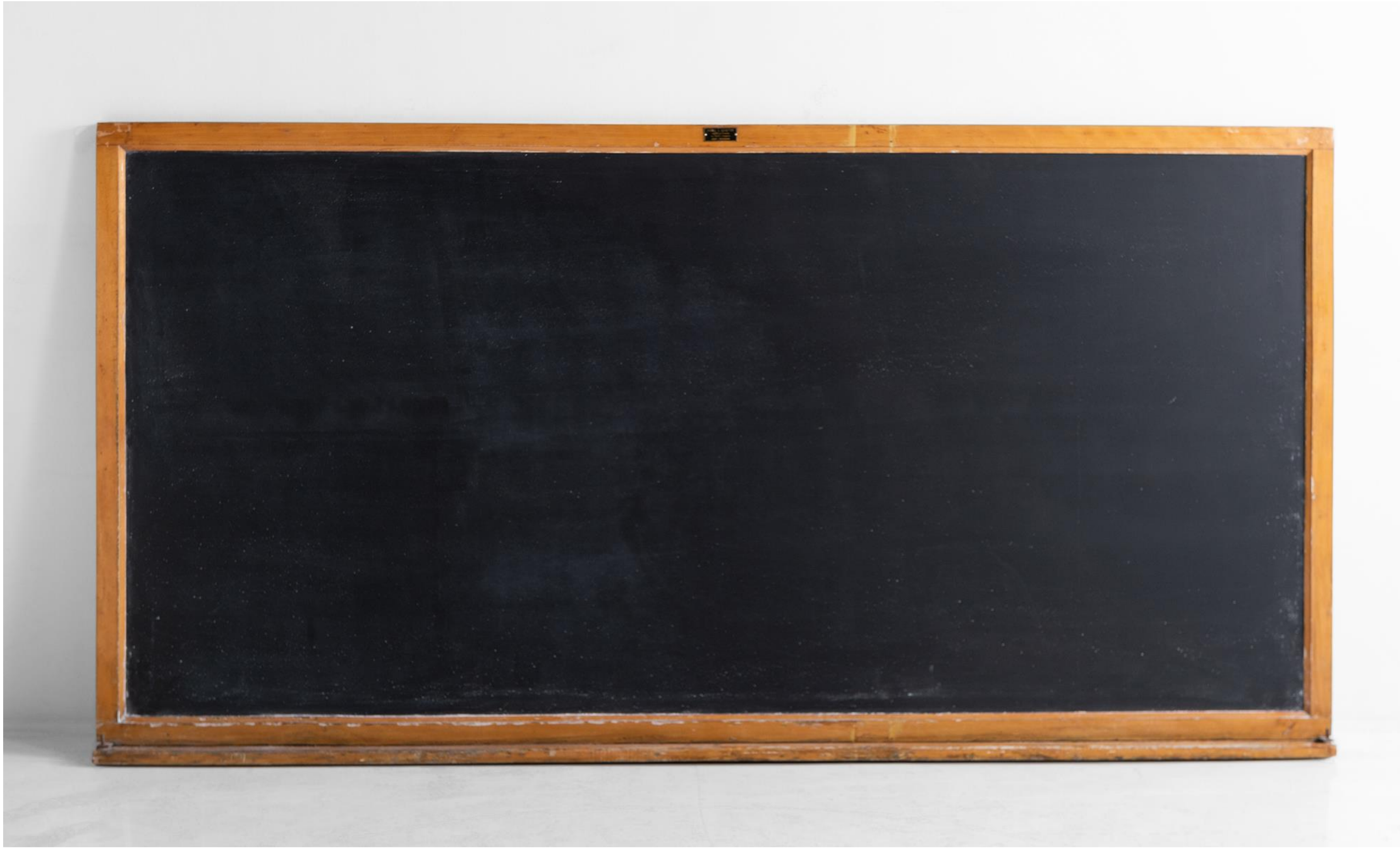


Figure 1.29
Travelers' tendencies at a regional airport.



To formulate a system of difference equations, let n represent the nth week of traveling and define

$$S_n = \text{the number of US Airways travelers in week } n$$
$$U_n = \text{the number of United Airlines travelers in week } n$$
$$A_n = \text{the number of American Airlines travelers in week } n$$



Formulating the system of difference equations, we have the following dynamical system:

$$
\begin{aligned}
S_{n+1} &= 0.75S_n + 0.20U_n + 0.40A_n \\
U_{n+1} &= 0.05S_n + 0.60U_n + 0.20A_n \\
A_{n+1} &= 0.20S_n + 0.20U_n + 0.40A_n
\end{aligned}
$$

Na forma geral o modelo poderia ser escrito como:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_{n+1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_n$$

Os estados x representam respectivamente o número de viajantes de US Airways, United e American:

$$\begin{Bmatrix} S \\ U \\ A \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

A matriz de interação é dada por:

$$[A] = \begin{bmatrix} 0.75 & 0.20 & 0.40 \\ 0.05 & 0.60 & 0.20 \\ 0.20 & 0.20 & 0.40 \end{bmatrix}$$

No equilíbrio, os estados em *n+1* são iguais aos estados em *n*, logo:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_n = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_n$$

Utilizando a matriz identidade:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_n = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_n$$

Deslocando tudo para o lado direito:

$$0 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_n - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_n = \begin{bmatrix} a_{11} - 1 & a_{12} & a_{13} \\ a_{21} & a_{22} - 1 & a_{23} \\ a_{31} & a_{32} & a_{33} - 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_n$$

Substituindo os valores do exemplo:

$$\begin{bmatrix} 0.75-1 & 0.20 & 0.40 \\ 0.05 & 0.60-1 & 0.20 \\ 0.20 & 0.20 & 0.4-1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_n = \begin{bmatrix} -0.25 & 0.20 & 0.40 \\ 0.05 & -0.40 & 0.20 \\ 0.20 & 0.20 & -0.60 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_n = 0$$

Analisando o sistema resultante:

$$\begin{bmatrix} -0.25 & 0.20 & 0.40 \\ 0.05 & -0.40 & 0.20 \\ 0.20 & 0.20 & -0.60 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_n = 0$$

Uma solução seriam os 3 estados nulos :

$$\begin{Bmatrix} S \\ U \\ A \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0$$

Será que existem estados não nulos que respeitam essa equação ?

Utilizando duas propriedades que não alteram as equações do sistema:

a) Multiplicação de linhas por um escalar

a) Soma (ou subtração de linhas)

para transformar a matriz numa matriz triangular.

$$L_2 = L_2 - \left(\frac{0.05}{-0.25}\right) * L_1 = L_2 + 0.2 * L_1$$

$$L_3 = L_3 - \left(\frac{0.20}{-0.25}\right) * L_1 = L_3 + 0.8 * L_1$$

$$\begin{bmatrix} -0.25 & 0.20 & 0.40 \\ 0.05 & -0.40 & 0.20 \\ 0.20 & 0.20 & -0.60 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_n = 0$$

Que resulta no novo sistema:

$$\begin{bmatrix} -0.25 & 0.20 & 0.40 \\ 0 & -0.36 & 0.28 \\ 0 & 0.36 & -0.28 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_n = 0$$

$$\begin{bmatrix} -0.25 & 0.20 & 0.40 \\ 0 & -0.36 & 0.28 \\ 0 & 0.36 & -0.28 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}_n = 0$$

Perceba que as duas últimas linhas são linearmente dependentes, logo qualquer solução onde :

$$0.36x_2 = 0.28x_3 \qquad x_2 = 0.77778x_3$$

respeita as duas últimas equações.

Da primeira equação pode-se isolar x_1 em relação a x_3 :

$$x_1 = \frac{1}{0.25}(0.2x_2 + 0.4x_3) = 4(0.2 * 0.77778x_3 + 0.4x_3) = 2.222x_3$$

$$x_1 = 2.222x_3$$

Um ponto de equilíbrio não nulo ocorre quando a relação entre os estados obedecer:

$$x_1 = 2.222x_3 \qquad S = 2.222A$$

ou

$$x_2 = 0.77778x_3 \qquad U = 0.77778A$$

Logo

$$A = 1000$$
$$S = 2222$$
$$U = 778$$

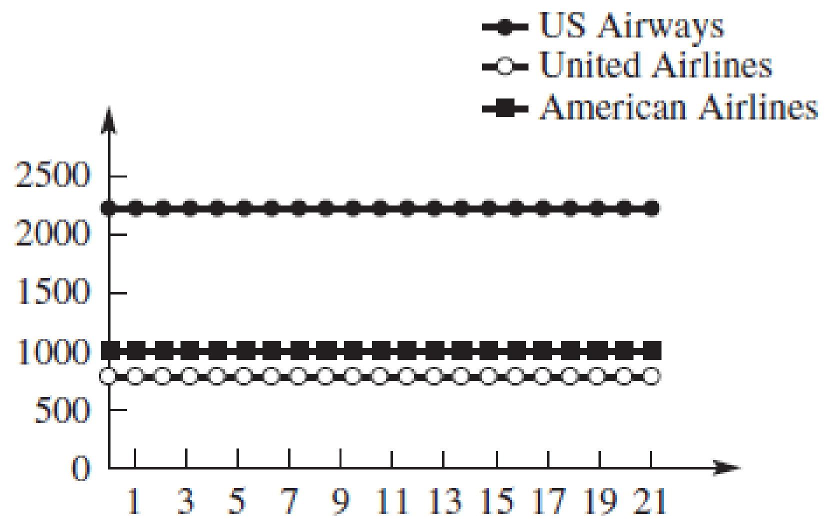
É um ponto de equilíbrio

Analisando a sensibilidade às condições iniciais e o comportamento em longo prazo para variações em torno do ponto de equilíbrio:

	US Airways	United Airlines	American Airlines
Case 1	2222	778	1000
Case 2	2720	380	900
Case 3	1000	1000	2000
Case 4	0	0	4000

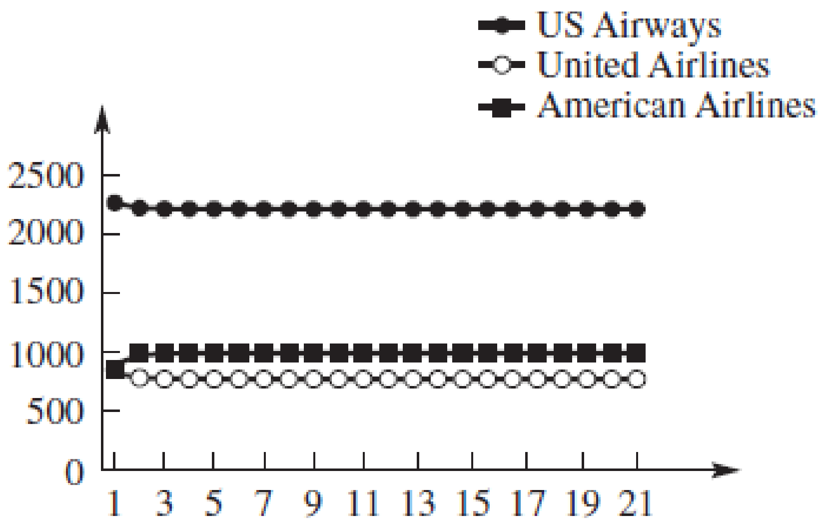
Ponto de equilíbrio

n	US Airways	United Airlines	American Airlines
0	2222	778	1000
1	2222.1	777.9	1000
2	2222.155	777.845	1000
3	2222.18525	777.81475	1000
4	2222.201888	777.7981125	1000
5	2222.211038	777.7889619	1000
6	2222.216071	777.783929	1000
7	2222.218839	777.781161	1000
8	2222.220361	777.7796385	1000
9	2222.221199	777.7788012	1000
10	2222.221659	777.7783407	1000
11	2222.221913	777.7780874	1000
12	2222.222052	777.777948	1000
13	2222.222129	777.7778714	1000
14	2222.222171	777.7778293	1000
15	2222.222194	777.7778061	1000
16	2222.222207	777.7777934	1000
17	2222.222214	777.7777863	1000
18	2222.222218	777.7777825	1000
19	2222.22222	777.7777804	1000
20	2222.222221	777.7777792	1000



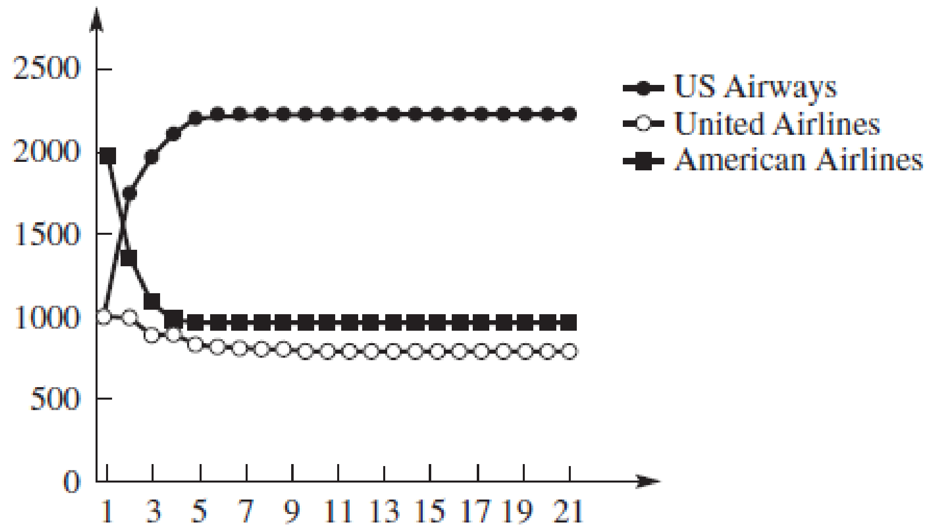
a. Case 1

n	US Airways	United Airlines	American Airlines
0	2272	828	900
1	2229.6	790.4	980
2	2222.28	781.72	996
3	2221.454	779.346	999.2
4	2221.6397	778.5203	999.84
5	2221.869835	778.162165	999.968
6	2222.022009	777.9843908	999.9936
7	2222.110825	777.8904549	999.99872
8	2222.160698	777.8395582	999.999744
9	2222.188333	777.8117186	999.9999488
10	2222.203573	777.7964376	999.9999898
11	2222.211963	777.7880391	999.999998
12	2222.216579	777.7834212	999.9999996
13	2222.219118	777.7808816	999.9999999
14	2222.220515	777.7794849	1000
15	2222.221283	777.7787167	1000
16	2222.221706	777.7782942	1000
17	2222.221938	777.7780618	1000
18	2222.222066	777.777934	1000
19	2222.222136	777.7778637	1000
20	2222.222175	777.777825	1000



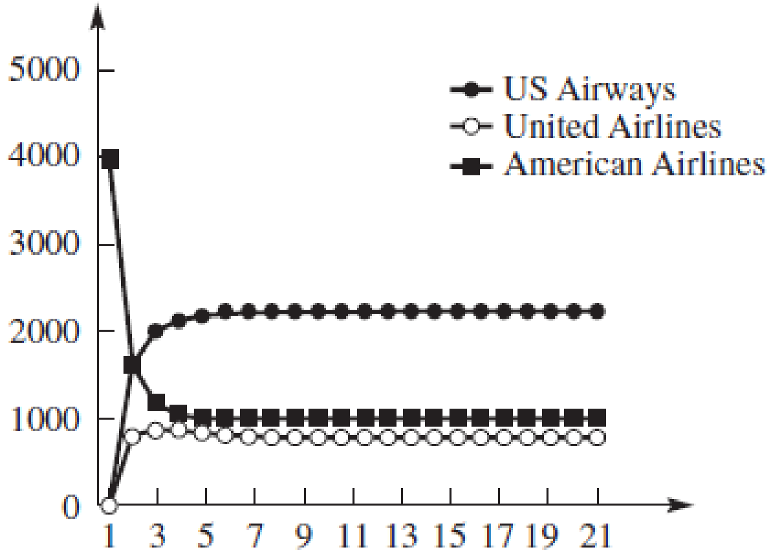
b. Case 2

n	US Airways	United Airlines	American Airlines
0	1000	1000	2000
1	1750	1050	1200
2	2002.5	957.5	1040
3	2109.375	882.625	1008
4	2161.75625	836.64375	1001.6
5	2189.285938	810.3940625	1000.32
6	2204.171266	795.7647344	1000.064
7	2212.306996	787.6802039	1000.013
8	2216.771408	783.2260321	1000.003
9	2219.224786	780.7747017	1000.001
10	2220.573735	779.4261627	1000
11	2221.315575	778.6844049	1000
12	2221.72357	778.2764257	1000
13	2221.947964	778.0520348	1000
14	2222.071381	777.9286192	1000
15	2222.139259	777.8607406	1000
16	2222.176593	777.8234073	1000
17	2222.197126	777.802874	1000
18	2222.208419	777.7915807	1000
19	2222.214631	777.7853694	1000
20	2222.218047	777.7819532	1000



c. Case 3

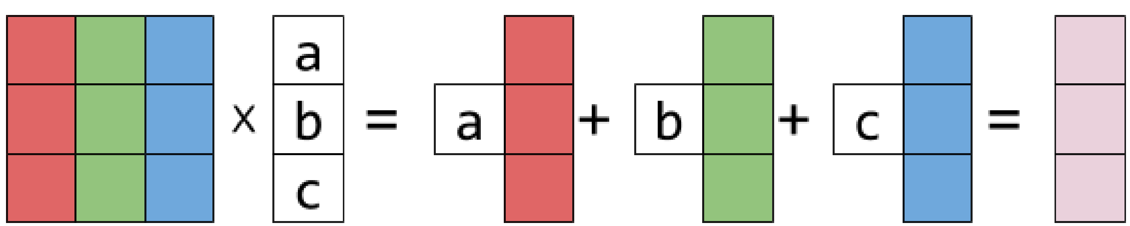
n	US Airways	United Airlines	American Airlines
0	0	0	4000
1	1600	800	1600
2	2000	880	1120
3	2124	852	1024
4	2173	822.2	1004.8
5	2196.11	802.93	1000.96
6	2208.0525	791.7555	1000.192
7	2214.467275	785.494325	1000.038
8	2217.964681	782.0276388	1000.008
9	2219.882111	780.1163533	1000.002
10	2220.935468	779.0642247	1000
11	2221.514569	778.4853697	1000
12	2221.833025	778.1669625	1000
13	2222.008166	777.9918312	1000
14	2222.104492	777.8955076	1000
15	2222.157471	777.8425292	1000
16	2222.186609	777.8133911	1000
17	2222.202635	777.7973651	1000
18	2222.211449	777.7885508	1000
19	2222.216297	777.7837029	1000
20	2222.218963	777.7810366	1000



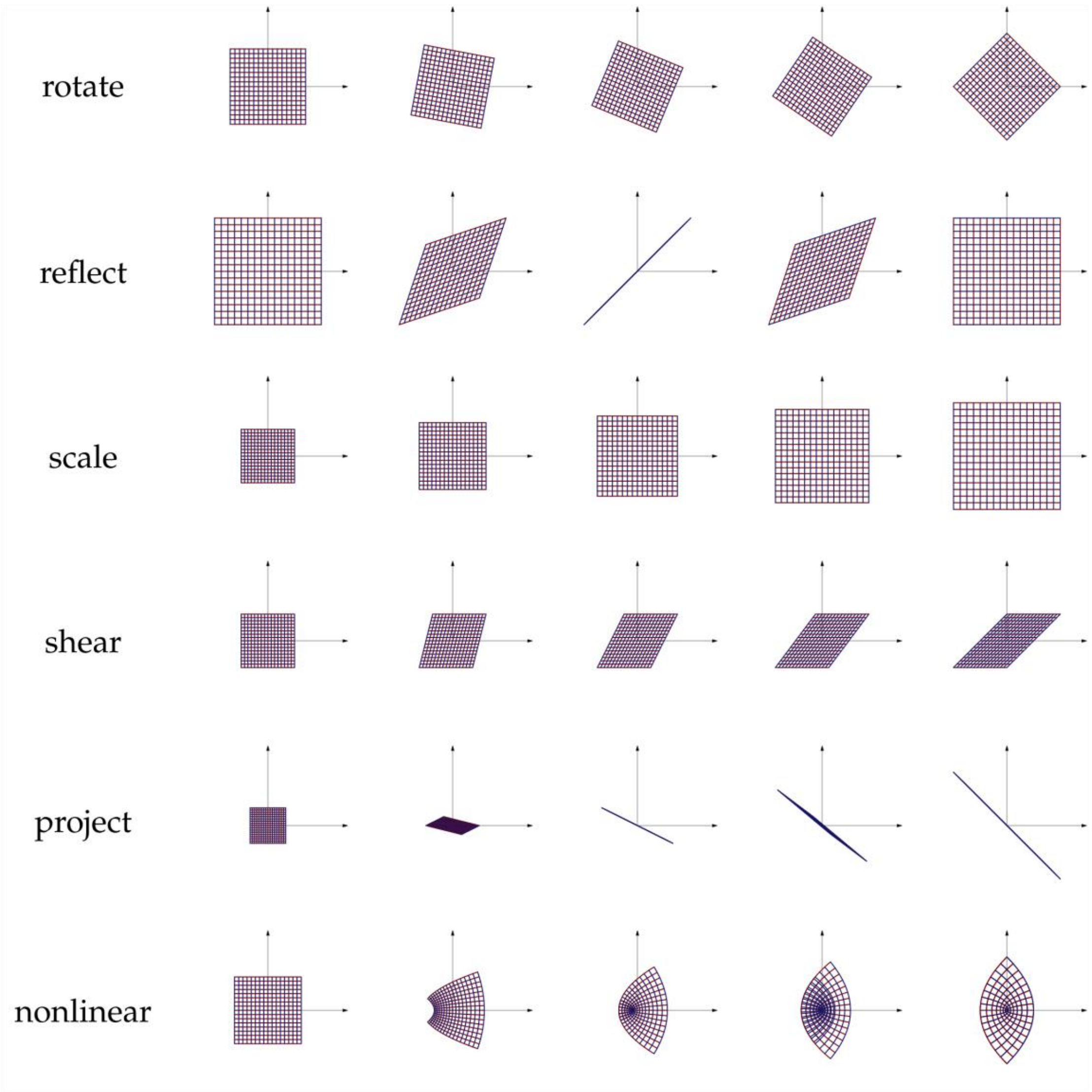
d. Case 4

- Matrizes podem ser interpretadas como transformações lineares entre espaços vetoriais (considere por simplicidade matrizes quadradas):

$$T(x+y) = T(x) + T(y)$$
$$T(k.x) = k.T(x)$$



- O produtos de matrizes por vetores produzem rotações e deformações



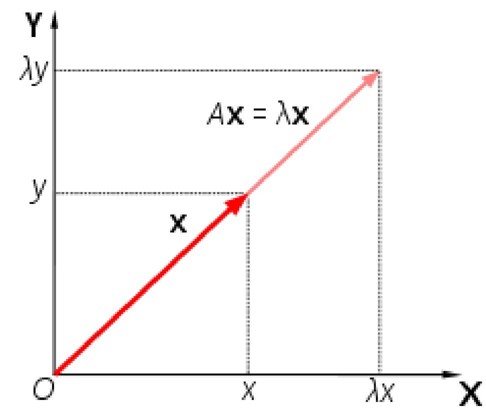
- Os autovetores de uma matriz são vetores que se multiplicados pela matriz sofrem apenas deformações (proporcionais aos respectivos autovetores)

eigenvalue equation:

$Ax = \lambda x$

or, $Ax - \lambda x = 0$

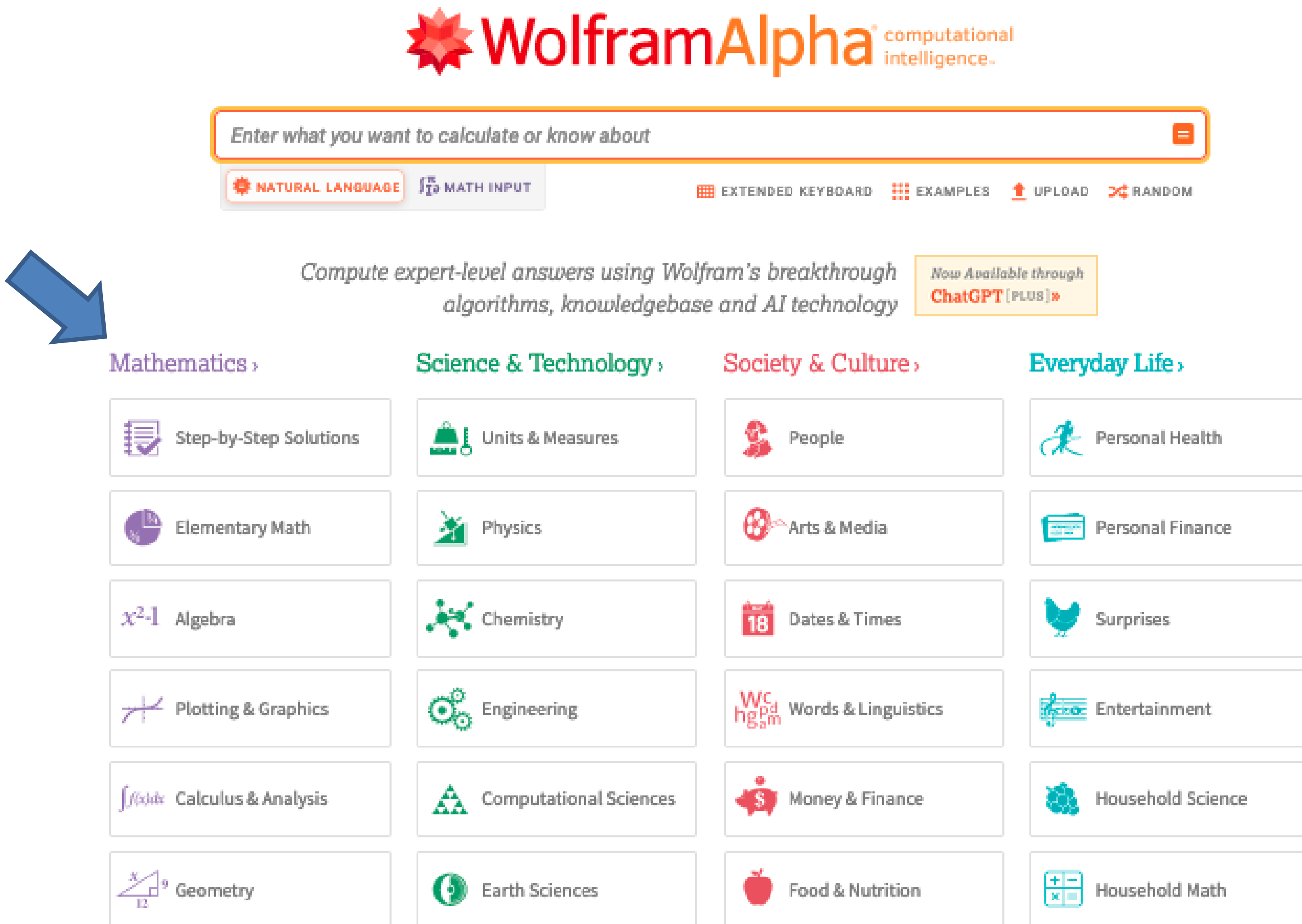
or, $(A - \lambda I)x = 0$ [where I is the identity matrix]



characteristic equation:

$(A - \lambda I)x = 0$ can have non-trivial solutions

when $\boxed{\det(A - \lambda I) = 0}$



https://www.wolframalpha.com/

Mathematics ›

Linear Algebra

Explore and compute properties of vectors, matrices and vector spaces.

Compute properties of a vector:

vector <3, -4>

Calculate properties of a matrix:

{{6, -7}, {0, 3}}

Determine whether a set of vectors is linearly independent:

Are (2, -1) and (4, 2) linearly independent?

More examples

Matrices

Explore various properties of matrices.

Calculate properties of a matrix:

{{6, -7}, {0, 3}}

Multiply matrices:

{{2, -1}, {1, 3}} . {{1, 2}, {3, 4}}

Row reduce a matrix:

row reduce {{2, 1, 0, -3}, {3, -1, 0, 1}, {1, 4, -2, -5}}

More examples

Eigenvalues & Eigenvectors

Calculate the eigensystem of a given matrix.

Compute the eigenvalues of a matrix:

eigenvalues {{4, 1}, {2, -1}}

Compute the eigenvectors of a matrix:

eigenvectors {{1, 0, 0}, {0, 0, 1}, {0, 1, 0}}

Compute the characteristic polynomial of a matrix:

characteristic polynomial {{4, 1}, {2, -1}}

WolframAlpha computational intelligence.

eigenvalues {{0.75, 0.2,0.4}, {0.05,0.6, 0.2},{0.2,0.2,0.4}}

NATURAL LANGUAGE | MATH INPUT | EXTENDED KEYBOARD | EXAMPLES | UPLOAD | RANDOM

compute input

Input

eigenvalues $\begin{pmatrix} 0.75 & 0.2 & 0.4 \\ 0.05 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.4 \end{pmatrix}$

Results

Decimal forms | Step-by-step solution

$\lambda_1 = 1$

$\lambda_2 = \frac{11}{20}$

$\lambda_3 = \frac{1}{5}$

Corresponding eigenvectors

Approximate forms | Step-by-step solution

$v_1 = \left(\frac{20}{9}, \frac{7}{9}, 1\right)$

$v_2 = (-1, 1, 0)$

$v_3 = \left(-\frac{4}{7}, -\frac{3}{7}, 1\right)$

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POWERED BY THE **WOLFRAM LANGUAGE**

EXAMPLE 5 *Discrete Epidemic Models*

Consider a disease that is spreading throughout the United States, such as the new flu. The Centers for Disease Control and Prevention is interested in knowing and experimenting with a model for this new disease before it actually becomes a real epidemic. Let us consider the population divided into three categories: susceptible, infected, and removed. We make the following assumptions for our model:

- No one enters or leaves the community, and there is no contact outside the community.
- Each person is susceptible S (able to catch this new flu); infected I (currently has the flu and can spread the flu); or removed R (already had the flu and will not get it again, which includes death).
- Initially, every person is either S or I.
- Once someone gets the flu this year, they cannot get the flu again.
- The average length of the disease is 5/3 weeks (1 and 2/3 weeks), over which time the person is deemed infected and can spread the disease.
- Our time period for the model will be per week.

Modelo S-I-R discreto

$$S_{n+1} = S_n - aI_n\, S_n$$

$$I_{n+1} = I_n - rI_n + aI_n\, S_n$$

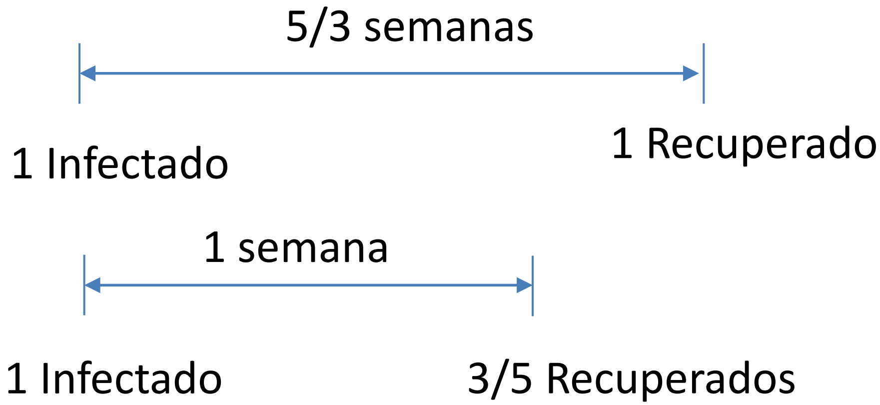
$$R_{n+1} = R_n + rI_n$$

A população suscetível se torna infectada pelo contato com infectados com uma taxa “a”

A população infectada se recupera com uma taxa “r” e aumenta pela infecção dos suscetíveis

A população de recuperados aumenta com recuperação dos infectados

Todas as constantes do modelo são positivas



$$R_{n+1} = R_n + rI_n$$

r= 3/5 = 0.6

Let's start our modeling process with $R(n)$. Our assumption for the length of time someone has the flu is 5/3 weeks. Thus, 3/5 or 60% of the infected people will be removed each week:

$$R(n+1) = R(n) + 0.6I(n)$$

The value 0.6 is called the *removal rate per week*. It represents the proportion of the infected persons who are removed from infection each week.

Para encontrar o parâmetro *a* é necessária alguma informação que não constava do enunciado original do exemplo.

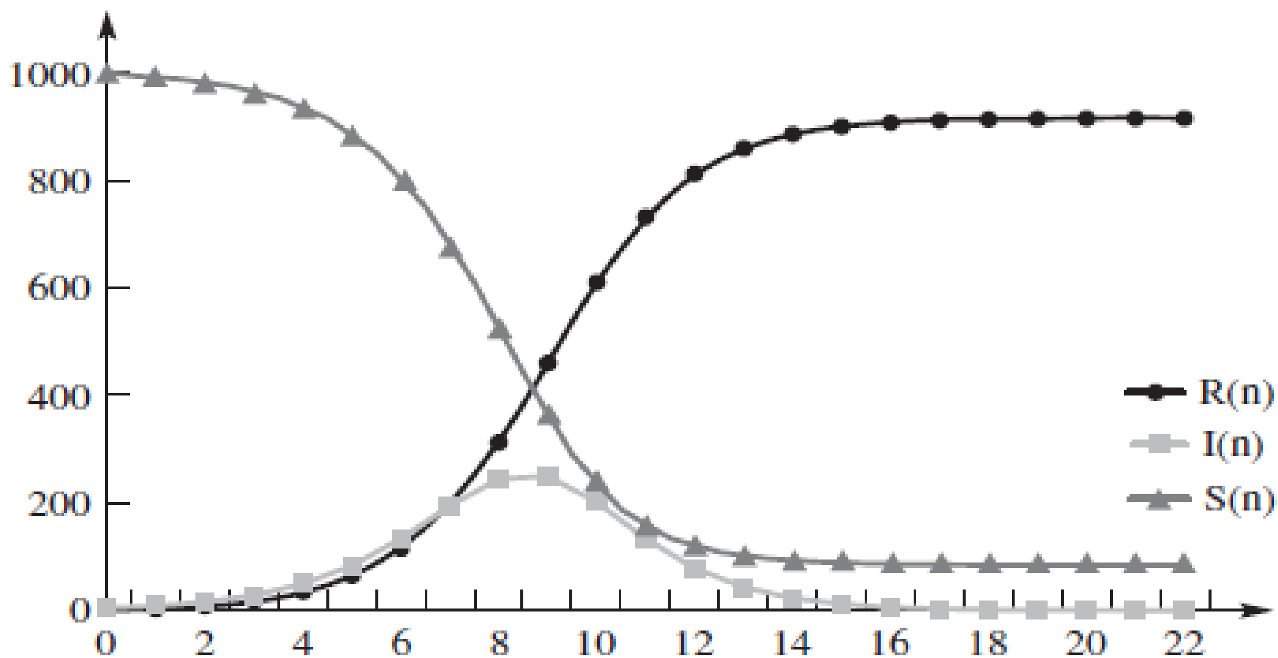
Let's illustrate as follows: Assume we have a population of 1000 students residing in the dorms. Our nurse found 5 students reporting to the infirmary initially: $I(0) = 5$ and $S(0) = 995$. After one week, the total number infected with the flu is 9. We compute a as follows:

$$
\begin{aligned}
I(0) &= 5, I(1) = I(0) - 0.6 * I(0) + aI(0) * S(0) \\
I(1) &= 9 = 5 - 3 + a * 5 * 995 \\
7 &= a(4975) \\
a &= 0.001407
\end{aligned}
$$

Modelo S-I-R discreto

$$\begin{cases} S_{n+1} = S_n - aI_n\, S_n \\ I_{n+1} = I_n - rI_n + aI_n\, S_n \\ R_{n+1} = R_n + rI_n \end{cases}$$

Week	$R(n)$	$I(n)$	$S(n)$
0	0	5	995
1	3	8.999825	988.0002
2	8.399895	16.11073	975.4894
3	18.06633	28.55649	953.3772
4	35.20023	49.72832	915.0714
5	65.03722	83.91682	851.046
6	115.3873	134.0505	750.5621
7	195.8176	195.1831	608.9993
8	312.9275	245.3182	441.7543
9	460.1184	250.6044	289.2772
10	610.481	202.241	187.2779
11	731.8256	134.1869	133.9874
12	812.3378	78.97173	108.6905
13	859.7208	43.66564	96.61352
14	885.9202	23.40195	90.67782
15	899.9614	12.34649	87.69211
16	907.3693	6.46194	86.16877
17	911.2465	3.368218	85.38533
18	913.2674	1.751936	84.98068
19	914.3185	0.910249	84.7712
20	914.8647	0.472668	84.66264
21	915.1483	0.245372	84.60633
22	915.2955	0.127358	84.57712



Exercício:

Our coupled model is

$$\begin{aligned} R(n+1) &= R(n) + 0.6I(n) \\ I(n+1) &= I(n) - 0.6I(n) + 0.001407I(n)S(n) \\ S(n+1) &= S(n) - 0.001407S(n)I(n) \\ I(0) &= 5,\ S(0) = 995,\ R(0) = 0 \end{aligned} \qquad (1.11)$$

The SIR model Equation (1.11), can be solved iteratively and viewed graphically. Lets iterate the solution and obtain the graph to observe the behavior to obtain some insights.

10. In model (1.11) determine the outcome with the following parameters changed:

a. Initially 5 are sick, and 15 are sick the next week.

b. The flu lasts 1 week.

c. The flu lasts 4 weeks.

d. There are 4000 students in the dorm; 5 are initially infected, and 30 more are infected the next week.

Fim Aula 06