Eletromagnetismo

Equações de Maxwell

- Em 50 anos, partimos de experimentos básicos à união entre eletricidade e magnetismo.
- Resultado levou a implementações e novas tecnologias.
- Mas o conjunto ainda não estava completo.
- Discrepâncias surgiam na teoria e em experimentos idealizados.

Resumindo o que temos até agora:

Coulomb: $\vec{F}_e = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$ (1785)

$$\Rightarrow \quad \vec{E} = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(r)}{r^2} \hat{r} \, dv$$

Biot-Savart: $\vec{B} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J} \times \hat{r}}{r^2} dv$ (1820)

$$\Rightarrow \boxed{\begin{array}{l} \nabla \cdot \vec{D} = \rho \quad ; \quad \vec{D} = \epsilon \vec{E} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} \\ \nabla \times \vec{H} = \vec{J} \quad ; \quad \vec{B} = \mu_0 \vec{H} \end{array}}$$

(Gauss, 1835)

(Faraday, 1831)

(Ampère - atribuído)

Resumindo o que temos até agora:

Coulomb: $\vec{F}_e = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$ (1785)

$$\Rightarrow \quad \vec{E} = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(r)}{r^2} \hat{r} \, dv$$

Biot-Savart: $\vec{B} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J} \times \hat{r}}{r^2} dv$ (1820) $= \frac{\mu_0}{4\pi} \int_C \frac{I d\vec{l} \times \hat{r}}{r^2}$

$I = \int \vec{J} \cdot d\vec{a}$

$$\Rightarrow \boxed{\begin{aligned} &\nabla \cdot \vec{D} = \rho \;;\quad \vec{D} = \epsilon \vec{E} \\ &\nabla \cdot \vec{B} = 0 \\ &\nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} \\ &\nabla \times \vec{H} = \vec{J} \;;\quad \vec{B} = \mu_0 \vec{H} \end{aligned}}$$

(Gauss, 1835)

(Faraday, 1831)

(Ampère - atribuído)

$$\Rightarrow \boxed{\begin{aligned} &\nabla\cdot\vec{D} = \rho \quad ; \quad \vec{D} = \varepsilon\vec{E} \\ &\nabla\cdot\vec{B} = 0 \\ &\nabla\times\vec{E} = -\frac{\partial}{\partial t}\vec{B} \\ &\nabla\times\vec{H} = \vec{J} \quad ; \quad \vec{B} = \mu_0\vec{H} \end{aligned}}$$

(Gauss, 1835)

(Faraday, 1831)

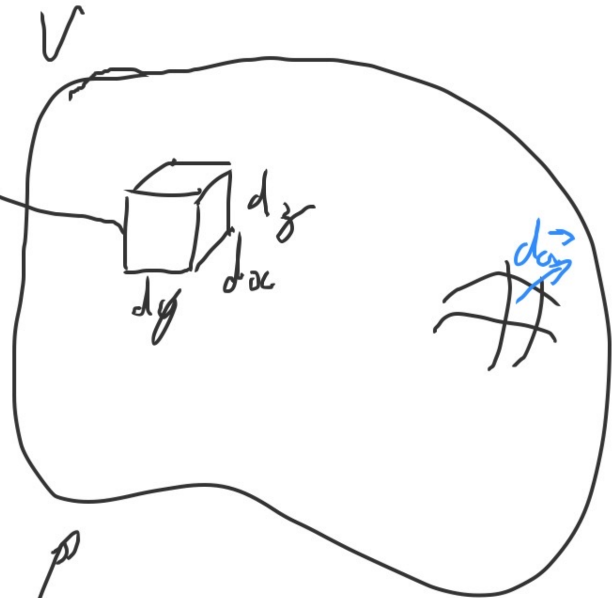
(Ampère - atribuído)

50 anos para estabelecer os fundamentos!

+ Equação de continuidade $\rightarrow$ $\nabla\cdot\vec{J} = -\frac{\partial}{\partial t}\rho$

+ Força de Lorentz (1895) $\rightarrow$ $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$

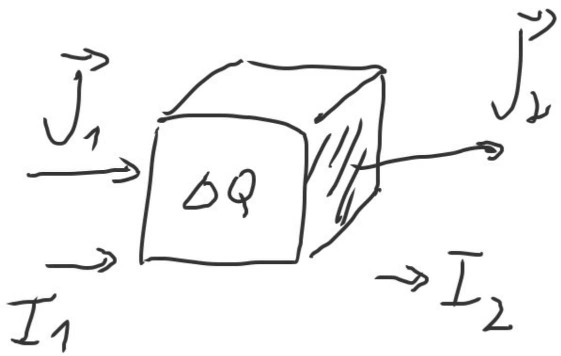
$$\nabla\cdot\vec{J} = -\frac{\partial}{\partial t}\rho$$



$$\int_V \nabla\cdot\vec{J}\cdot dv = -\int_V \frac{\partial}{\partial t}\rho\, dv$$

$$\oint_S \vec{J}\cdot d\vec{a} = -\frac{\partial}{\partial t}\int_V \rho\, dv$$

$$\oint_S \vec{J}\cdot d\vec{a} = -\frac{d}{dt} Q$$



Mas algo está faltando...

Se calculamos: $\nabla\cdot(\nabla\times\vec{B}) = \mu_0 \nabla\cdot\vec{J}$

Lembra que $\vec{B} = \nabla\times\vec{A}$; $\nabla\cdot\vec{B} = \nabla\cdot(\nabla\times\vec{A}) = 0$?

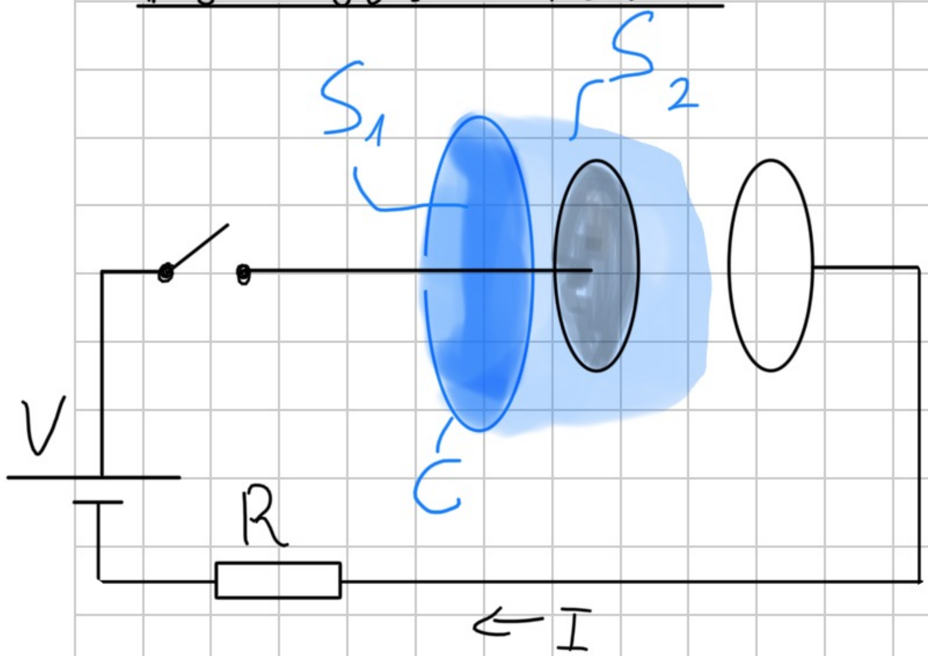
$$\nabla\cdot(\nabla\times\vec{B}) = 0 \quad , \quad \text{mas} \quad \nabla\cdot\vec{J} = -\frac{\partial}{\partial t}\rho \quad \underset{0}{\nabla}$$

Note que $\nabla\cdot(\nabla\times\vec{E}) = -\nabla\cdot\left(\frac{\partial}{\partial t}\vec{B}\right) = -\frac{\partial}{\partial t}(\nabla\cdot\vec{B})$

$$\nabla\cdot(\nabla\times\vec{E}) = 0 \quad ; \quad \nabla\cdot\vec{B} = 0$$

<u>Matematicamente inconsistente</u>

Por outro lado:



Campo $\vec{B}$ na curva C (ampereana) independe da superfície S

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I$$

Em S_1, $I > 0$, mas em S_2, $I = 0$!

O que falta?

Como $\nabla \cdot \vec{J} = -\frac{\partial}{\partial t}\rho$; $\rho = \epsilon_0 [\nabla \cdot \vec{E}]$

$$\nabla \cdot \vec{J} = -\frac{\partial}{\partial t}\epsilon_0 (\nabla \cdot \vec{E}) = -\nabla \cdot \left(\epsilon_0 \frac{\partial}{\partial t}\vec{E}\right)$$

$$\therefore \quad \nabla \cdot \left(\vec{J} + \epsilon_0 \frac{\partial}{\partial t}\vec{E}\right) = 0$$

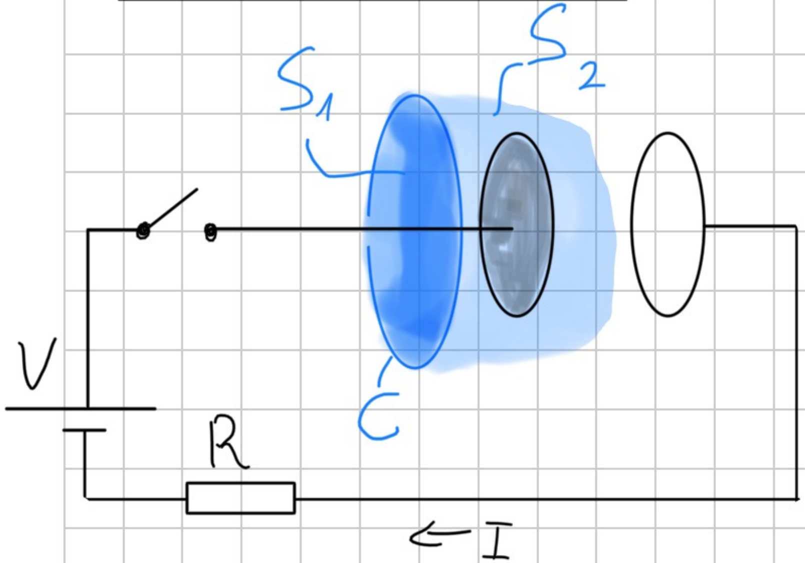
$$\nabla \cdot (\nabla \times \vec{B}) = \nabla \cdot \vec{J} + \nabla \cdot \left(\epsilon_0 \frac{\partial}{\partial t}\vec{E}\right)$$

Maxwell reescreve a relação como

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\vec{E}$$

Corrente de deslocamento de Maxwell $\vec{J}_d = \epsilon_0 \frac{\partial}{\partial t}\vec{E}$

Por outro lado:



Campo $\vec{B}$ na curva C (ampereana) independe da superfície S

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I$$

No capacitor: $E = \dfrac{Q}{\epsilon_0 A} \Rightarrow \dfrac{\partial}{\partial t} E = \dfrac{1}{\epsilon_0 A} \dfrac{\partial}{\partial t} Q = \dfrac{I}{\epsilon_0 A}$

$$\int_{S_1} \mu_0 \vec{J} \, d\vec{a} = \int_{S_2} \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{E} \cdot d\vec{a}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\mu_0 I = \mu_0 \epsilon_0 \frac{I}{\epsilon_0 A} \cdot A$$

Equações de Maxwell

$$\nabla \cdot \vec{D} = \rho$$ (Gauss, 1835)

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$ (Faraday, 1831)

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}\vec{D}$$ (Ampère-Maxwell, 1863)

$+$

Relacões constitutivas

$$\vec{D} = \epsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

+ Equação de continuidade → $\nabla \cdot \vec{J} = -\frac{\partial}{\partial t}\rho$

+ Força de Lorentz (1.895) → $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

Coulomb : 1785

Marcos: Pilha (Volta) 1800

Iluminação (Humphry Davi), ~1806

Motor (Jedlik) 1828

Gerador (Faraday) 1831

Telégrafo – 1837 (Cooke & Wheatstone)

→ 1851 (Morse)

Iluminação pública ~1880

Potenciais: Vimos que podemos usar

$\vec{B} = \nabla \times \vec{A}$ (se $\nabla \cdot \vec{B} = 0$) ; $\vec{E} = -\nabla V$ (se $\nabla \times \vec{E} = 0$)

Note que $\nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} = -\frac{\partial}{\partial t} (\nabla \times \vec{A}) = -\nabla \times \left(\frac{\partial}{\partial t} A \right)$

$$\Rightarrow \nabla \times \left(\vec{E} + \frac{\partial}{\partial t} \vec{A} \right) = 0$$

$\therefore \vec{E} + \frac{\partial}{\partial t} \vec{A} = -\nabla V \rightarrow$ recuperando o formalismo!

$\therefore \vec{E} = -\nabla V - \frac{\partial}{\partial t} \vec{A} \rightarrow$ voltando ao caso estático a $\vec{E} = -\nabla V$

Ótimo para $\nabla \cdot \vec{B} = 0$ $\quad \nabla \times \vec{E} = -\frac{\partial}{\partial t} \vec{B}$

Como ficam as equações restantes, com fontes?

$$\nabla \cdot \vec{D} = \rho \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t}\vec{D}$$

Para simplificar, vamos trabalhar com o vácuo: $\vec{D} = \epsilon_0 \vec{E}$

$$\nabla \cdot \vec{D} = \rho \Rightarrow \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \vec{E} = -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = \frac{\rho}{\epsilon_0}$$

Equação de Poisson modificada: $\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\frac{\rho}{\epsilon_0}$

$$\vec{H} = \frac{\vec{B}}{\mu_0} \Rightarrow \nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{E}$$

$$\nabla \times \left(\nabla \times \vec{A}\right) = \mu_0 \vec{J} - \mu_0 \epsilon_0 \left[\frac{\partial}{\partial t}(\nabla V) + \frac{\partial^2}{\partial t^2} \vec{A} \right]$$

Como $\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

$$\left(\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{A}\right) - \nabla \left(\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} V \right) = -\mu_0 \vec{J}$$

Forma alternativa (e feia) das eqs. de Maxwell

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot A) = -\frac{\rho}{\epsilon_0}$$

$$\left(\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2}\vec{A}\right) - \nabla\left(\nabla \cdot \vec{A} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} V\right) = -\mu_0 \vec{J}$$

$$\vec{B} = \nabla \times \vec{A} \quad ; \quad \vec{E} = -\nabla V - \frac{\partial}{\partial t}\vec{A}$$

Mas dá para melhorar: Ajustando os potenciais

→ Transformações de Calibre

$$\vec{A}' = \vec{A} + \vec{\alpha} \quad ; \quad V' = V + \beta$$

$$\vec{B} = \nabla \times \vec{A}' = \nabla \times \vec{A} + \nabla \times \vec{\alpha} \Rightarrow \quad \nabla \times \vec{\alpha} = 0 \rightarrow \vec{\alpha} = \nabla \lambda$$

$$\vec{A}' = \vec{A} + \nabla \lambda$$

$$\vec{E} = -\nabla V' - \frac{\partial}{\partial t}\vec{A}' = -\nabla V - \nabla \beta - \frac{\partial}{\partial t}\vec{A} - \frac{\partial}{\partial t}\nabla \lambda$$

$$= -\nabla . V - \frac{\partial}{\partial t}\vec{A} - \nabla\left(\beta + \frac{\partial}{\partial t}\lambda\right)$$

$$\Rightarrow \nabla\left(\beta + \frac{\partial}{\partial t}\lambda\right) = 0 \qquad \beta + \frac{\partial}{\partial t}\lambda = f(t)$$ (independente da posição)

$$\vec{E} = -\nabla V' - \frac{\partial}{\partial t}\vec{A}' = -\nabla V - \nabla\beta - \frac{\partial}{\partial t}\vec{A} - \frac{\partial}{\partial t}\nabla\lambda$$

$$= -\nabla . V - \frac{\partial}{\partial t}\vec{A} - \nabla\left(\beta + \frac{\partial}{\partial t}\lambda\right)$$

$$\Rightarrow \nabla\left(\beta + \frac{\partial}{\partial t}\lambda\right) = 0 \qquad \beta + \frac{\partial}{\partial t}\lambda = f(t)$$ (independente da posição)

Simplificando $f(t) = 0 \rightarrow \beta = -\frac{\partial}{\partial t}\lambda$

Liberdade de ajustar $\lambda(\vec{r}, t)$,

$$\vec{A}' = \vec{A} + \nabla\lambda \quad ; \quad V' = V - \frac{\partial}{\partial t}\lambda$$

Compensando os dois lados

Calibre de Coulomb: Adotando $\nabla \cdot \vec{A} = 0$

$$\nabla^2 V + \frac{\partial}{\partial t}\cancelto{0}{(\nabla \cdot A)} = -\frac{\rho}{\epsilon_0}$$ → Poisson

$$\left(\nabla^2 \vec{A} - \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{A}\right) - \nabla\left(\cancelto{0}{\nabla \cdot \vec{A}} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} V\right) = -\mu_0 \vec{J}$$

Potencial escalar segue instantaneamente a carga ✓

Potencial vetor permanece feio...

Calibre de Lorentz : Adotando $\nabla\cdot\vec{A} = -\mu_0\epsilon_0 \frac{\partial}{\partial t} V$

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla\cdot A) = -\frac{\rho}{\epsilon_0} \rightarrow \nabla^2 V - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2} V = -\frac{\rho}{\epsilon_0}$$

$$\left(\nabla^2\vec{A} - \mu_0\epsilon_0 \frac{\partial}{\partial t}\vec{A}\right) - \nabla\left(\underbrace{\nabla\cdot\vec{A} + \mu_0\epsilon_0 \frac{\partial}{\partial t} V}_{0}\right) = -\mu_0\vec{J}$$

V e $\vec{A}$ desacoplam !

$$\left(\nabla^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\right) V = -\frac{\rho}{\epsilon_0} \qquad \square^2 V = -\frac{\rho}{\epsilon_0}$$

$$\Rightarrow$$

$$\left(\nabla^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_0\vec{J} \qquad \square^2\vec{A} = -\mu_0\vec{J}$$

D' Alambertiano

$\vec{r}$ e t entram em pé de igualdade !

Forças de Lorentz

$$\vec{F} = \frac{d}{dt}\vec{p} = q\,(\vec{E} + \vec{v}\times\vec{B})$$

$$= q\left[-\nabla V - \frac{\partial}{\partial t}\vec{A} + \vec{v}\times(\nabla\times\vec{A})\right]$$

Como $\nabla(\vec{v}\cdot\vec{A}) = \vec{v}\times(\nabla\times\vec{A}) + (\vec{v}\cdot\nabla)\vec{A}$ (pois $\vec{v} = \vec{v}(t)$)

$$\frac{d}{dt}\vec{p} = -q\Big[\underbrace{\frac{\partial}{\partial t}\vec{A} + (\vec{v}\cdot\nabla)\vec{A}}_{\frac{d}{dt}\vec{A}} + \nabla\,(V - \vec{v}\cdot\vec{A})\Big]$$

$\frac{d}{dt}\vec{A}$, pois $d\vec{A} = \vec{A}(\vec{r} + \vec{v}dt, t + dt) - \vec{A}(\vec{r}, t)$

$$= \left(\frac{\partial}{\partial x}\vec{A}\right) v_x\,dt + \left(\frac{\partial}{\partial y}\vec{A}\right) v_y\,dt$$

$$+ \left(\frac{\partial}{\partial z}\vec{A}\right) v_z\,dt + \left(\frac{\partial}{\partial t}\vec{A}\right) dt$$

Forças de Lorentz

$$\vec{F} = \frac{d}{dt}\vec{p} = q\,(\vec{E} + \vec{v} \times \vec{B})$$

$$= q\left[-\nabla V - \frac{\partial}{\partial t}\vec{A} + \vec{v} \times (\nabla \times \vec{A})\right]$$

como $\nabla(\vec{v}\cdot\vec{A}) = \vec{v} \times (\nabla \times \vec{A}) + (\vec{v}\cdot\nabla)\vec{A}$ (pois $\vec{v} = \vec{v}(t)$)

$$\frac{d}{dt}\vec{p} = -q\Big[\underbrace{\frac{\partial}{\partial t}\vec{A} + (\vec{v}\cdot\nabla)\vec{A}}_{\frac{d}{dt}\vec{A}} + \nabla\,(V - \vec{v}\cdot\vec{A})\Big]$$

$\frac{d}{dt}\vec{A}$, pois $d\vec{A} = \vec{A}(\vec{r} + \vec{v}dt, t+dt) - \vec{A}(\vec{r}, t)$

$$= \left(\frac{\partial}{\partial x}\vec{A}\right)\underbrace{v_x\,dt}_{dx} + \left(\frac{\partial}{\partial y}\vec{A}\right)\underbrace{v_y\,dt}_{dy}$$

$$+ \left(\frac{\partial}{\partial z}\vec{A}\right)\underbrace{v_z\,dt}_{dz} + \left(\frac{\partial}{\partial t}\vec{A}\right)dt = (\vec{v}\cdot\nabla)\vec{A}\cdot dt + \frac{\partial}{\partial t}\vec{A}\cdot dt$$

$d\vec{f}(x,y) =$

$$\frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy$$

$$\frac{d}{dt}\vec{p} = -q\left[\underbrace{\frac{\partial}{\partial t}\vec{A} + (\vec{v}\cdot\nabla)\vec{A}}_{\frac{d}{dt}\vec{A}} + \nabla\,(V - \vec{v}\cdot\vec{A})\right]$$

$\frac{d}{dt}\vec{A}$, pois $d\vec{A} = \vec{A}(\vec{r}+\vec{v}dt,\, t+dt) - \vec{A}(\vec{r},t)$

$$= \left(\frac{\partial}{\partial x}\vec{A}\right) v_x\, dt + \left(\frac{\partial}{\partial y}\vec{A}\right) v_y\, dt$$

$$+ \left(\frac{\partial}{\partial z}\vec{A}\right) v_z\, dt + \left(\frac{\partial}{\partial t}\vec{A}\right) dt$$

$$\Rightarrow \frac{d}{dt}\left(\vec{p} + q\vec{A}\right) = -\nabla\left[q\left(V - \vec{v}\cdot\vec{A}\right)\right]$$

$$\frac{d}{dt}\vec{p}_{can} = -\nabla U$$ → potencial acoplado à velocidade

↳ momento canônico