2020-1, "STATPHYS", AULA 07

OBJETIVOS: VOLTAR A DISCUTIR TRANSFORMAÇÕES DE V.A.'s, INDICANDO ASPECTOS RELEVANTES PARA SIMULAÇÕES ESTOCÁSTICAS.

ONDE ESTAMOS: 1.2 ELEMENTOS DE PROBABILIDADE

INÍCIO DA AULA

* TRANSFORMAÇÕES DE V.A.'s

(i) PROBABILISTAS, "UM PRA UM"

$$\boxed{Y = f(X)}$$

$$y = f(x)$$

$$dy = f'(x)dx$$

$$1 = \int \rho_Y(y)\,dy = \int \underbrace{\rho_Y[\overset{y}{\overbrace{f(x)}}]\,\overset{dy/dx}{\overbrace{f'(x)}}}_{\rho_X(x)\,?}dx$$

$$\rho_X(x) = \rho_Y[f(x)]\cdot\left|\frac{dy}{dx}\right|$$

$\swarrow$ $x = x(y)$

$$\boxed{\rho_Y(y) = \rho_X[x(y)]\cdot\left|\frac{dx}{dy}\right|}$$

◇ EXEMPLO: $X \sim U(0,1)$, $Y = -\frac{1}{\lambda} \log X$ ⟶ $\lambda > 0$

$$y = -\frac{1}{\lambda} \log x \quad \Leftrightarrow \quad x(y) = e^{-\lambda y}$$

$$\rho_Y(y) = \rho_X[x(y)] \cdot \left| \frac{dx}{dy} \right|$$

$$= \left| -\lambda e^{-\lambda y} \right| = \lambda e^{-\lambda y}, \; y > 0$$

$x \in (0,1) \Leftrightarrow y > 0$

$\therefore \boxed{Y \sim \mathcal{E}(\lambda)}$ EXPONENCIAL

$\{x_i\} \downarrow \{y_i\}$ $\quad i = 1, \dots, M$

IMPORTANTE!
SIMULAÇÕES!!!

$$\boxed{y_i = -\frac{1}{\lambda} \log(x_i)} \quad \diamond$$

(ii) PROBABILISTAS, "MUITOS PARA MUITOS"

$\vec{X} = (X_1, \dots, X_n)$ E $\vec{Y} = (Y_1, \dots, Y_n)$, VETORES DE V.A.s

PARA $i = 1, \dots, n$, $Y_i = y_i(\vec{X})$, $X_i = x_i(\vec{Y})$

$$\boxed{\rho_{\vec{Y}}(\vec{y}) = \rho_{\vec{X}}[\vec{x}(\vec{y})] \cdot \left| \frac{\partial(x_1, \dots, x_n)}{\partial(y_1, \dots, y_n)} \right|}$$

◆ MOSTRE QUE, SE $X_1, X_2 \sim U(0,1)$ FOREM INDEPENDENTES, TAMBÉM O SERÃO

$$\begin{cases} Z_1 = \sqrt{-2 \log X_1} \cdot \cos(2\pi X_2) \\ Z_2 = \sqrt{-2 \log X_1} \cdot \sin(2\pi X_2) \end{cases}$$

, COM $Z_1, Z_2 \sim N(0,1)$.

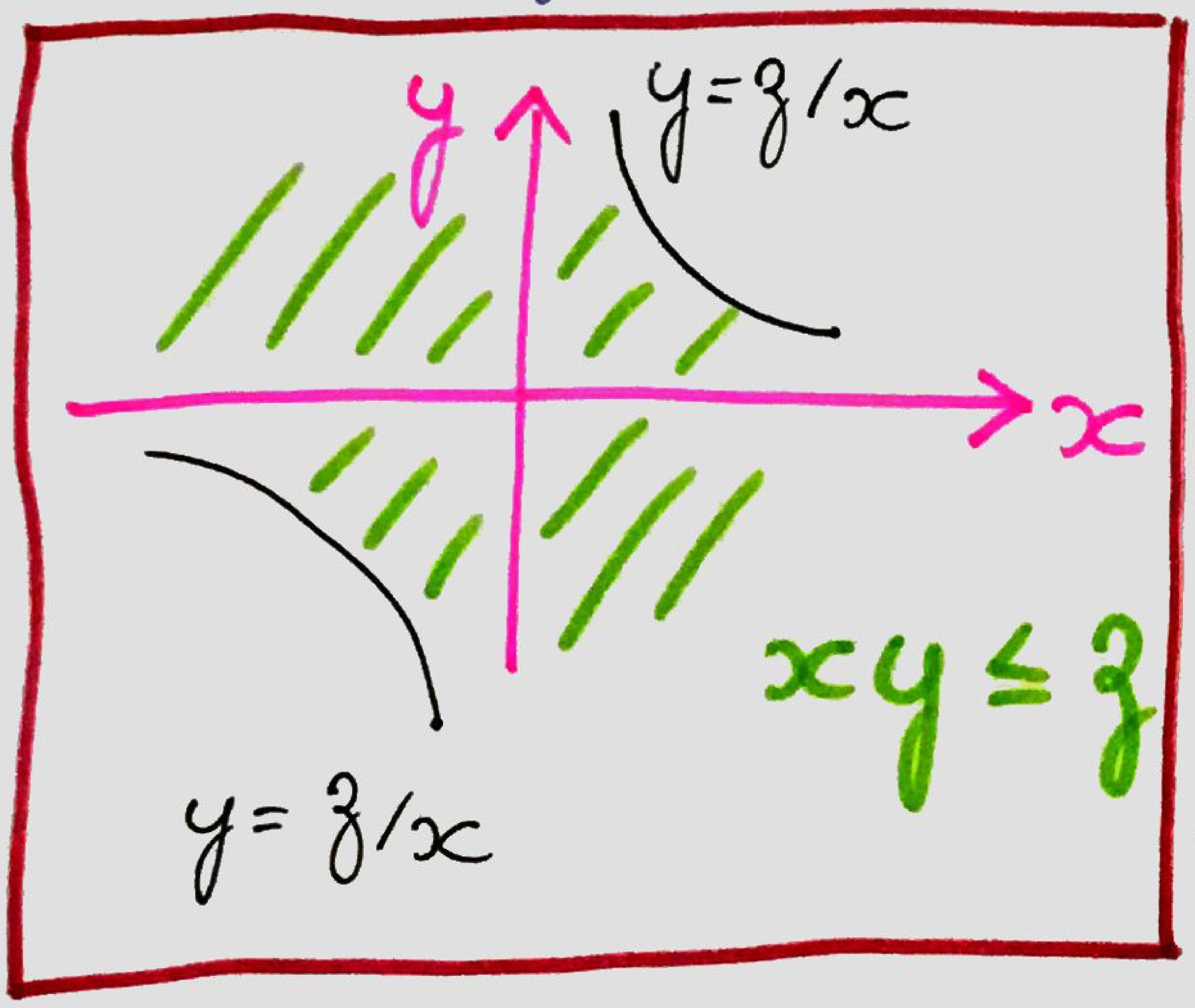
(iii) PROBABILISTAS, "MAIS PARA MENOS": CUMULATIVA?

◇ EXEMPLO: SE $Z = X.Y$,

$$\boxed{\rho_Z(z) = \int_{-\infty}^{+\infty} \frac{1}{|x|} \rho_{X,Y}\left(x, \frac{z}{x}\right) dx}, \quad \text{POIS}$$

$$F_Z(z) = \mathbb{P}(Z \leq z) = \iint_{xy \leq z} \rho_{X,Y}(x,y)\, dx\, dy =$$

$$= \int_{-\infty}^{0} \left[\int_{z/x}^{\infty} \rho_{X,Y}(x,y)\, dy\right] dx + \int_{0}^{\infty} \left[\int_{-\infty}^{z/x} \rho_{X,Y}(x,y)\, dy\right] dx =$$



$$y \to u \equiv xy$$

$$= \int_{-\infty}^{0} \left[\int_{z}^{-\infty} \rho_{X,Y}\left(x, \frac{u}{x}\right) \frac{du}{x}\right] dx$$

$$+$$

$$\int_{0}^{\infty} \left[\int_{-\infty}^{z} \rho_{X,Y}\left(x, \frac{u}{x}\right) \frac{du}{x}\right] dx =$$

$$= \int_{-\infty}^{z} du \left\{ \int_{-\infty}^{0} \frac{1}{-x} \rho_{X,Y}\left(x, \frac{u}{x}\right) dx + \int_{0}^{\infty} \frac{1}{x} \rho_{X,Y}\left(x, \frac{u}{x}\right) dx \right\} =$$

$$= \int_{-\infty}^{z} \underbrace{\left[\int_{-\infty}^{+\infty} \frac{1}{|x|} \rho_{X,Y}\left(x, \frac{u}{x}\right) dx\right]}_{= \rho_Z(z)} du$$

(iv) FÍSICOS, DELTA DE DIRAC

$$Y = f(X) \leadsto \rho_Y(y) = \int dx\, \rho_X(x)\, \delta[y - f(x)]$$

* NORMALIZAÇÃO

$$\int dy\, \rho_Y(y) = \int dx\, \rho_X(x) \left[\int dy\, \delta[y - f(x)] \right]$$
$$= \int dx\, \rho_X(x) = 1$$

* CARACTERÍSTICA

$$\phi_Y(k) = \int dy\, e^{iky} \rho_Y(y)$$
$$= \int dx\, \rho_X(x) \left\{ \int dy\, e^{iky}\, \delta[y - f(x)] \right\}$$
$$= \int dx\, \rho_X(x)\, e^{ikf(x)}$$
$$= \langle e^{ikf(X)} \rangle = \langle e^{ikY} \rangle$$

* PROPRIEDADES DA DELTA

a. $\int f(x)\, \delta(x)\, dx = f(0)$

b. $\int f(x)\, \delta(x-a)\, dx = f(a)$

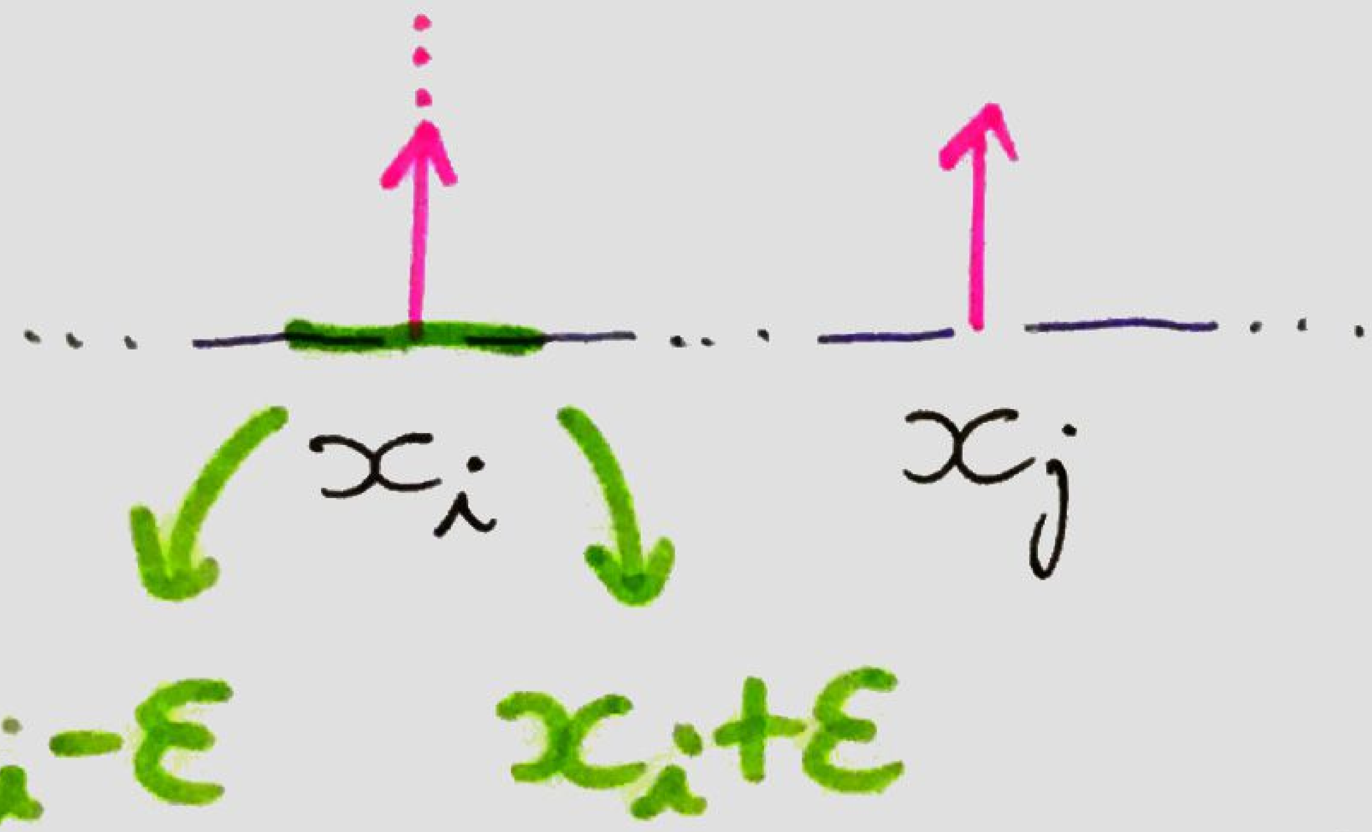
c. $\int f(x) . \delta(c.x)\, dx = \frac{1}{|c|} f(0)$, $c \neq 0$

d. $\int f(x)\, \delta[z - g(x)]\, dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}$,

ONDE CADA $x_i = x_i(z)$ TAL QUE $z = g[x_i(z)]$.

e. $\delta(x) = \frac{1}{2\pi} \int dk \, e^{ikx}$

□ DEMONSTRAÇÃO DE ⓓ



$$\int f(x)\, \delta[z - g(x)]\, dx =$$

$$= \sum_i \lim_{\varepsilon \to 0} \int_{x_i - \varepsilon}^{x_i + \varepsilon} dx\, f(x)\, \delta\Big\{ \overbrace{z - g(x_i)}^{=0} - \underbrace{g'(x_i)}_{= \text{CTE}} \cdot (x - x_i) \Big\} =$$

$$= \sum_i \frac{f(x_i)}{|g'(x_i)|} \quad \square$$

◇ EXEMPLOS

ⓘ $Y = cX$ $(c \neq 0)$ $\Rightarrow \boxed{\rho_Y(y) = \frac{1}{|c|} \rho_X(y/c)}$

↳ MOSTRE!

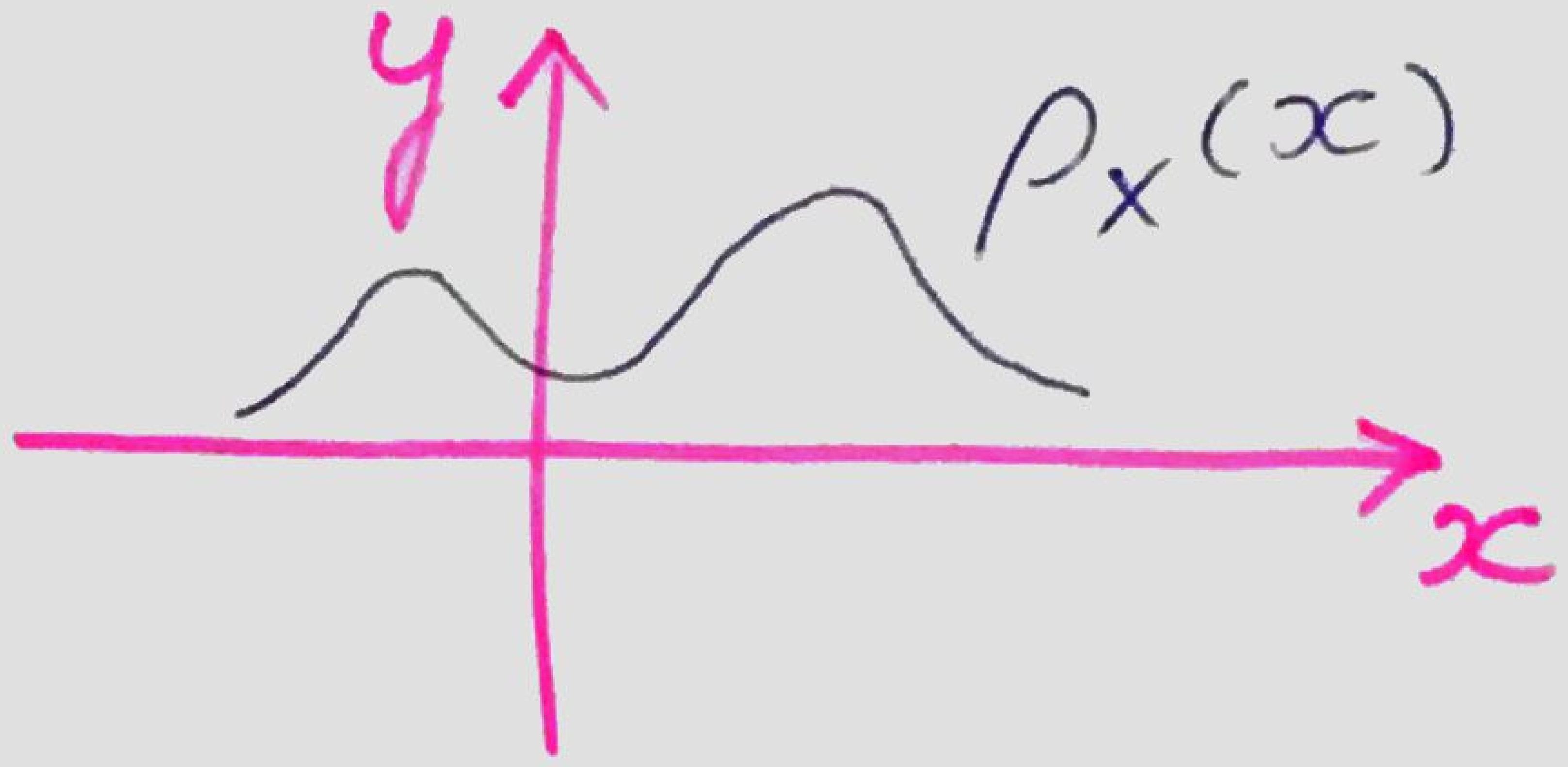
ⓘⓘ $Y = F_X(X)$, $F_X(x) = \mathbb{P}(X \leq x)$

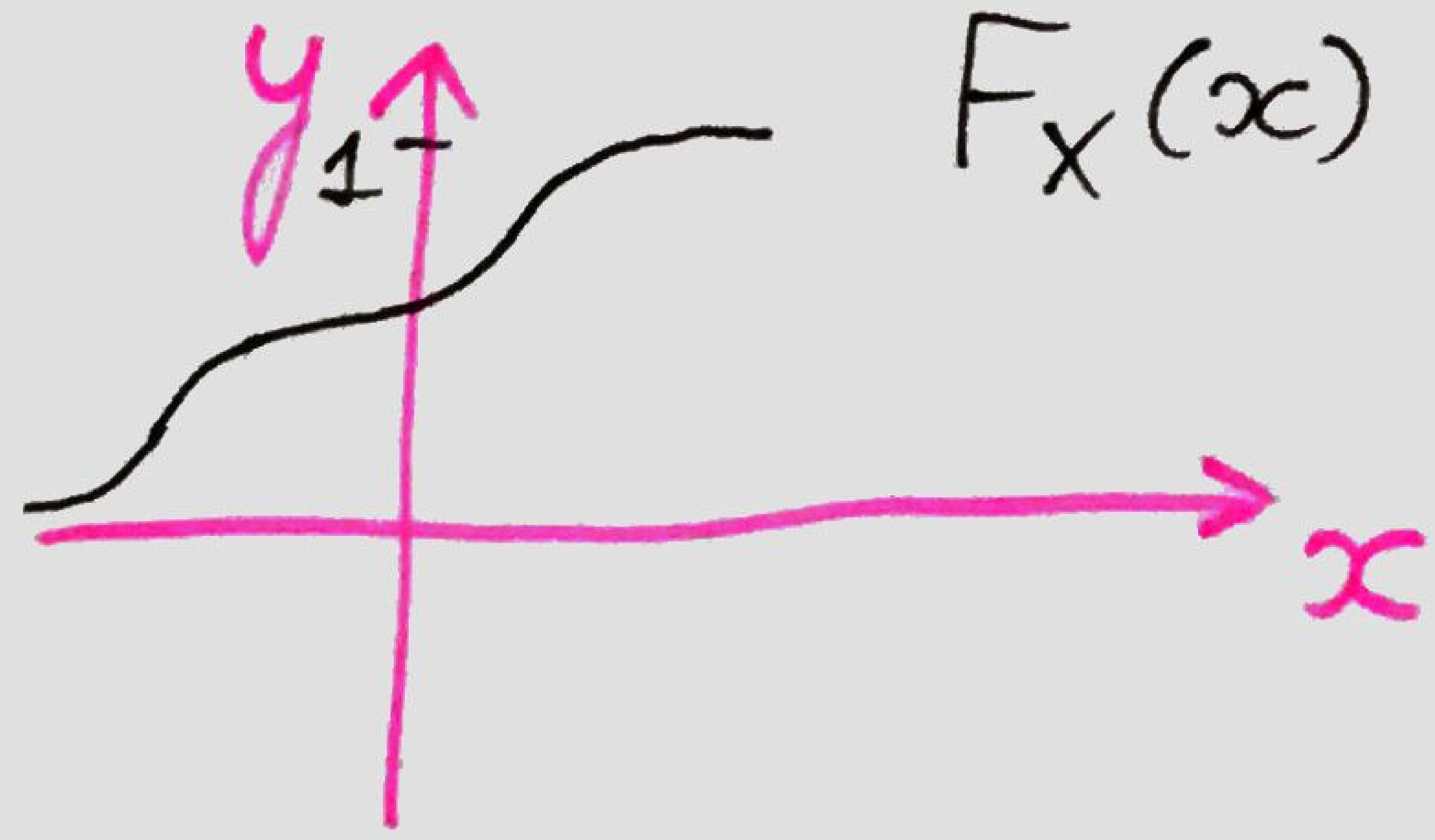
$$\rho_Y(y) = \int dx\, \rho_X(x)\, \delta[y - F_X(x)] =$$

$$= \frac{\rho_X[F_X^{-1}(y)]}{|F_X'[F_X^{-1}(y)]|} = 1 \; ,$$

POIS $F_X(x)$ CRESCE MONOTONICAMENTE,

$x_i(y) = F_X^{-1}(y)$ É A ÚNICA RAIZ.





PARA X <u>QUALQUER</u>, $Y \equiv F_X(X)$ É UNIFORME!!!

INVERTENDO, DADO Y UNIFORME,

X <u>QUALQUER</u> VEM DE $X = F_X^{-1}(Y)$.

ANALITICAMENTE DIFÍCIL