PGF5112 - Plasma Physics I

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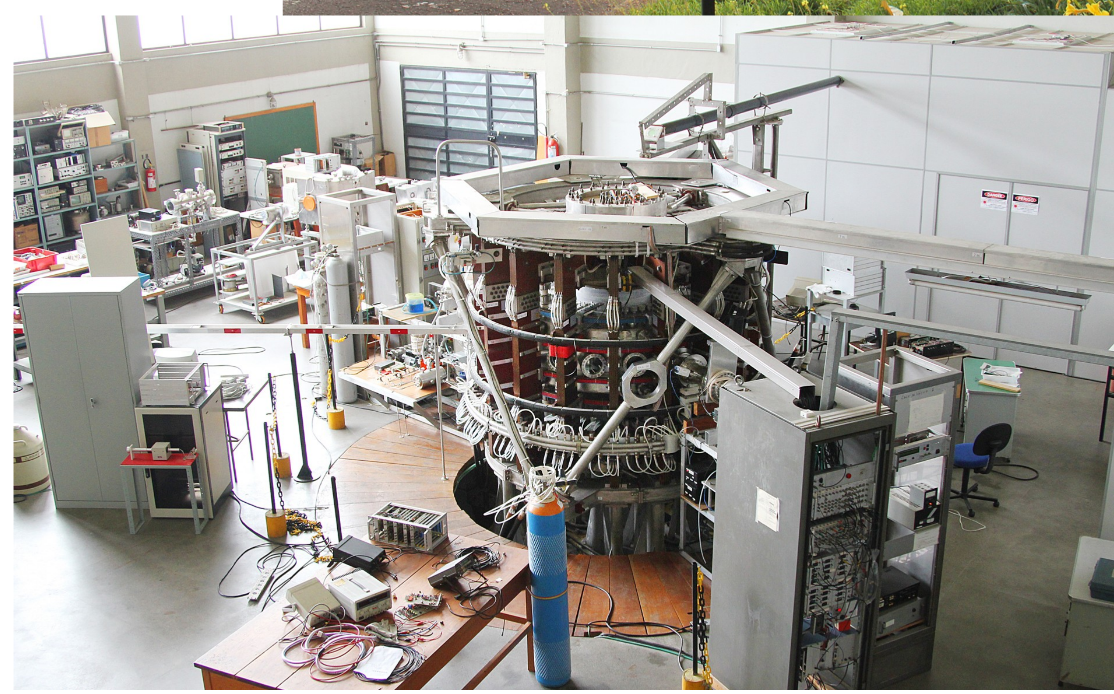

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SCIENTIA VINCES

- **Tokamak engineering**
  - *Magnetic forces*
  - *Toroidal field coils*
  - *The central solenoid*
  - *Poloidal field coils*
  - *The vertical plasma instability and the RZIP model*

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Magnetic forces: the virtual work formalism

- **Establishing a magnetic field requires energy and this is a direct consequence of Faraday's induction law**

$$\epsilon = -\frac{d\Phi}{dt}$$

- **If a power supply of voltage V is connected to a circuit, then the current in the circuit can be expressed as $V + \epsilon = RI$, where ϵ accounts for the presence of an induced EMF**

- **The work done by the power supply to displace an amount of charge $dq = Idt$ is**

$$Vdq = VIdt = RI^2dt - \epsilon Idt = RI^2dt + Id\Phi$$

- **While the RI^2dt term represents the irreversible conversion of electrical energy into heat, the $Id\Phi$ term represents the work done by the power supply against the induced EMF. Therefore, the amount of work done ON the system to modify its magnetic field is**

$$dW_{ext} = \sum_{j=1}^{N} I_j d\Phi_j$$

Magnetic forces: the virtual work formalism

- **From classical electrodynamics, we know that the magnetic energy stored in a linear system is given by**

$$U = \frac{1}{2}\sum_{j=1}^{N} I_j \Phi_j$$

- **Suppose that one allows part of the system to move under the action of a magnetic force, but at constant currents. Then, the work done BY the force is**

$$dW = \mathbf{F} \cdot d\mathbf{r}$$

- **Under this circumstance, the work done BY the system has two contributions**

$$dW = dW_{ext} - dU$$

  - *Here, dU is the variation of the magnetic energy of the system and dW_{ext} is the work done BY the power supplies to keep the currents constant*

- **If the geometry of the system is modified, but the currents remain the same,**

$$dU = \frac{1}{2}\sum_{j=1}^{N} I_j d\Phi_j = \frac{dW_{ext}}{2}$$

Magnetic forces: the virtual work formalism

- **Since $dW_{ext} = 2dU$, one also has that $dW = dU$. Therefore, since the work done BY de magnetic force is $dW = \mathbf{F} \cdot d\mathbf{r} = dU$ one has that**

$$dU = \mathbf{F} \cdot d\mathbf{r} \quad \rightarrow \quad \mathbf{F} = \nabla U$$ *(Note that there is NO minus sign)*

- **If the system is forced to rotate around some axis, them we also have that**

$$dU = \boldsymbol{\tau} \cdot d\boldsymbol{\theta} \quad \rightarrow \quad \boldsymbol{\tau} = \nabla_{\boldsymbol{\theta}} U$$ *(Note that there is NO minus sign)*

- **The magnetic energy of a conductor with current I and self-inductance L is**

$$U = \frac{1}{2} L I^2$$

while the magnetic energy of, for example, a pair of conductors is

$$U = \frac{1}{2} \left(L_1 I_1^2 + L_2 I_2^2 \pm 2 M_{12} I_1 I_2 \right)$$ *(Here, M_{12} is the mutual inductance)*

- **Therefore, the force and torque that cause the magnetic energy of the system to decrease (keeping the current constant) are**

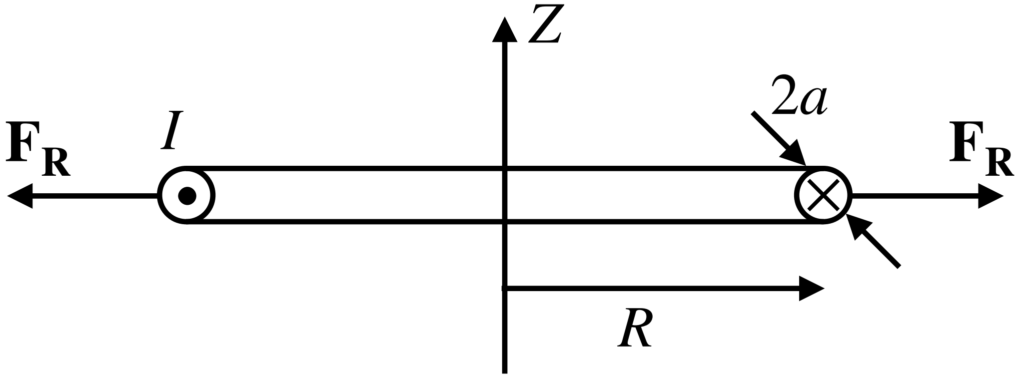
$$F_i = \frac{\partial U}{\partial x_i} \quad \text{and} \quad \tau_i = \frac{\partial U}{\partial \theta_i}$$

Magnetic forces: the hoop force

- **The self-inductance of a circular (DC) current loop with $R \gg a$ is**

$$L = \mu_0 R \left[\ln\left(\frac{8R}{a}\right) - \frac{7}{4} \right]$$

  - *Therefore, the magnetic force F_i acting on the system due to a change on the variable x_i is given by*



$$F_i = \frac{\partial}{\partial x_i}\left(\frac{1}{2} L I^2\right) = \frac{I^2}{2}\frac{\partial L}{\partial x_i}$$

  - *Supposing that the diameter of the wire ($2a$) does not change, the only force acting on a circular current loop is*

$$F_R = \frac{I^2}{2}\frac{\partial L}{\partial R} = \frac{\mu_0 I^2}{2}\left[\ln\left(\frac{8R}{a}\right) - \frac{3}{4} \right]$$

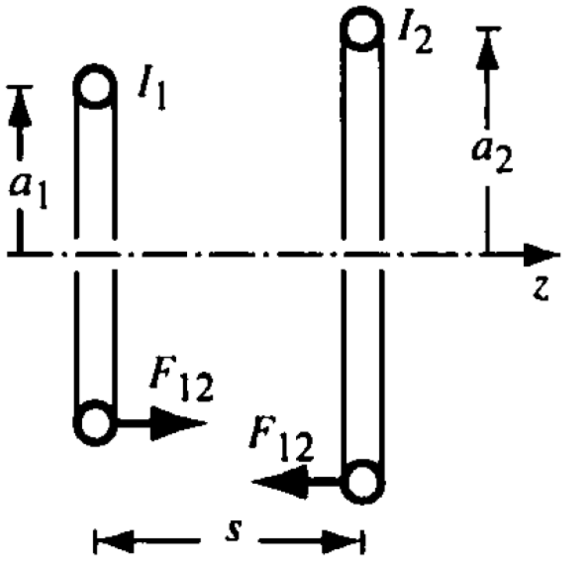
This is the so-called hoop force

  - *This force points radially outwards and it tends to increase the loop radius*

Magnetic forces: the virtual work formalism

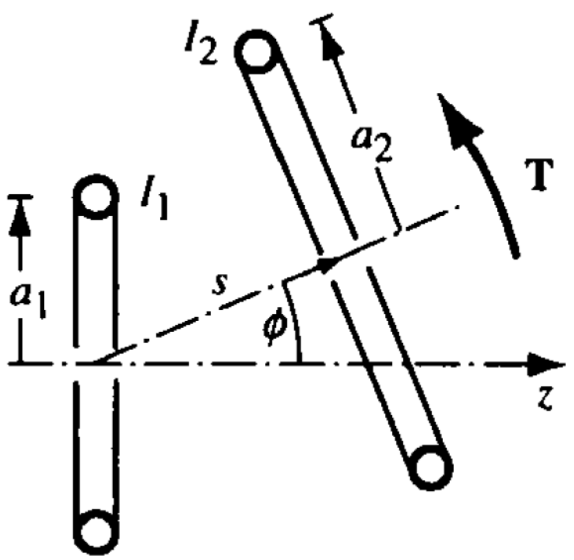
- **The force between two coaxial circular current loops**
  - *Supposing that the diameter of the wires and the radii of the current loops (a_1 and a_2) do not change, the only force acting on the circular current loops is due to a change in their separation distance (s):*

$$F_s = \frac{1}{2}\frac{\partial}{\partial s}\left(L_1 I_1^2 + L_2 I_2^2 \pm 2M_{12} I_1 I_2\right) = \pm I_1 I_2 \frac{\partial M_{12}}{\partial s}$$



- **The torque between two inclined circular current loops**
  - *Supposing that the diameter of the wires and the radii of the current loops (a_1 and a_2) do not change, the only torque acting on the circular current loops is due to a change in their orientation angle (ϕ):*

$$\tau_\phi = \frac{1}{2}\frac{\partial}{\partial \phi}\left(L_1 I_1^2 + L_2 I_2^2 \pm 2M_{12} I_1 I_2\right) = \pm I_1 I_2 \frac{\partial M_{12}}{\partial \phi}$$



Exercise

- **The mutual inductance between two coaxial circular current loops of radius r_1 and r_2 , and separation distance d, is given by**

$$M_{12} = \mu_0 \sqrt{r_1 r_2} \left[\left(\frac{2}{k} - k \right) K(k^2) - \frac{2}{k} E(k^2) \right] \qquad \text{with} \qquad k^2 = \frac{4 r_1 r_2}{(r_1 + r_2)^2 + d^2}$$

where $K(k^2)$ and $E(k^2)$ are the complete elliptic integrals of first and second kind, respectively:

$$K(k^2) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \qquad\qquad E(k^2) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \phi}\, d\phi$$

Expanding these expressions up to 2nd order in k^2 ($d \gg r_1$ and $d \gg r_2$) yields

$$M_{12} = \frac{\mu_0 \pi r_1^2 r_2^2}{2d^3} \left(1 + 3\frac{r_1 r_2}{d^2} + \frac{75}{8} \frac{r_1^2 r_2^2}{d^4} + \dots \right)$$

Calculate the force between two coaxial circular current loops up to 2nd order and show that the force is attractive for currents running in the same direction

- **Tokamak engineering**
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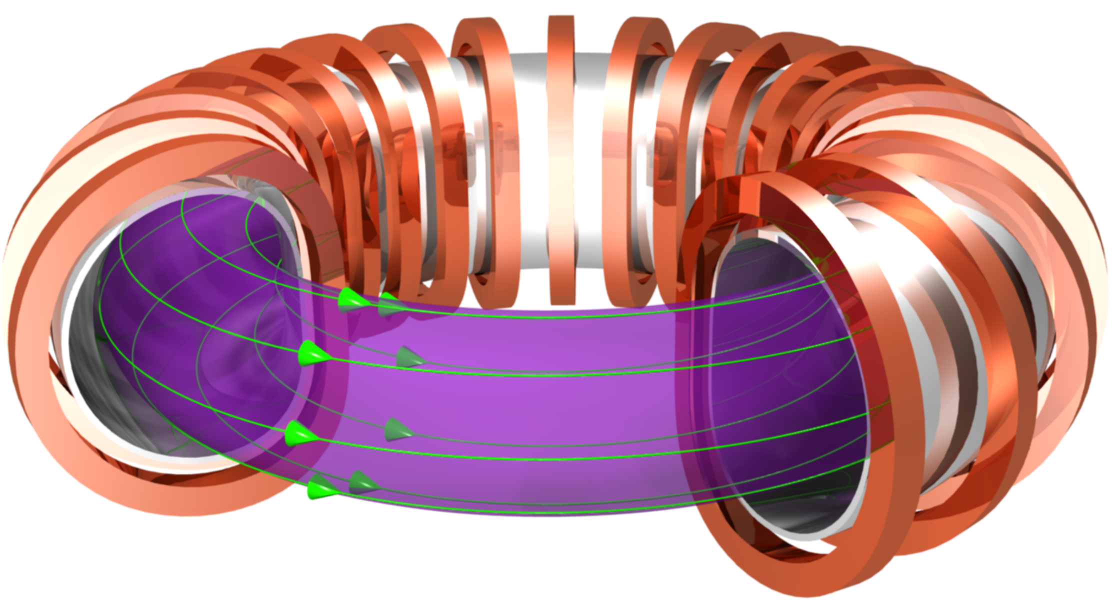
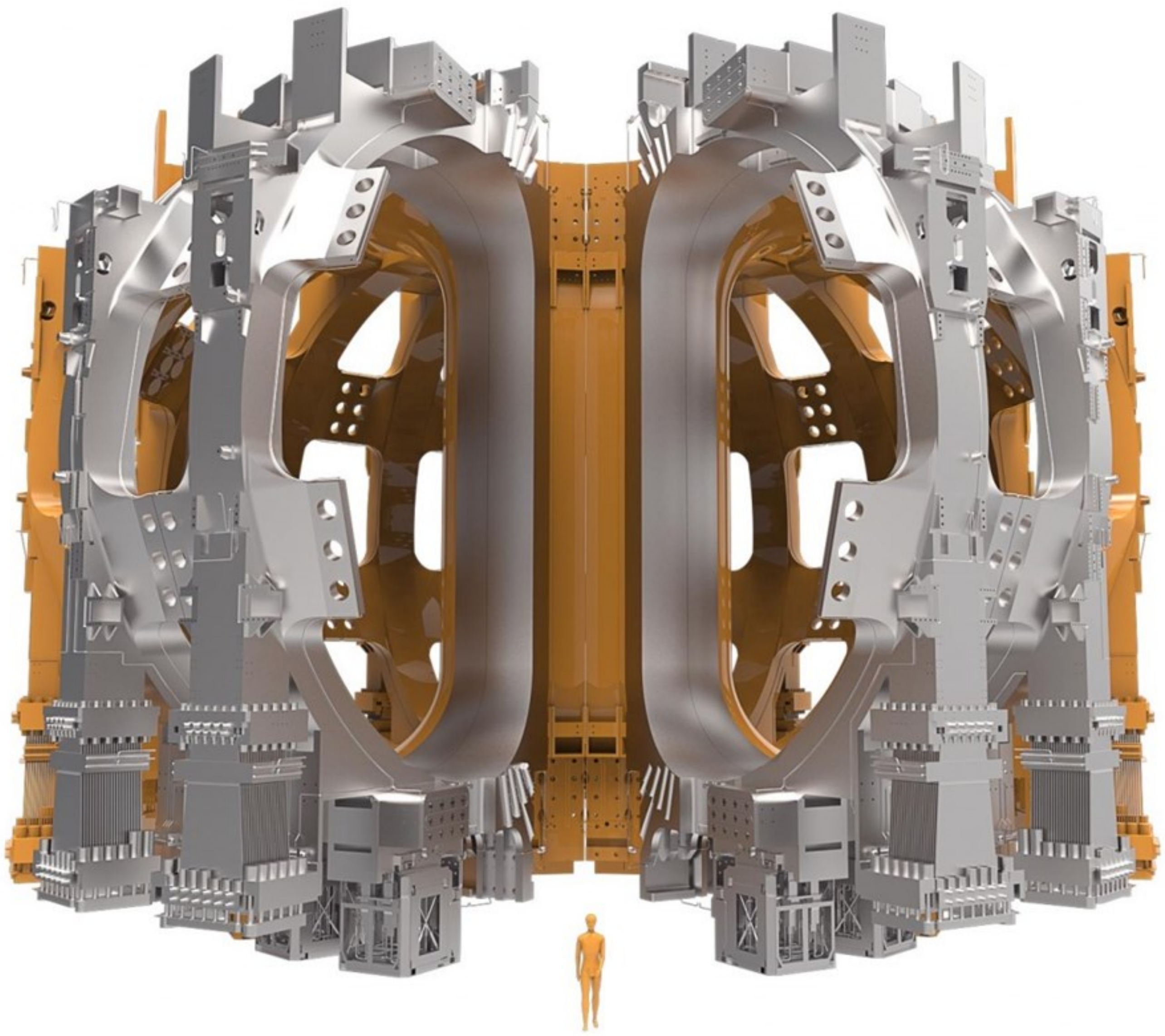
Toroidal field coils

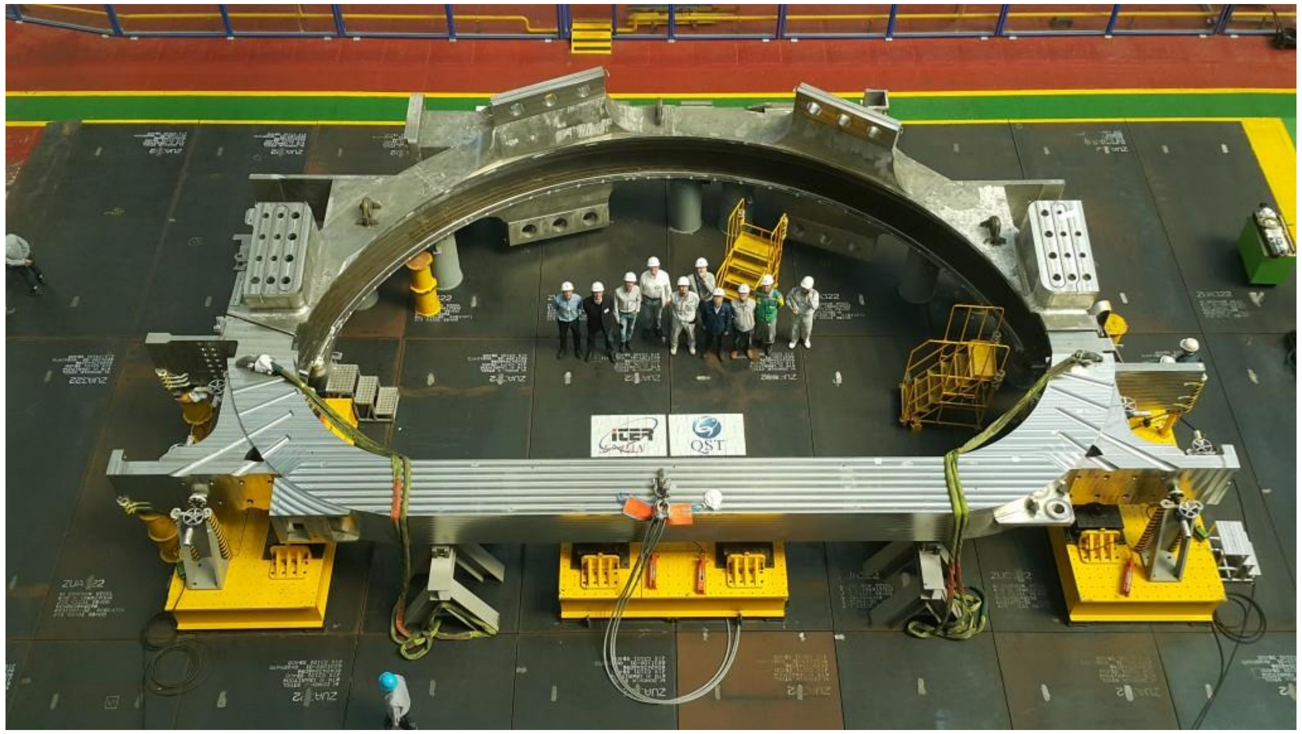
- **The toroidal field coils produce the plasma confining magnetic field, which is the quantity the has the strongest impact on the fusion power to be produced**
  - *The fusion power in a tokamak is* $P_{\mathrm{fus}} \propto B_0^4 V$





The toroidal field coils of ITER

- **The list of applicable superlatives about the ITER toroidal field coils is long**
  - *The toroidal field coils are the largest and most powerful superconducting magnets ever designed, with a stored magnetic energy of 41 GJ*
  - *Together, they weigh in at over 6,000 tonnes*
  - *They required the production of 500 tonnes of Nb_3Sn superconducting strand (100,000 km) required for the toroidal field superconducting cables*



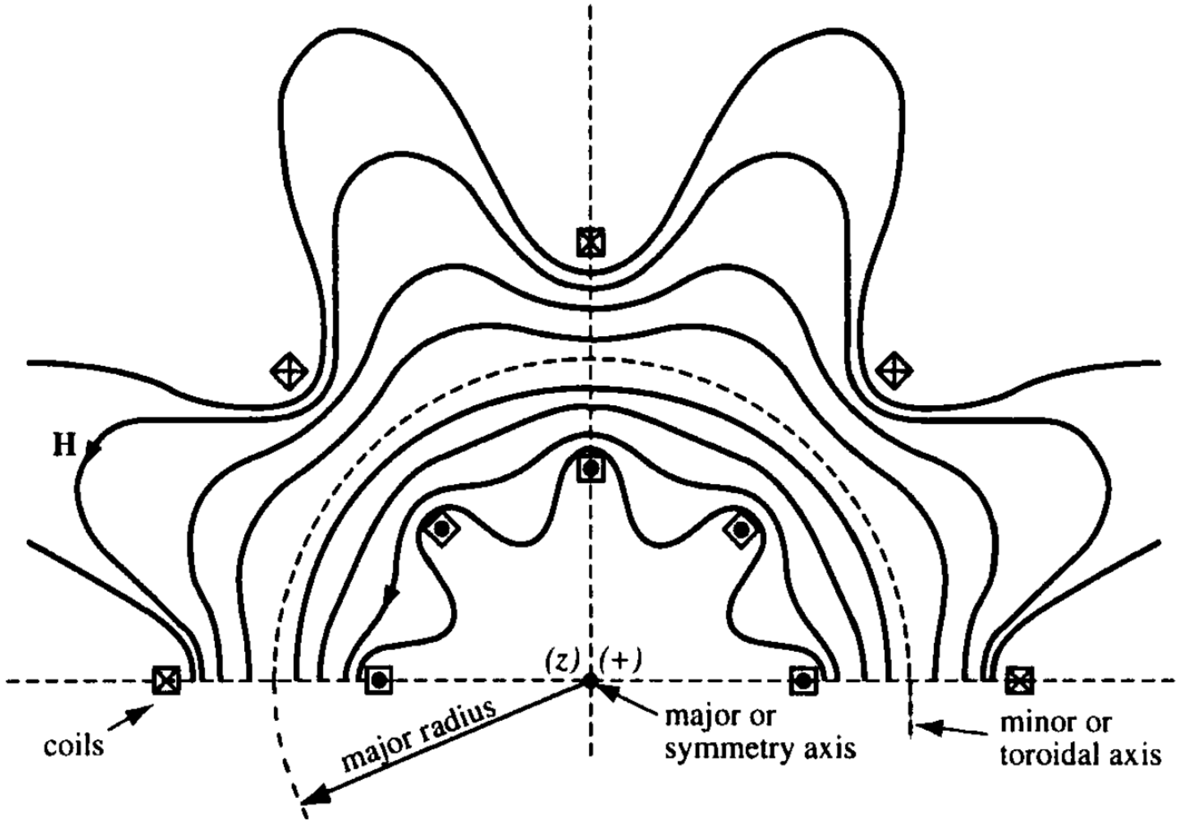
Toroidal field ripple due to the discrete number of coils

- **Due to the discrete number of toroidal field coils, the magnetic field close to the coils is not purely toroidal. This fluctuation is called ripple**
  - *Toroidal field ripple increases plasma losses and, therefore, decreases confinement and plasma performance*

- **Magnetic field lines at the equatorial plane (Z = 0) view from the top**
  - *Field line trajectories can be found by integrating the field line equations*

$$\mathbf{B} \times d\mathbf{l} = 0$$

$$\frac{dR}{B_R} = \frac{Rd\phi}{B_\phi} \qquad \frac{dZ}{B_Z} = \frac{Rd\phi}{B_\phi}$$



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- **Magnetic field lines at the equatorial plane (Z = 0) view from the top**
  - *Field line trajectories can be found by integrating the field line equations*

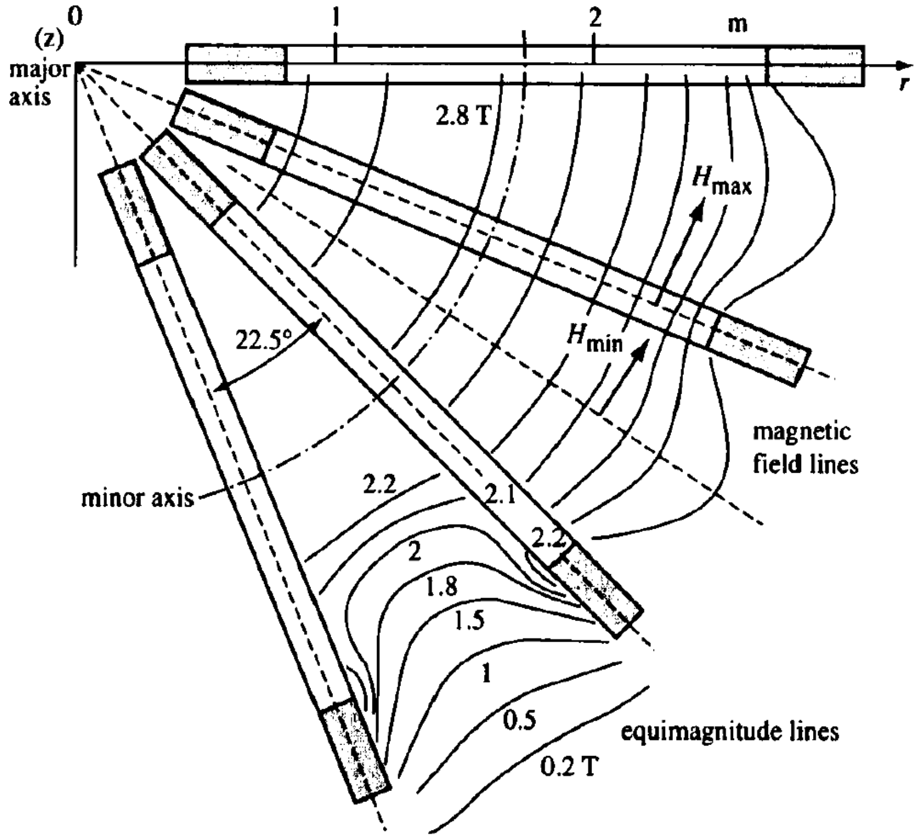
- **The magnetic field ripple can be characterized be the parameter**

$$\epsilon = \frac{H_{max} - H_{min}}{H_0} = \frac{B_{max} - B_{min}}{B_0}$$

where

$$H_0 = \frac{H_{max} + H_{min}}{2} \qquad B_0 = \frac{B_{max} + B_{min}}{2}$$

  - *Acceptable values of field ripple are* $\epsilon < 0.1\,\%$

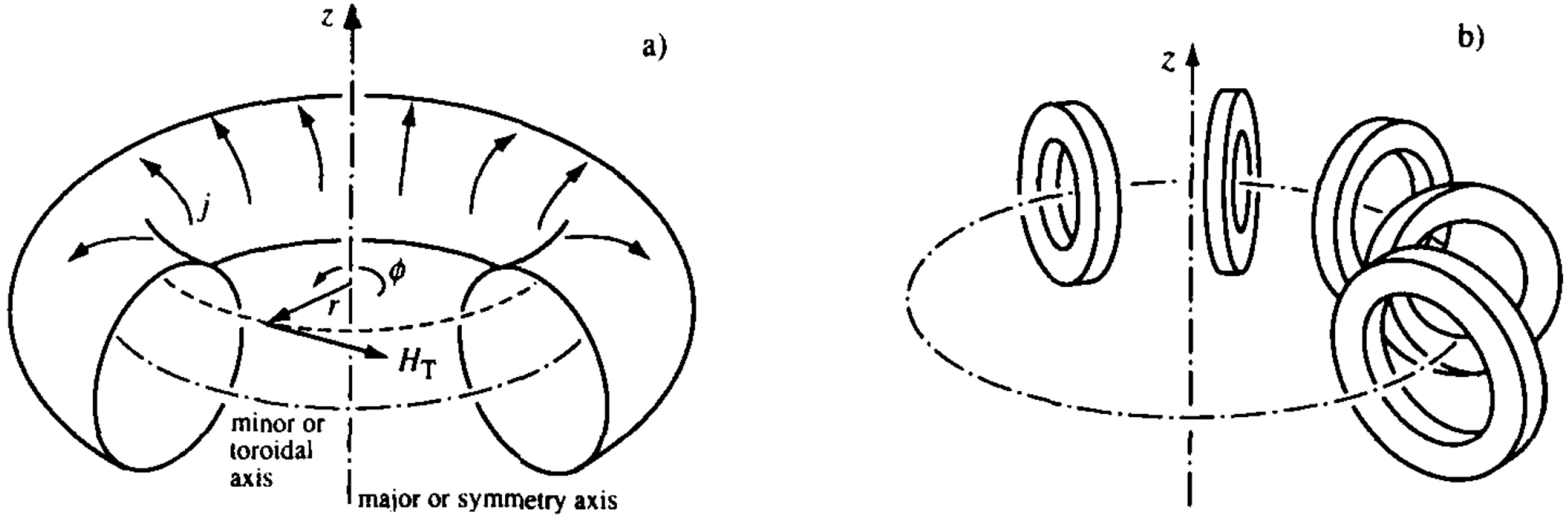


Toroidal field coil forces: the centering force

- **The inductance of a evenly wound toroidal sheet magnet of circular radius $\bar{a}$ is**

$$L = \mu_0 N^2 \left(R_0 - \sqrt{R_0^2 - \bar{a}^2} \right)$$

  - *When the toroidal magnet is made of single coils, this expression must be corrected. However, this effect can be neglected when $N > 10$*



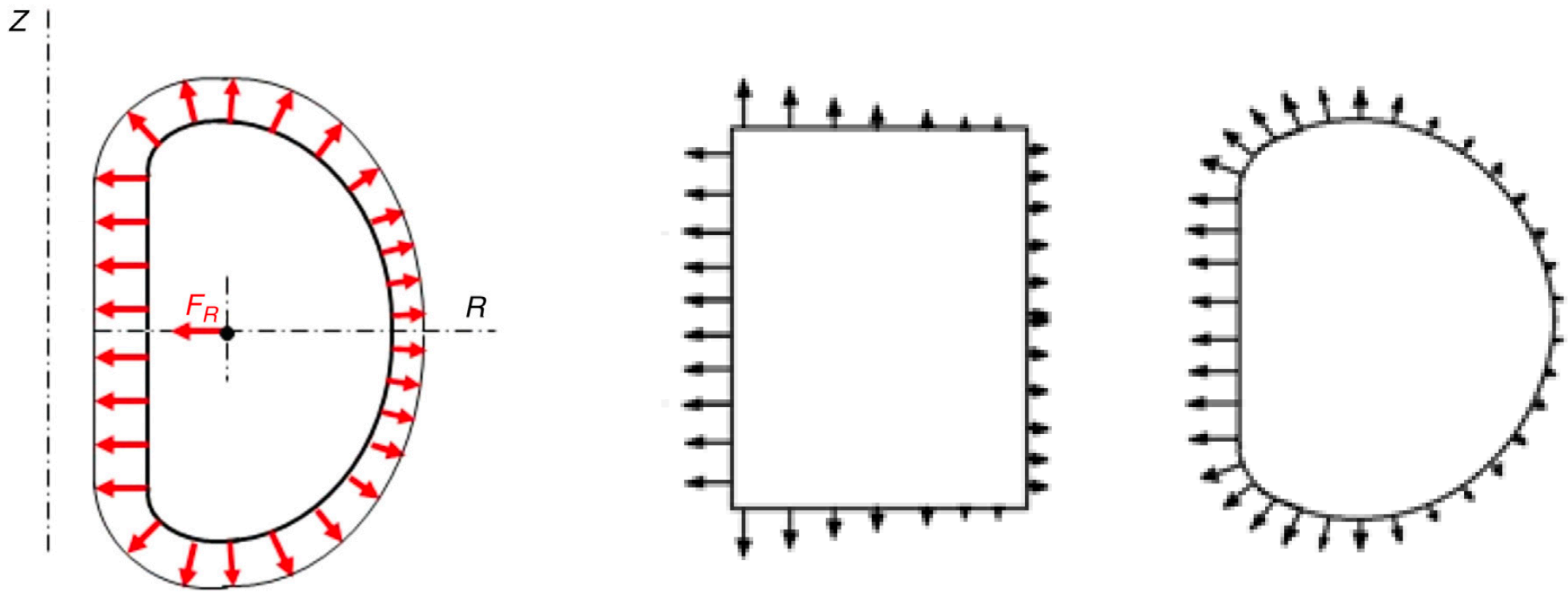
- **The force on a toroidal field coils can be calculated as**

$$F_R\Big|_{\text{per coil}} = \frac{I^2}{2N}\frac{\partial}{\partial R_0}\left[\mu_0 N^2 \left(R_0 - \sqrt{R_0^2 - \bar{a}^2} \right)\right] = -\frac{\mu_0 N I^2}{2}\left(\frac{R_0}{\sqrt{R_0^2 - \bar{a}^2}} - 1 \right)$$

Centering Force

Toroidal field coil forces: the centering force

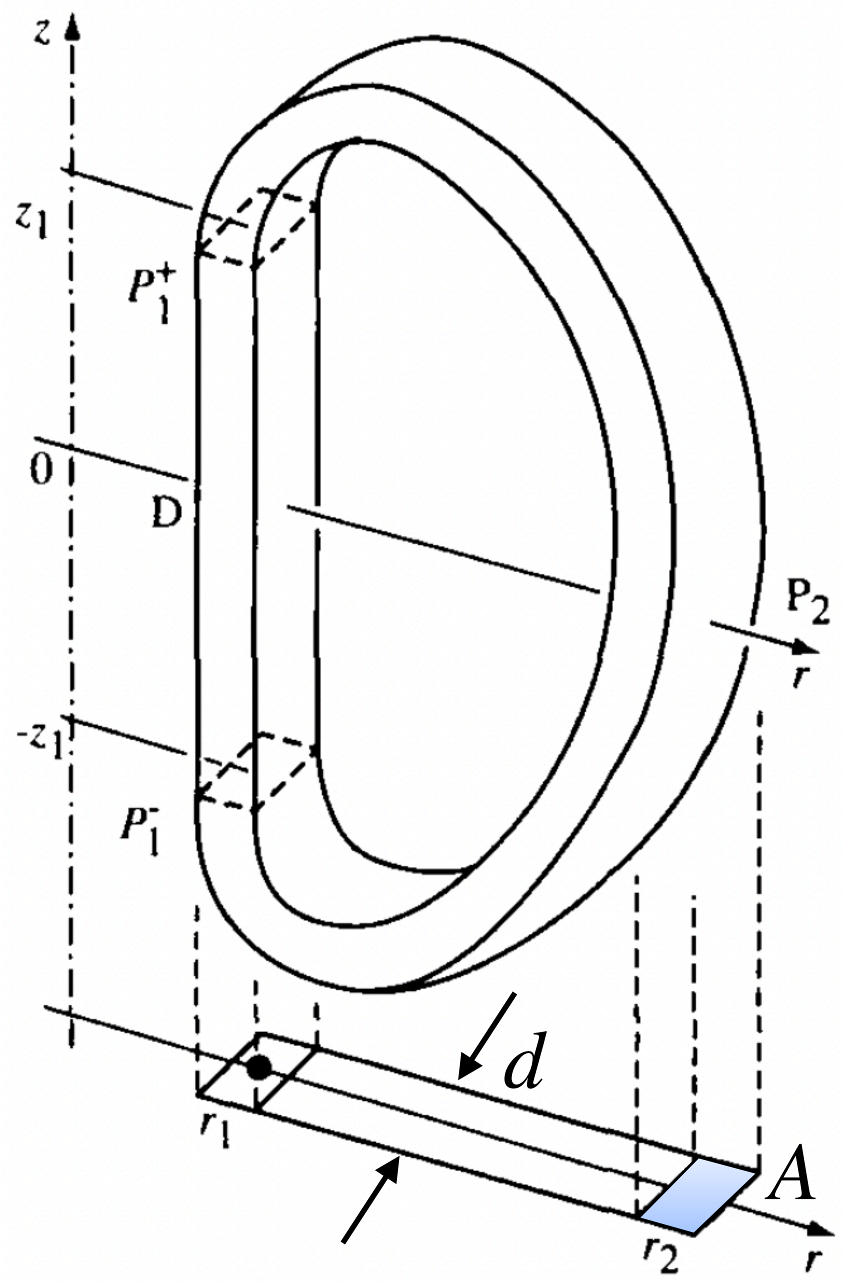
- **In tokamaks, there is a centering force acting on each toroidal field coils**
  - *Depending on the shape of the TF coils, regions with large mechanical stress can arise*



- **Is there a shape that makes the TF coils to have a constant stress?**
  - *YES, that is why TF coils in large tokamaks have a D-shape*

Toroidal field coil forces: the D-shape

- **To find the TF coil shape that leads to constant stress, let's consider a coil with cross section area A and thickness (in the toroidal direction) d**



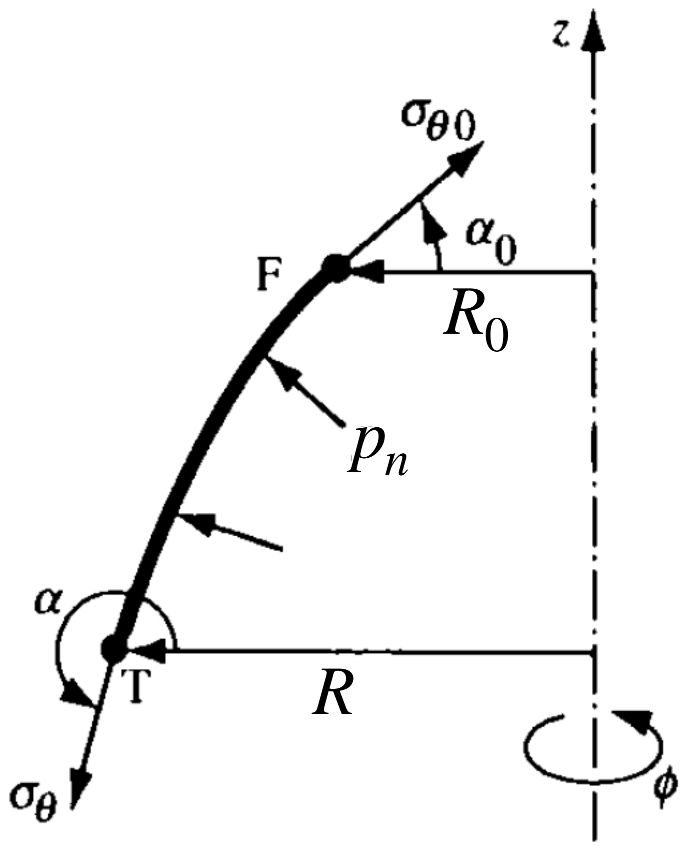
Toroidal field coil forces: the D-shape

- **To find the TF coil shape that leads to constant stress, let's consider a coil with cross section area A and thickness (in the toroidal direction) d**

- **One can show that, in equilibrium, the vertical component of the total tangential stress along the coil, acting on F and T,**

$$S_Z = A\sigma_{\theta 0} \sin \alpha_0 - A\sigma_\theta \sin \alpha$$

must balance the vertical component of the normal pressure load p_n



$$F_Z = \int_s^{s_0} p_z(s)\, d\, ds = d \int_s^{s_0} p_n \cos(\alpha)\, ds = d \int_R^{R_0} p_n(R) dR$$

  - *The last integral, with $dR = ds\, \cos(\alpha)$, is independent of the integration path, i.e. the coil shape*
  - *Here, s is a coordinate along the coil*
  - *If one imposes that R_0 be the radial position of the maximum heigh of the coil, i.e. the position where $dZ/dR = \tan \alpha_0 = 0$, then $S_Z = -A\sigma_\theta \sin \alpha$*

Toroidal field coil forces: the D-shape

- **Taking the normal pressure as being caused by the toroidal field**

$$p = p_1 \frac{R_1}{R} = \frac{B_1^2 R_1}{2\mu_0 R}$$

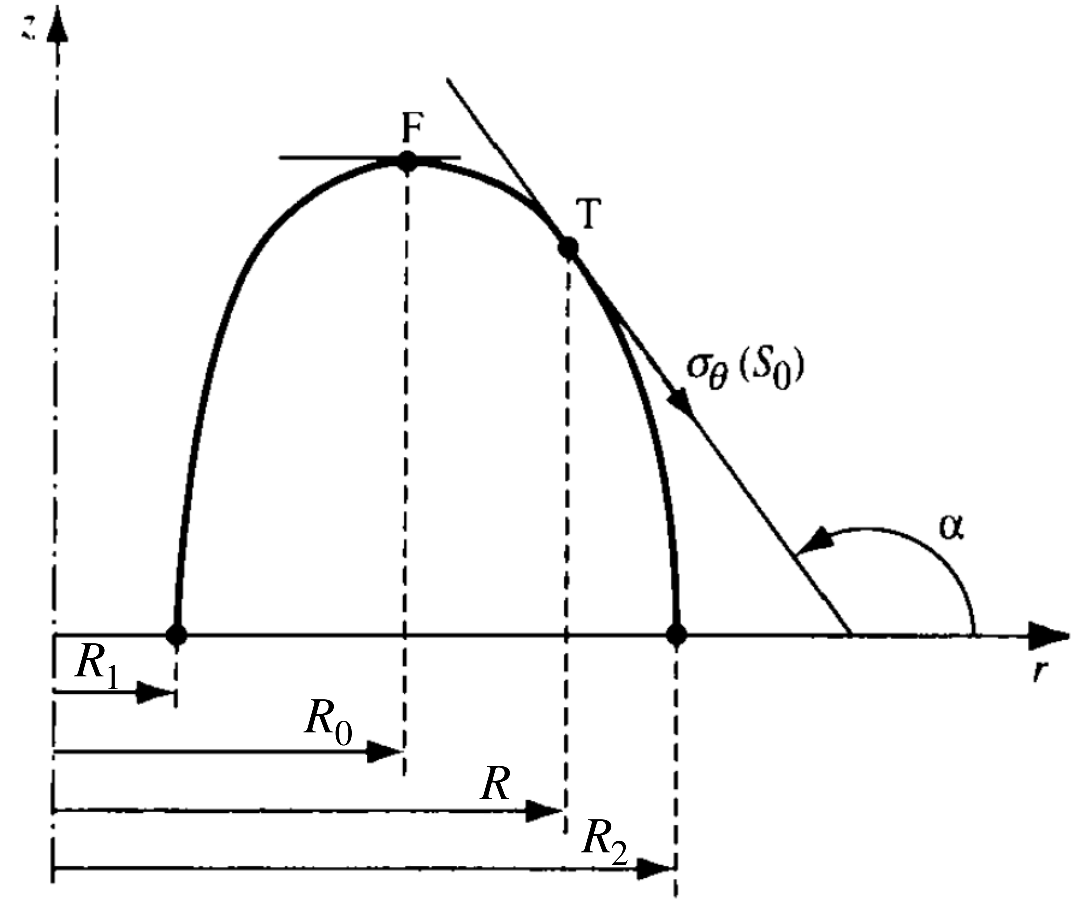
(Results from the $\mathbf{J} \times \mathbf{B}$ *force in a toroidal current sheet)*

the vertical component of the normal load is

$$F_Z = p_1 R_1 \, d \ln \frac{R_0}{R}$$

- **To find the curve $Z = Z(R)$ that gives constant $\sigma_\theta(R, Z)$, let's impose that, in equilibrium, one has $F_Z + S_Z = 0$, and that $A\sigma_\theta = A\sigma_0 = S_0$ is constant, therefore,**

$$\sin \alpha = \frac{p_1 R_1 d}{S_0} \ln \frac{R_0}{R}$$



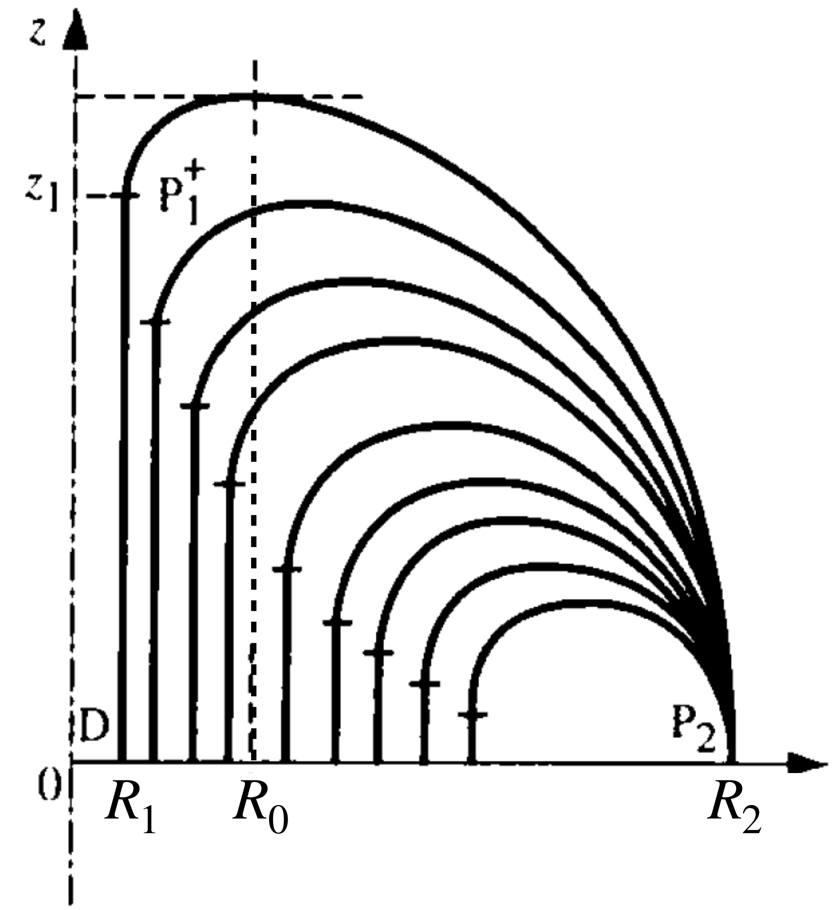
Toroidal field coil forces: the D-shape

- **Using that $dZ/dR = \tan\alpha$ and also the trigonometric identity**

$$\sin\alpha = \frac{\pm\tan\alpha}{\sqrt{1+\tan^2\alpha}}$$

one can show that

$$\frac{dZ}{dR} = \frac{\pm\ln(R_0/R)}{\sqrt{k^2 - \ln^2(R_0/R)}} \qquad k = \frac{S_0}{p_1 R_1 d}$$



- **The condition $\left.\frac{dZ}{dR}\right|_{R\to R_1} = \left.\frac{dZ}{dR}\right|_{R\to R_2} = \infty$ determines**

$$\ln^2(R_0/R_1) = \ln^2(R_0/R_2) = \ln^2(R_2/R_0)$$

$$\frac{R_0}{R_1} = \frac{R_2}{R_0} \;\rightarrow\; R_0 = \sqrt{R_1R_2} \;\rightarrow\; k = \ln(\sqrt{R_2/R_1}) \;\rightarrow\; S_0 = p_1R_1d\,\ln(\sqrt{R_2/R_1})$$

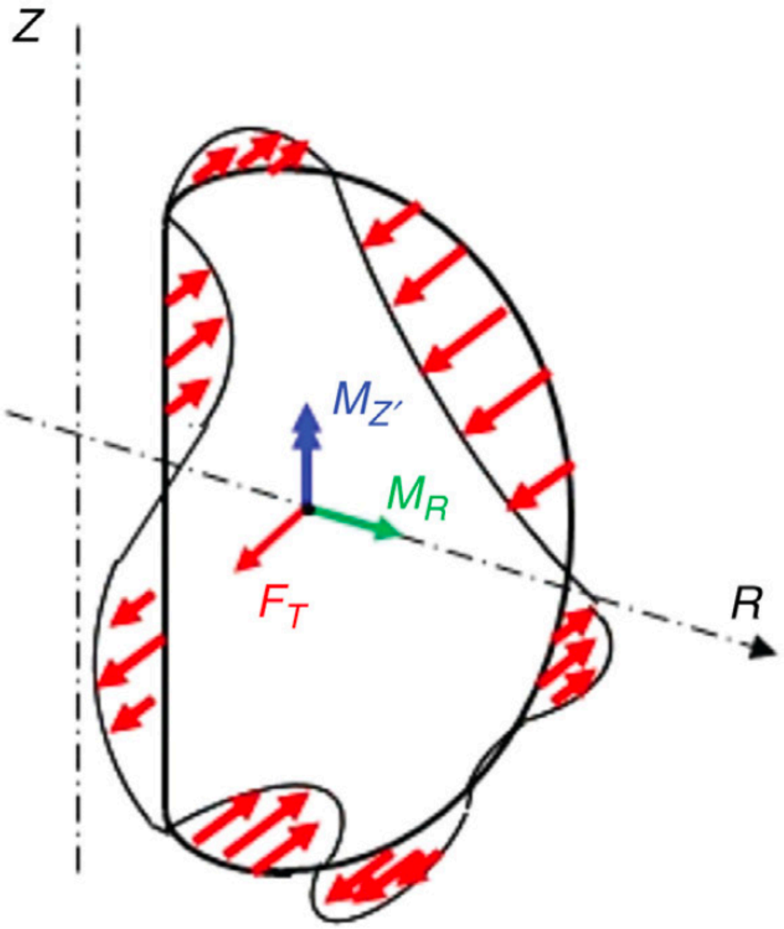
- **Finally, the shape of the shell (or TF coil) is** $Z(R) = \pm\int_{R_1\ or\ R_2}^{R} \frac{\ln(R_0/R')}{\sqrt{k^2-\ln^2(R_0/R')}}dR'$

Toroidal field coil forces: out-of-plane forces

- **Toroidal field coils are also under the action of forces perpendicular to the plane of the coil**

- **These out-of-plane forces are due to the interaction between the poloidal magnetic field from the plasma current and the toroidal field coil current**
  - *The specific details of the forces depends on the plasma scenario*

- **Imbalance between these toroidal forces can cause a toroidal force ($\mathbf{F_T}$), which leads to a vertical torque ($\mathbf{M_Z}$)**
  - *This imbalance can also cause a torque in the radial direction ($\mathbf{M_R}$)*



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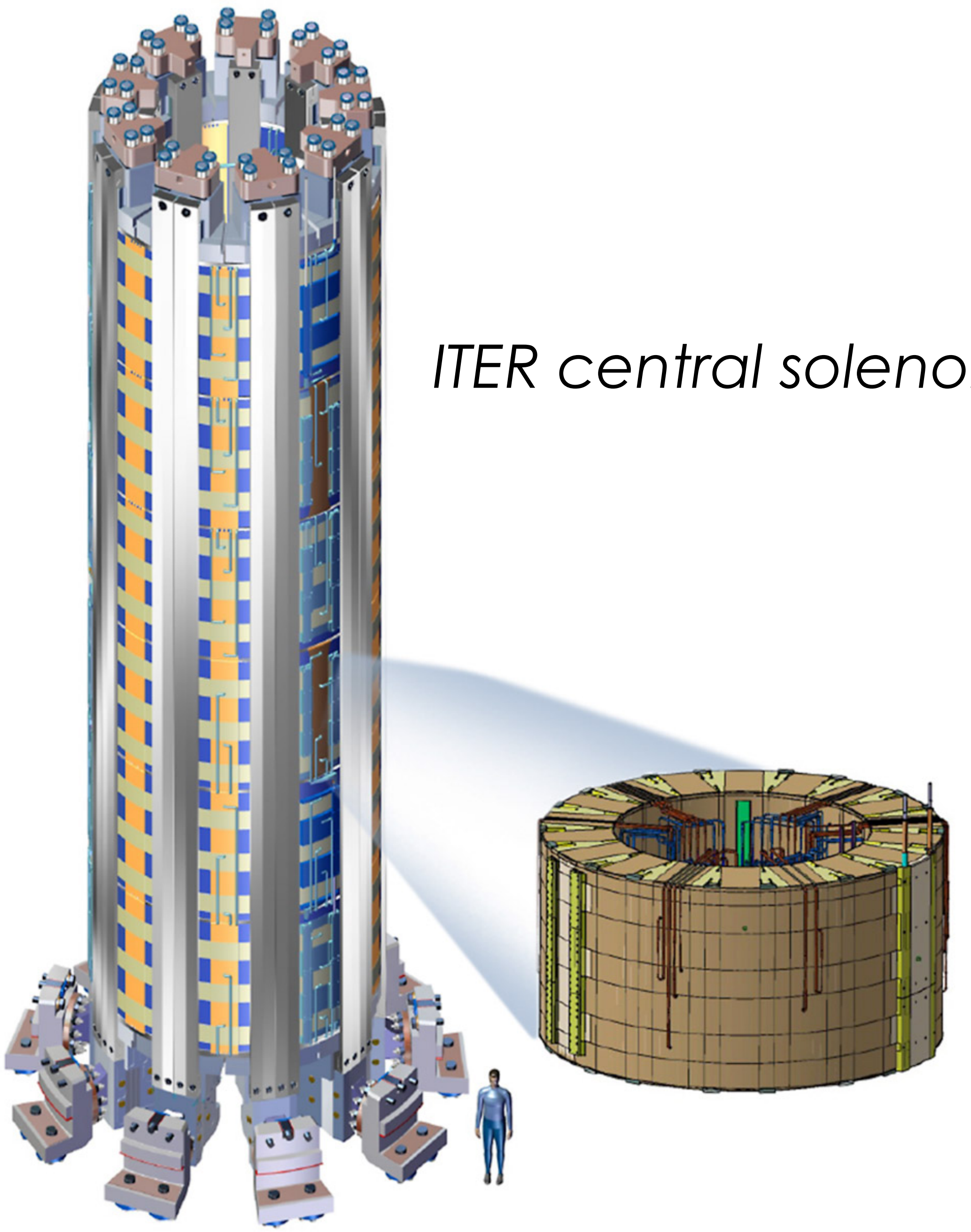
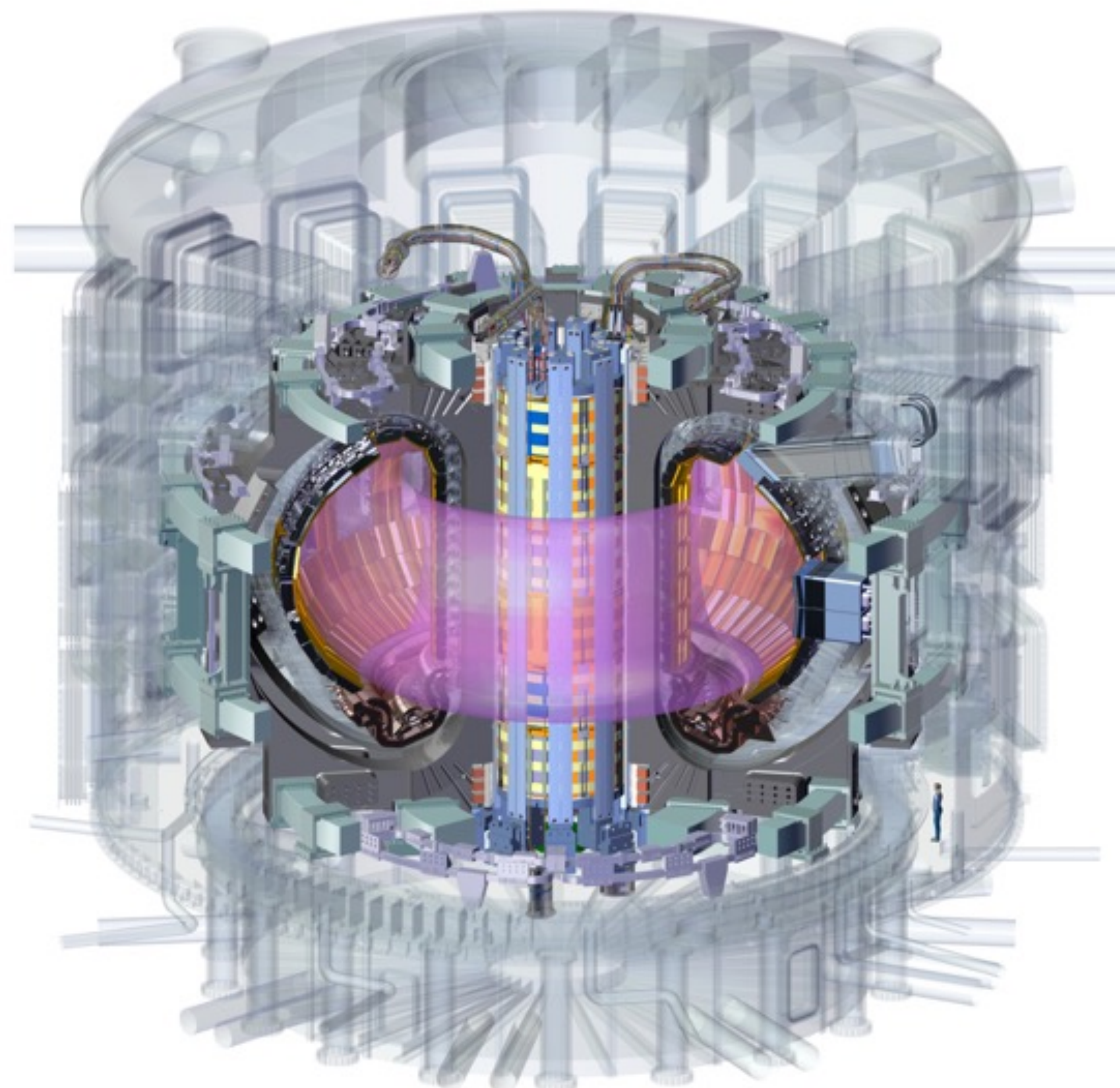
The central solenoid

- **The central solenoid is responsible for plasma breakdown and for driving the plasma current**



ITER central solenoid



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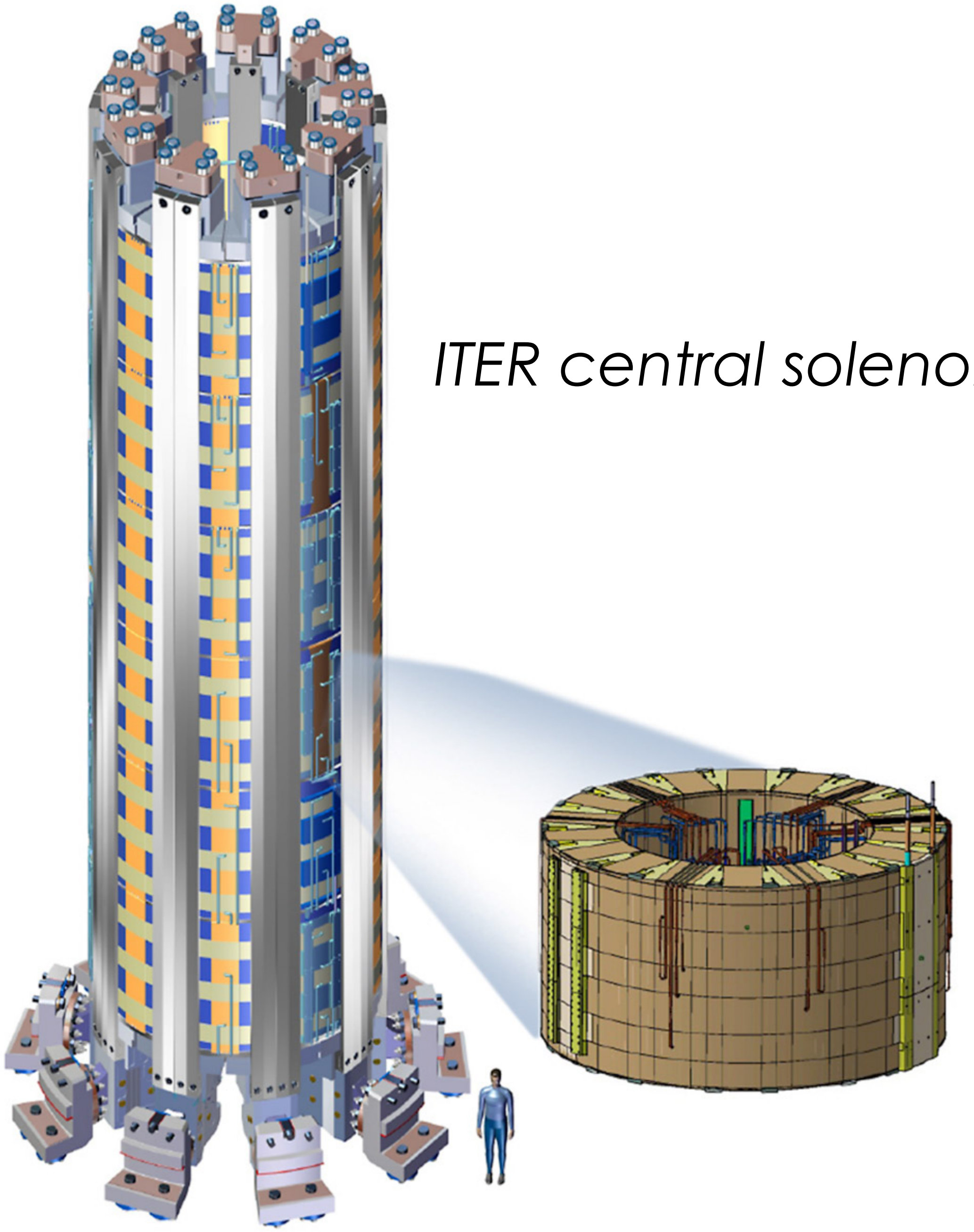


ITER central solenoid

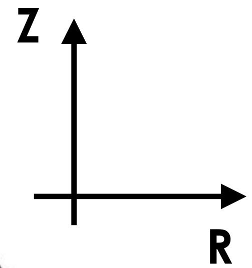
A module of the ITER CS being manufactured at General Atomics, USA

The central solenoid is responsible for driving the plasma current

- **The plasma current depends on the OH coil current ramp rate**

- **From Faraday's induction law**

$\{R, \phi, Z\}$



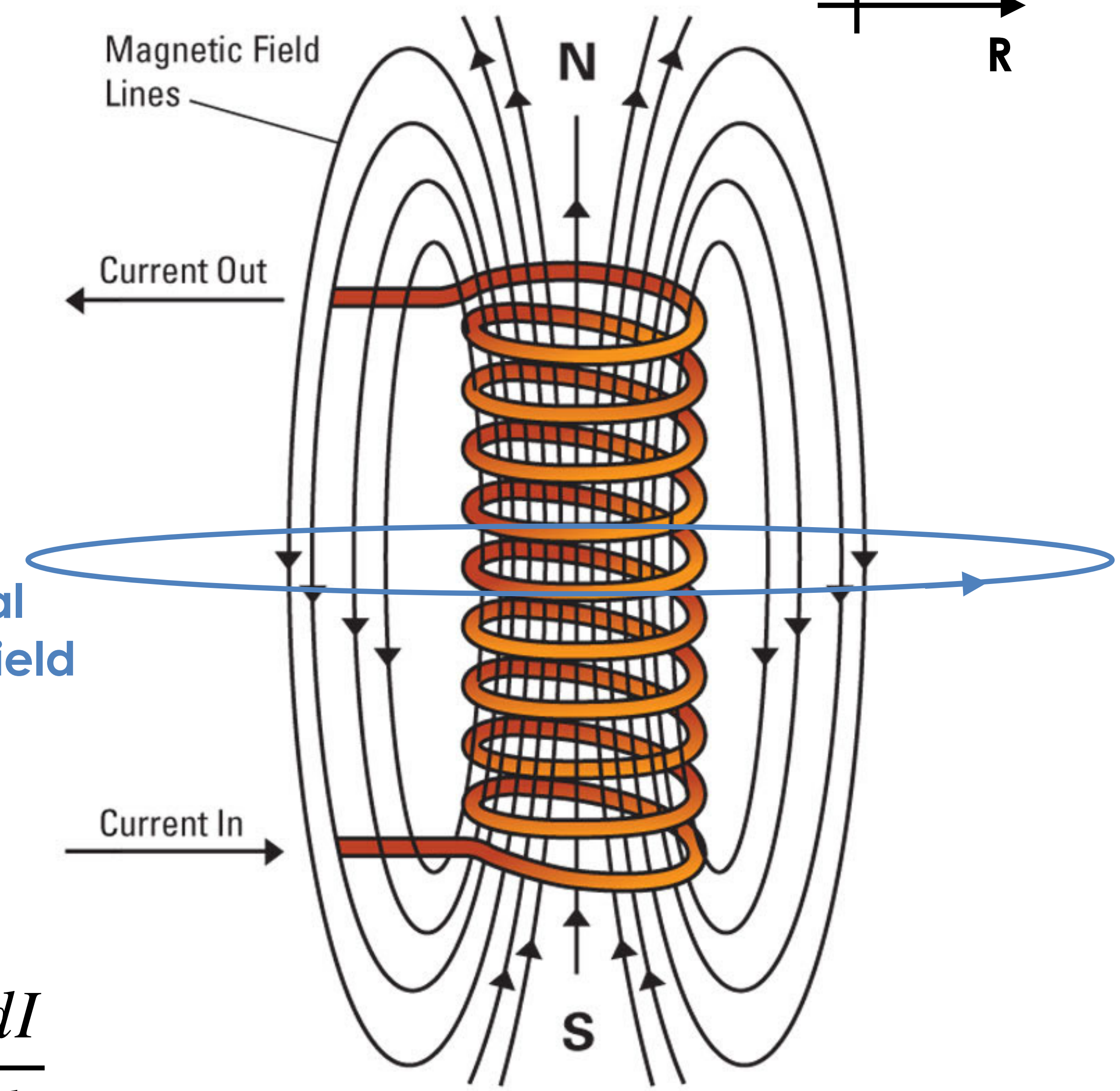
$$\oint_l \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B} \cdot \hat{\mathbf{n}} dS$$

$$\int_0^{2\pi} E_\phi(R)\hat{\mathbf{e}}_\phi \cdot \hat{\mathbf{e}}_\phi R d\phi = -\frac{d}{dt}\int B_Z \hat{\mathbf{e}}_\mathbf{Z} \cdot \hat{\mathbf{e}}_\mathbf{Z} dS$$

$$2\pi R E_\phi(R) = -\frac{d}{dt}\int \frac{\mu_0 N I}{L} dS$$

$$2\pi R E_\phi(R) = V = -\frac{\mu_0 N A}{L}\frac{dI}{dt}$$

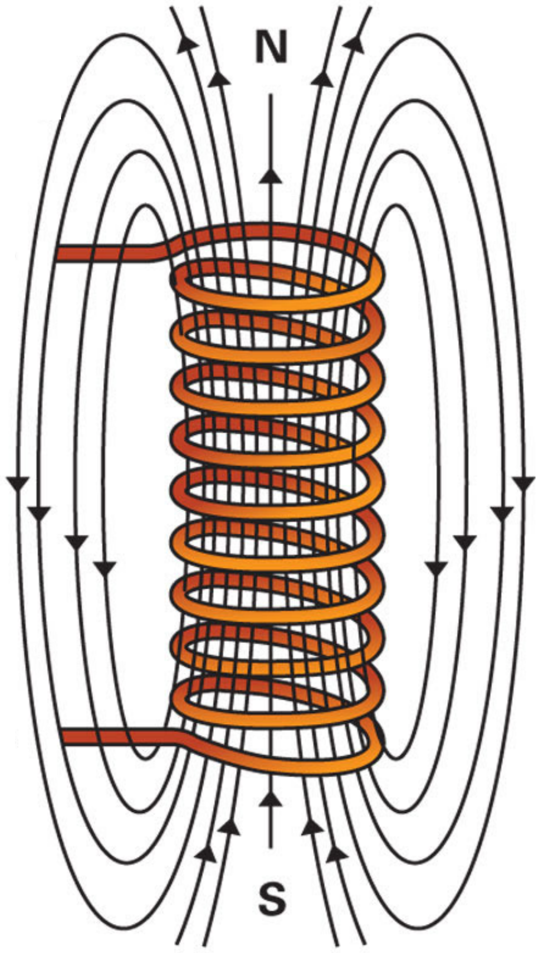
$$V = R_{plasma} I_{plasma} \qquad \rightarrow \qquad I_{plasma} = -\frac{\mu_0 N A}{R_{plasma} L}\frac{dI}{dt}$$

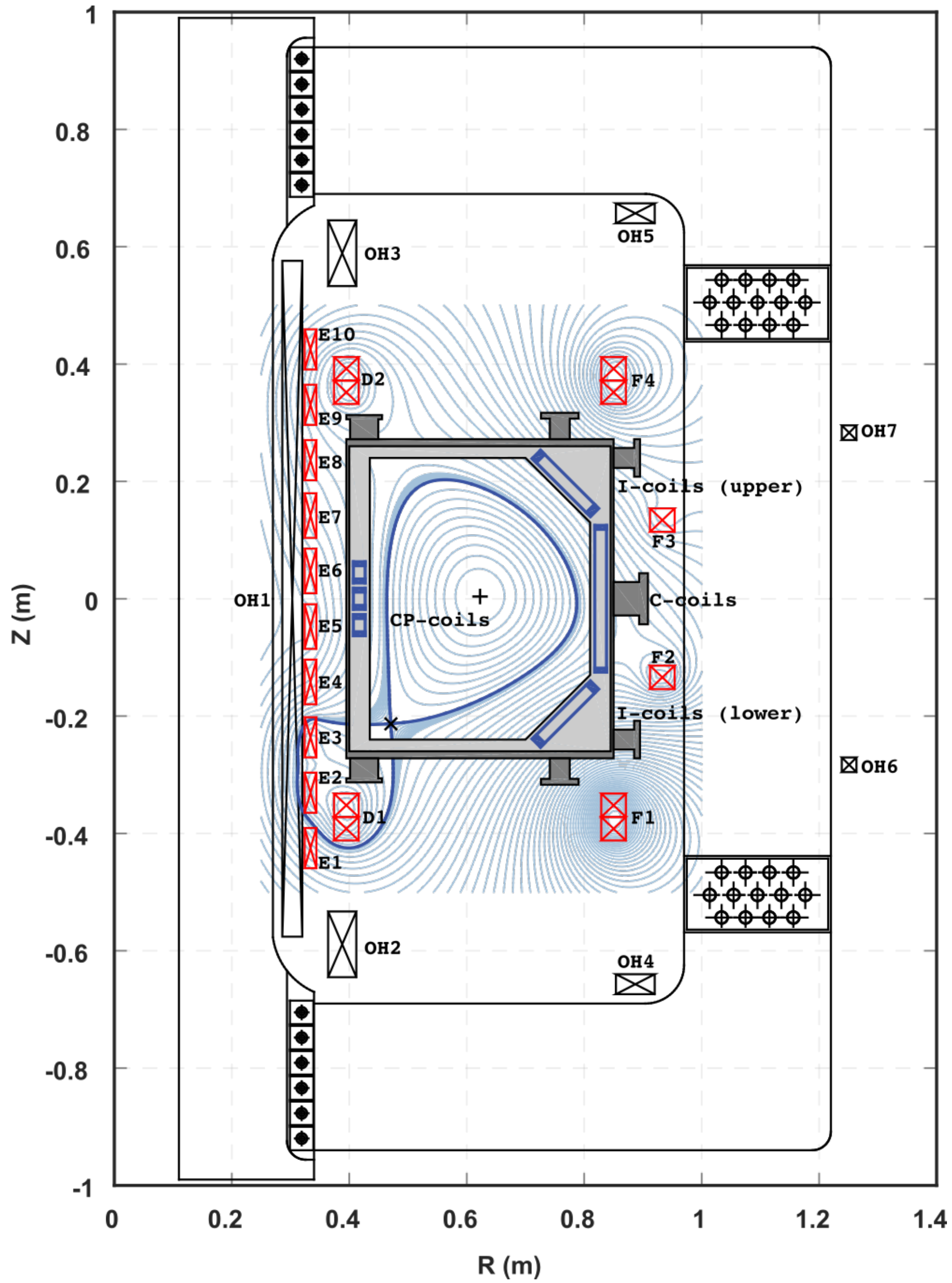


Toroidal electric field

The need for stray field compensation coils

- **The magnetic field lines leaving the ends of the CS tend to pass through the plasma region, causing the induced electric field to decrease**
  - *These (stay) fields are compensated by some additional OH coils, called compensation coils, which guide the field lines to close outside the plasma*





The central solenoid and the flux swing

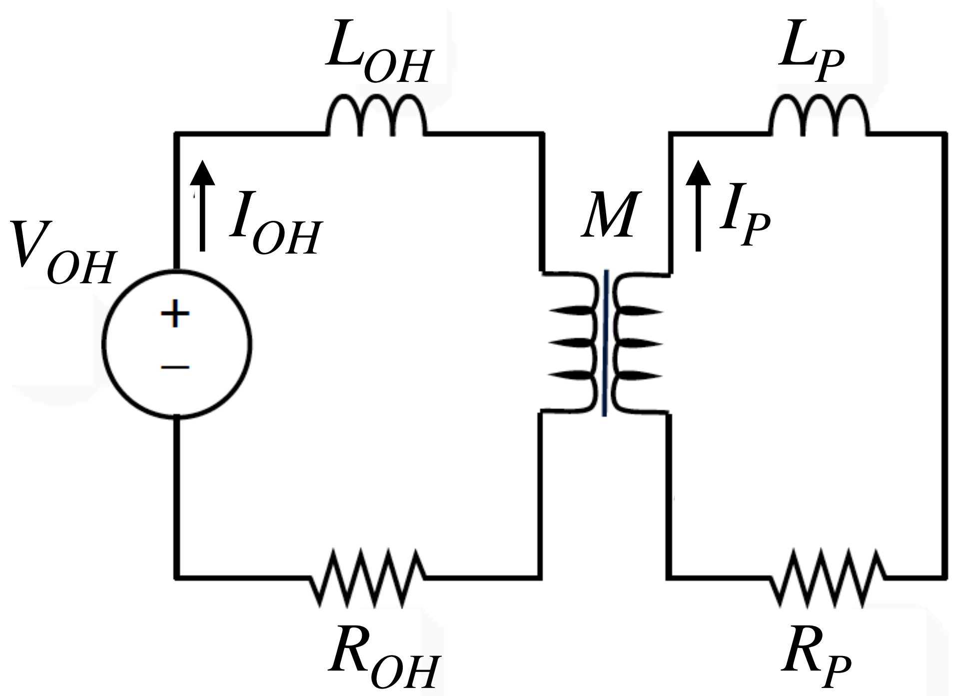
- **An important quantity related to the design of a central solenoid in pulsed tokamaks is the amount of magnetic flux (the flux swing) it can produce**

$$V_{\rm loop} = -\frac{d\Phi}{dt} \quad \rightarrow \quad d\Phi = -V_{\rm loop}dt \quad \rightarrow \quad \Delta\Phi = -\int_0^{\Delta t} V_{\rm loop}(t)dt \quad \textbf{or} \quad \Delta\Phi = -V_{\rm loop}\Delta t$$

- **The coupling between the central solenoid and the plasma can be modeled, in a very simplified way, by the electric circuit below**
  - *Here, $M > 0$ means that the I_{OH} and the I_P flow in the same toroidal direction*

$$V_{OH} = R_{OH}I_{OH} + L_{OH}\frac{dI_{OH}}{dt} + M\frac{dI_P}{dt}$$

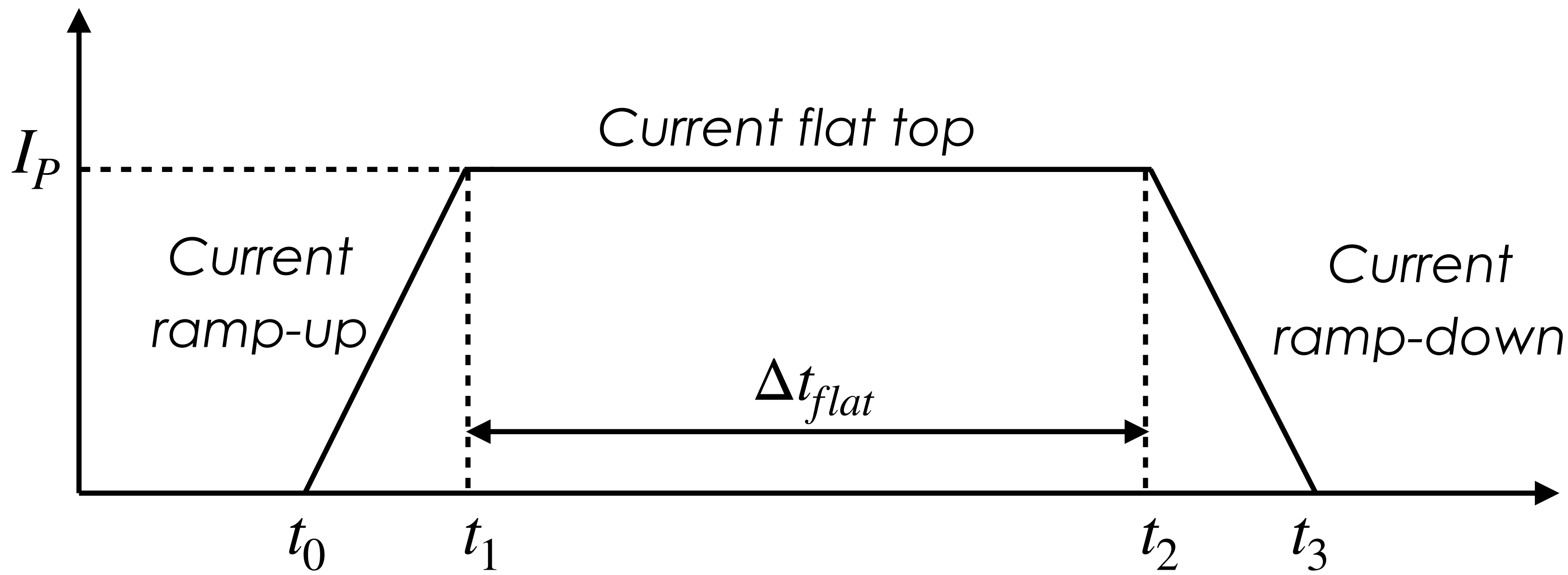
$$0 = R_P I_P + L_P\frac{dI_P}{dt} + \underbrace{M\frac{dI_{OH}}{dt}}_{-V_{\rm loop}}$$



- **What is the power supply voltage temporal evolution needed to sustain a certain pre-programmed plasma current time trace?**

The central solenoid and the flux swing

- **Let's suppose we want the plasma current to change as indicated below**



  - *The current ramp-up should not be too fast: triggering of resistive instabilities*
  - *The current ramp-up should not be too slow: consumption of the flux swing*

- **The current ramp-up phase**
  - *Integrating the second circuit equation:*

$$\Delta I_{OH,up} = -\frac{1}{M}\left[L_P \Delta I_P + \int_{t_0}^{t_1} R_P(t')I_P(t')dt'\right] \approx -\frac{2L_P\Delta I_P}{M} \quad \rightarrow \quad \Delta\Phi_{up} = M\Delta I_{OH,up} = -2L_P\Delta I_P$$

  - *First term: corresponds to the work needed to change the plasma current*
  - *Second term: resistive flux swing consumption (assumed to be about $L_P\Delta I_P$)*

The central solenoid and the flux swing

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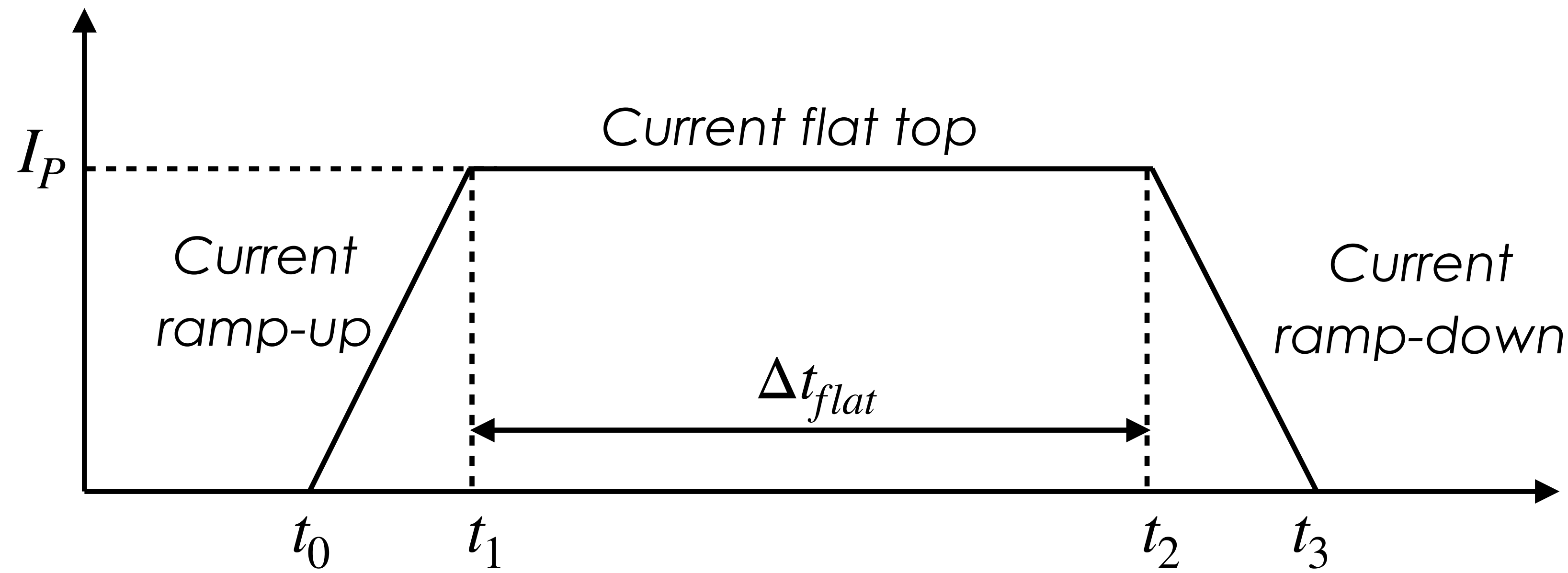


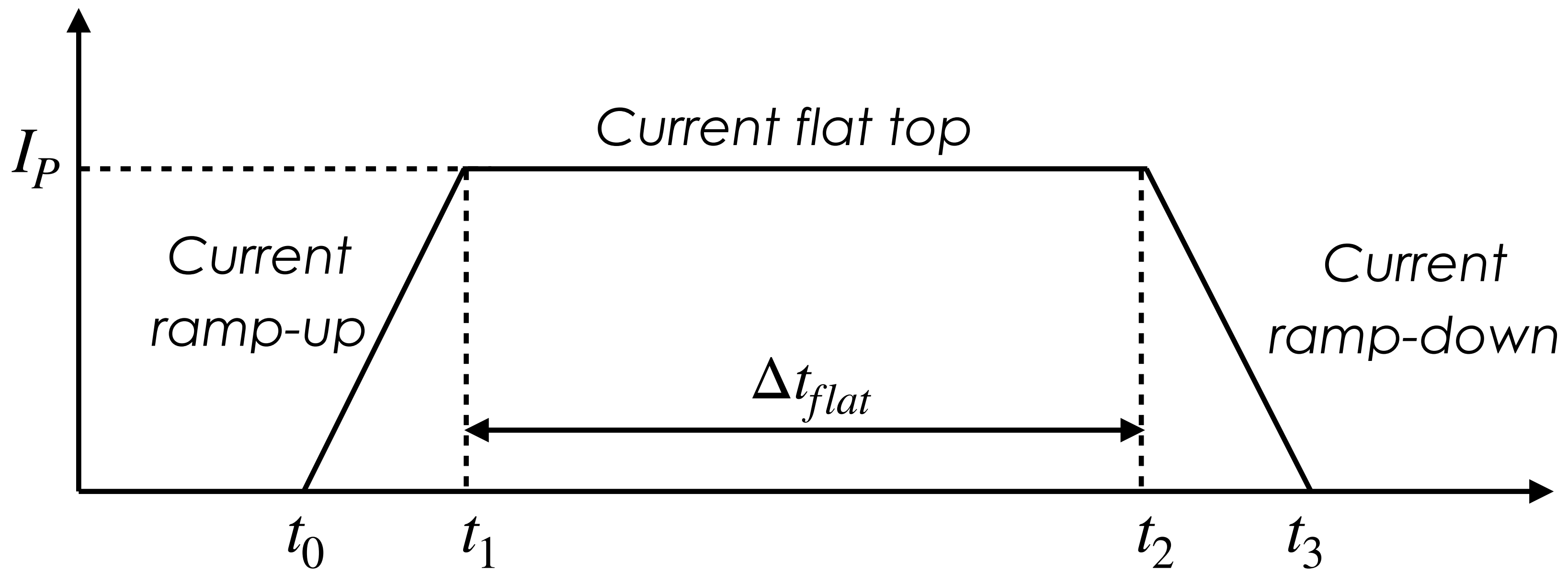
- **The current flat-top phase**
  - *From the loop voltage (with I_P constant and, therefore, V_{loop} constant):*

$$V_{\text{loop}} = -M\frac{dI_{OH}}{dt} \quad \rightarrow \quad \Delta I_{OH,flat} = -\frac{V_{\text{loop}}\Delta t_{flat}}{M} \quad \rightarrow \quad \Delta\Phi_{flat} = M\Delta I_{OH,flat} = -V_{\text{loop}}\Delta t_{flat}$$

  - *From the last expression, flux swing is sometimes also expressed in volt-seconds*

The central solenoid and the flux swing

- **Let's suppose we want the plasma current to change as indicated below**



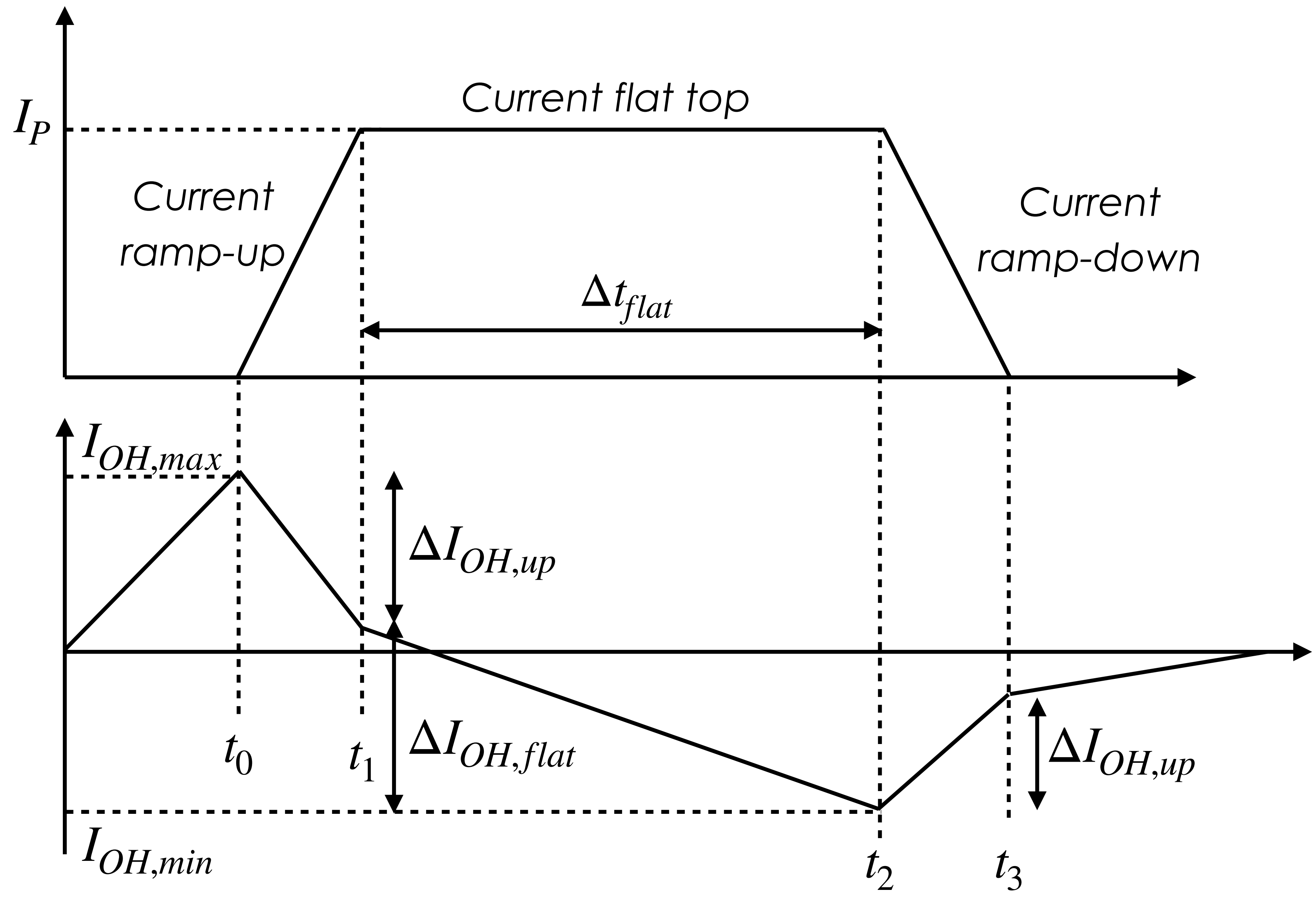
- **The current ramp-down phase**
  - *Again, from integrating the second circuit equation:*

$$\Delta I_{OH,down} = -\frac{1}{M}\left[L_P \Delta I_P + \int_{t_0}^{t_1} R_P(t')I_P(t')dt'\right] \approx -\frac{L_P \Delta I_P}{M} \quad \rightarrow \quad \Delta\Phi_{down} = M\Delta I_{OH,down} = -L_P \Delta I_P$$

  - *First term: corresponds to the work needed to change the plasma current*
  - *The second term is now much smaller compared to the ramp-up phase as the plasma is now hot and, consequently, its resistance is very low*

The central solenoid and the flux swing

- **Time traces of the OH current and the plasma current**



  - *The OH coil must be magnetized before the discharge*
  - *The OH current reverses direction to reduce its peak value*

The central solenoid and the flux swing

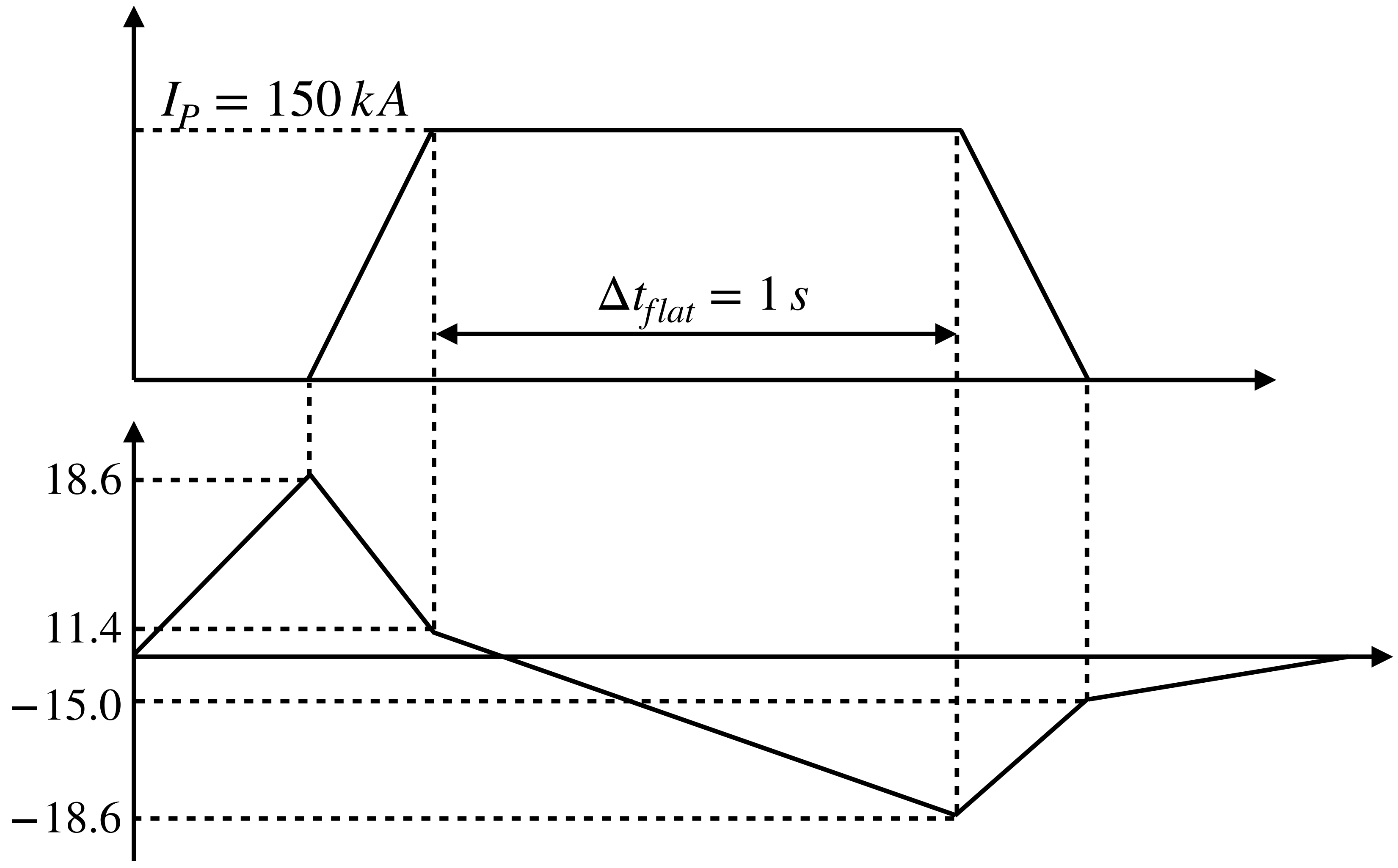
- **Example: TCABR**
  - *Let's take* $R_0 = 0.6\,m$, $a = 0.18\,m$, $M = 50\,\mu H$, $I_P = 150\,kA$, $V_{\text{loop}} = 1.5\,V$, $\Delta t_{flat} = 1\,s$

$$L_P = \mu_0 R_0 \left[\ln\left(\frac{8R_0}{a}\right) - \frac{7}{4}\right] = 1.2\,\mu H$$

$$\Delta I_{OH,up} = -\frac{2L_P \Delta I_P}{M} = -7.2\,kA$$

$$\Delta I_{OH,flat} = -\frac{V_{\text{loop}} \Delta t_{flat}}{M} = -30\,kA$$

$$\Delta I_{OH,down} = -\frac{L_P \Delta I_P}{M} = +3.6\,kA$$



$$I_{OH,max} = I_{OH,min} = \frac{\Delta I_{OH,up} + \Delta I_{OH,flat}}{2} = 18.6\,kA$$

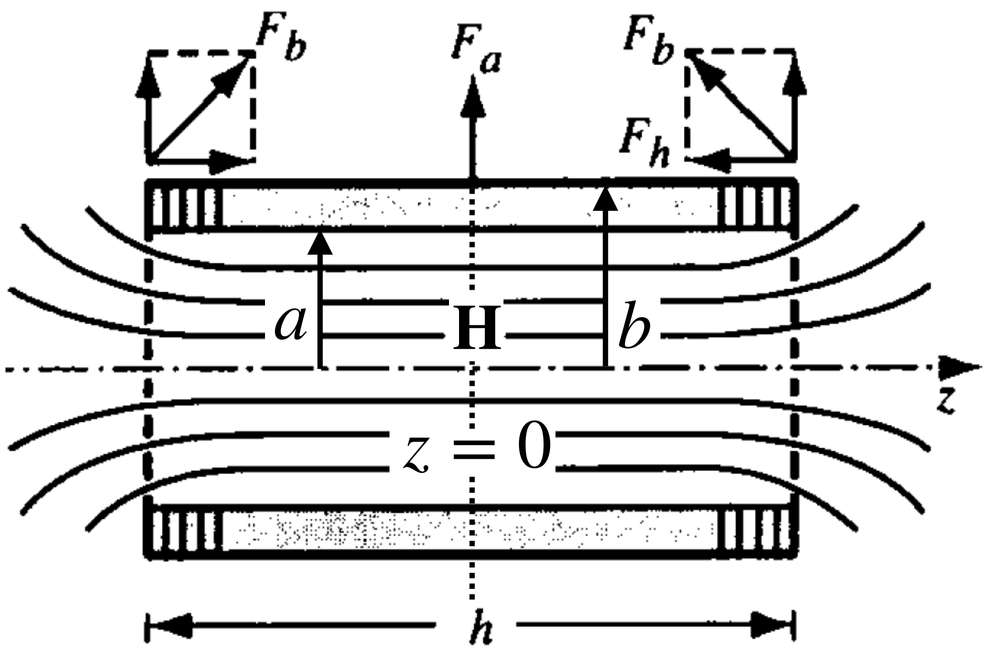
$$\Delta\Phi_{swing} = \Delta\Phi_{up} + \Delta\Phi_{flat} = 1.86\ V \cdot s$$

The forces acting on a solenoid

- **The forces action on a solenoid can also be found using the previous method**
  - *The self-inductance of a solenoid of inner radius a, outer radius b, length h and N turns is given by*

$$L = \frac{\mu_0 N^2 \pi a^2}{h} K_L$$

where the correction factor $K_L = K_L(h/2a)$ is due to its finite length (boundary effect)

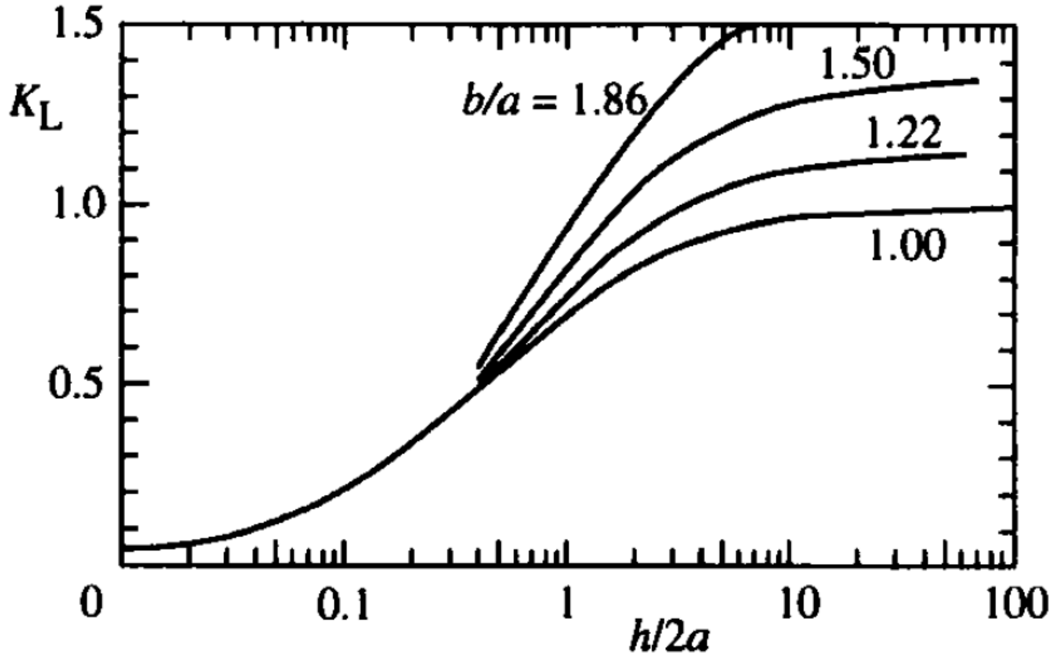


- **Let's define a parameter $x \equiv h/2a$ and calculate**

$$\frac{dK_L}{da} = \frac{dK_L}{dx}\frac{dx}{da} = -\frac{h}{2a^2}\frac{dK_L}{dx}$$

  - *Note that, for $x \gg 1$ we have*

$$\frac{dK_L}{dx} \ll \frac{K_L}{x}$$



The forces acting on a solenoid

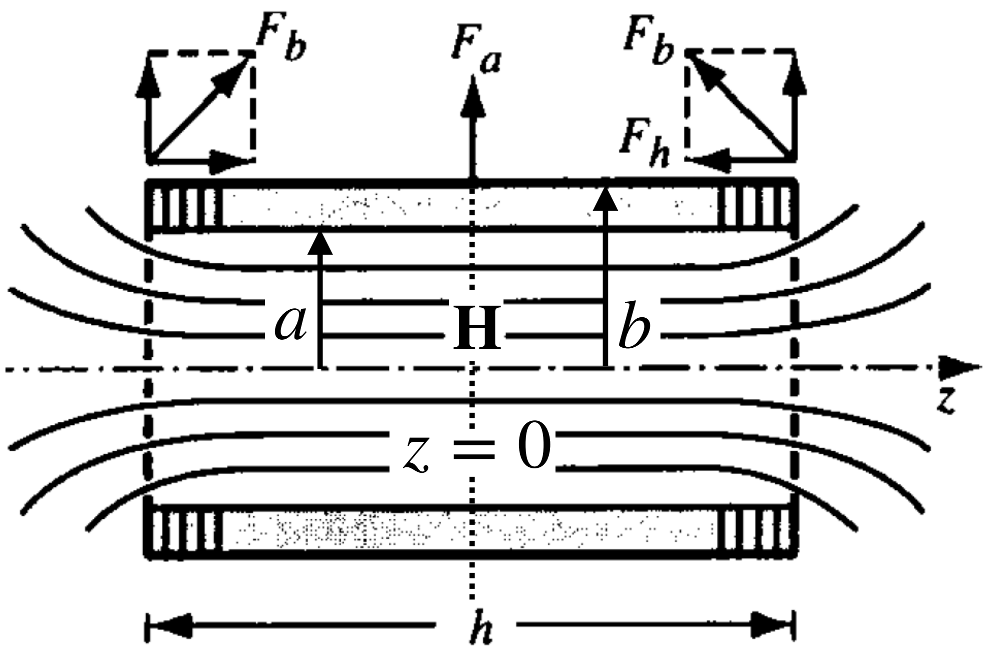
- **The hoop force acting on a solenoid is calculated as**

$$F_a = \frac{I^2}{2}\frac{\partial L}{\partial a} = \frac{I^2}{2}\frac{\partial}{\partial a}\left(\frac{\mu_0 N^2 \pi a^2}{h} K_L\right)$$

$$F_a = \frac{\mu_0 N^2 \pi I^2}{2}\left(\frac{K_L}{x} - \frac{1}{2}\frac{dK_L}{dx}\right)$$

  - *For solenoids with* $x \gg 1$

$$F_a \approx \frac{\mu_0 N^2 \pi a K_L I^2}{h}$$



- **Similarly, the axial (compressional) force acting on a solenoid is calculated as**

$$F_h = \frac{I^2}{2}\frac{\partial L}{\partial h} = \frac{I^2}{2}\frac{\partial}{\partial h}\left(\frac{\mu_0 N^2 \pi a^2}{h} K_L\right) = -\frac{\mu_0 N^2 \pi a I^2}{4h}\left(\frac{K_L}{x} - \frac{\partial K_L}{\partial x}\right)$$

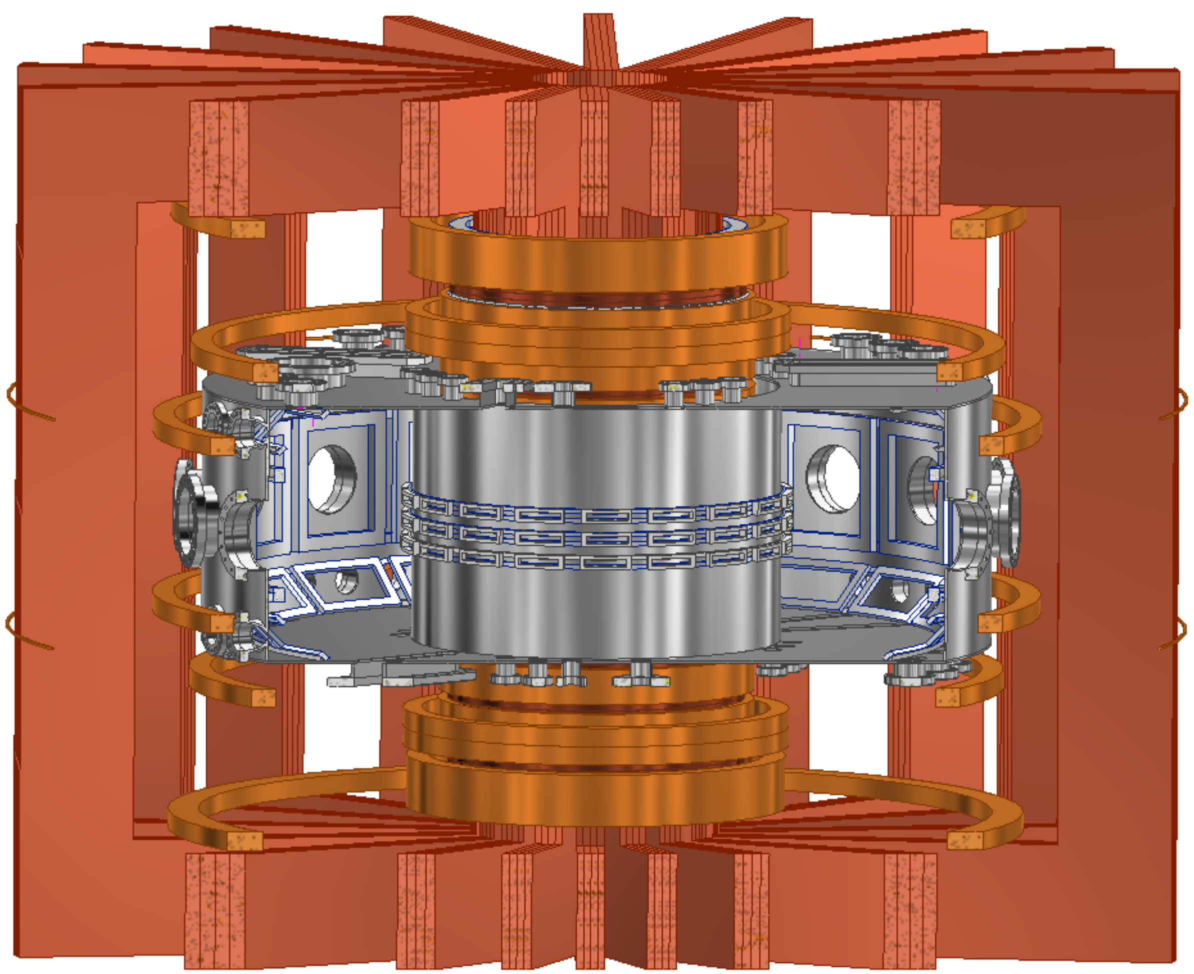
  - *For solenoids with* $x \gg 1$, *and defining the density of turns as* $n = N/h$:

$$F_h = -\frac{\mu_0 n^2 \pi a^2 I^2 K_L}{2}$$

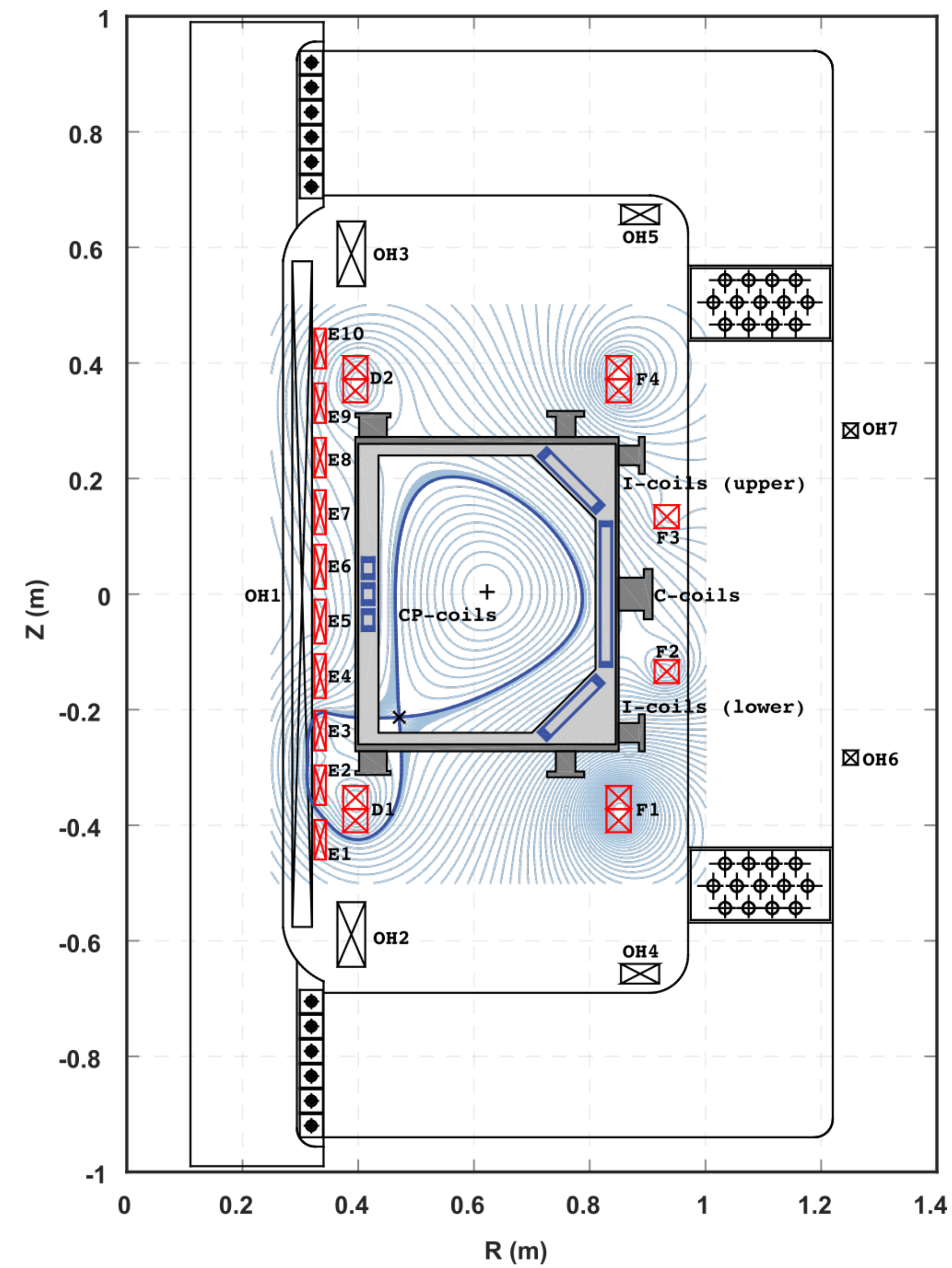
- **Tokamak engineering**
  - *Magnetic forces*
  - *Toroidal field coils*
  - *The central solenoid*
  - *Poloidal field coils*
  - *The vertical plasma instability and the RZIP model*

Poloidal field coils

- **Poloidal field coils (PF) are responsible for shaping the plasma boundary and also for controlling the plasma position**

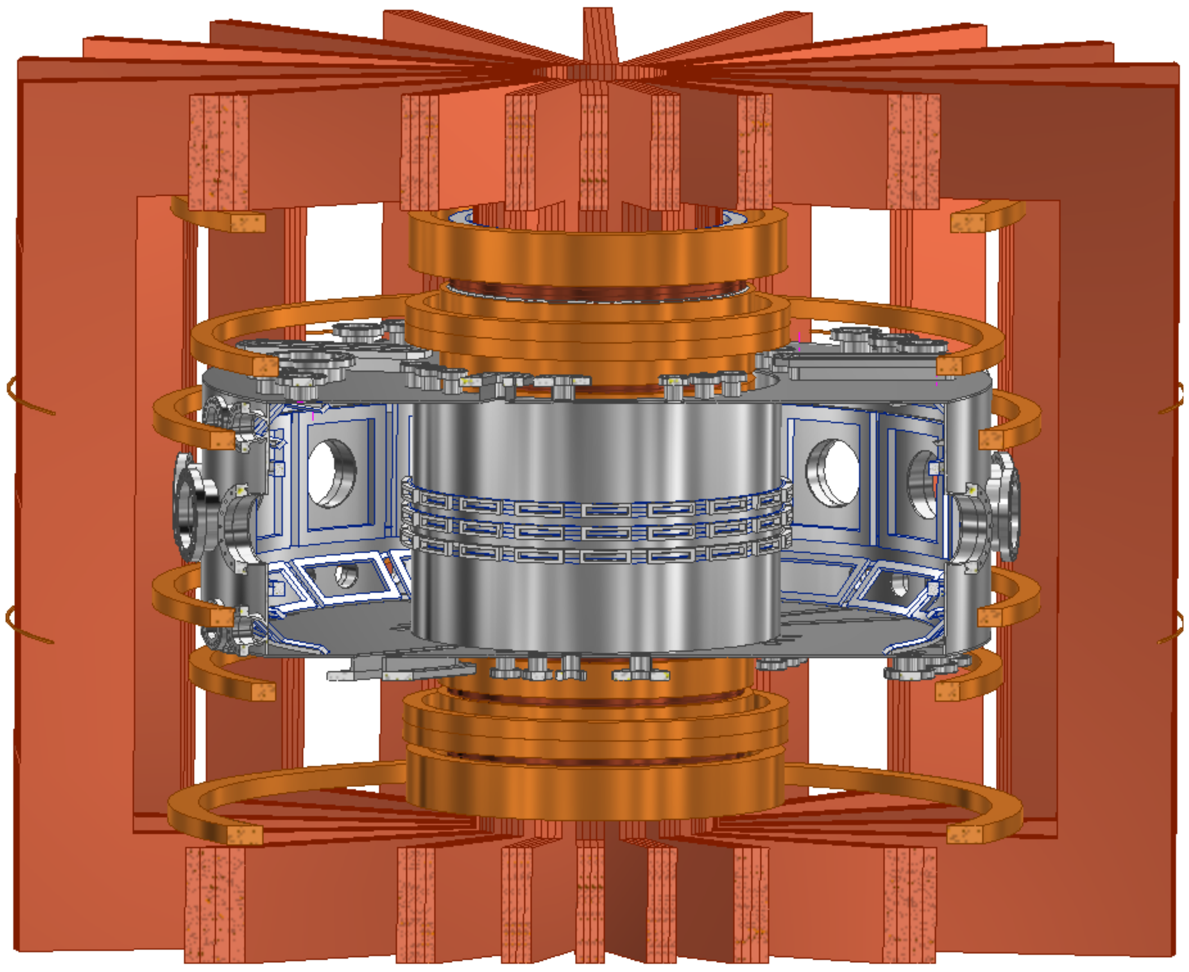


TCABR PF coils

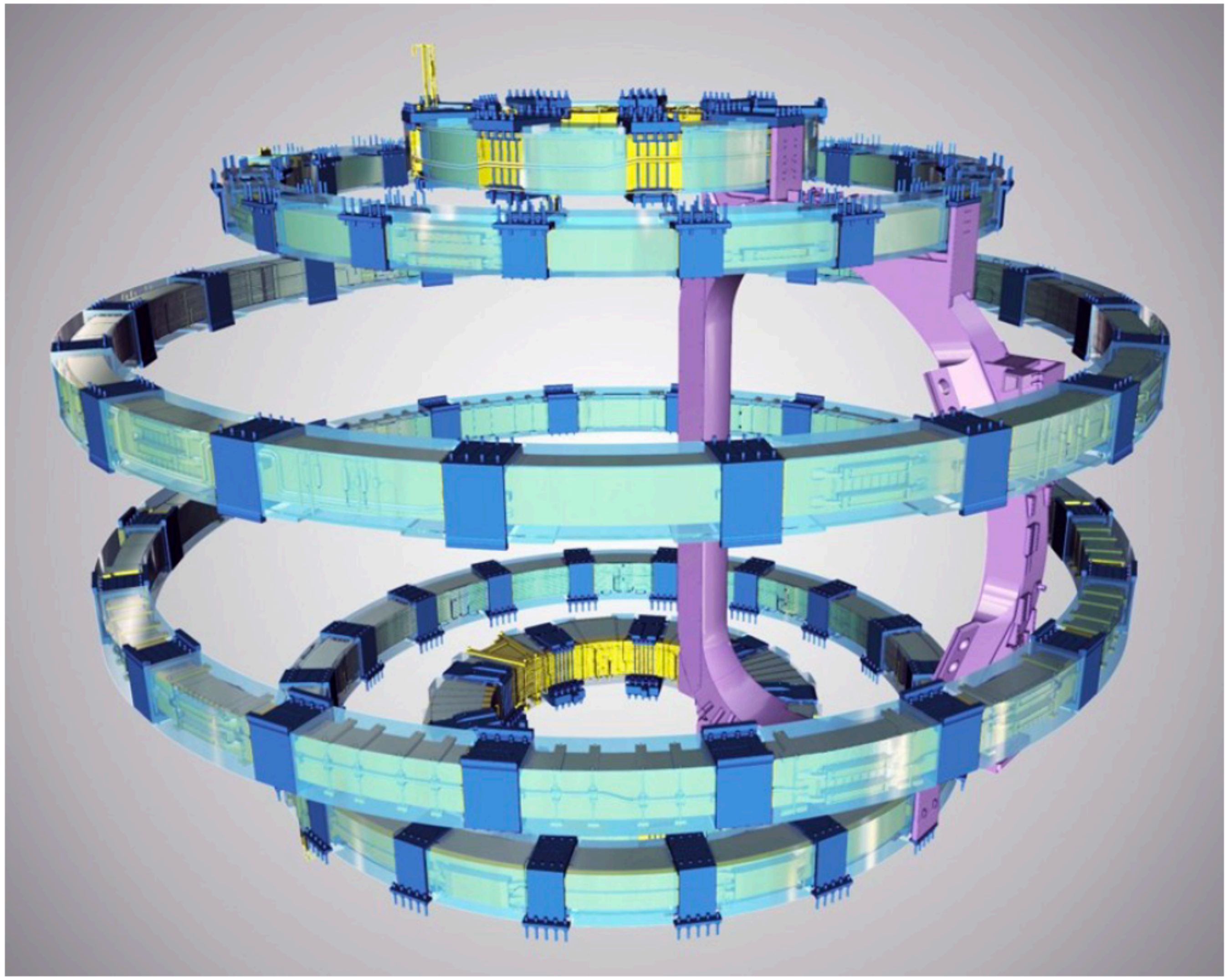


Poloidal field coils

- **Poloidal field coils (PF) are responsible for shaping the plasma boundary and also for controlling the plasma position**



TCABR PF coils



ITER PF coils

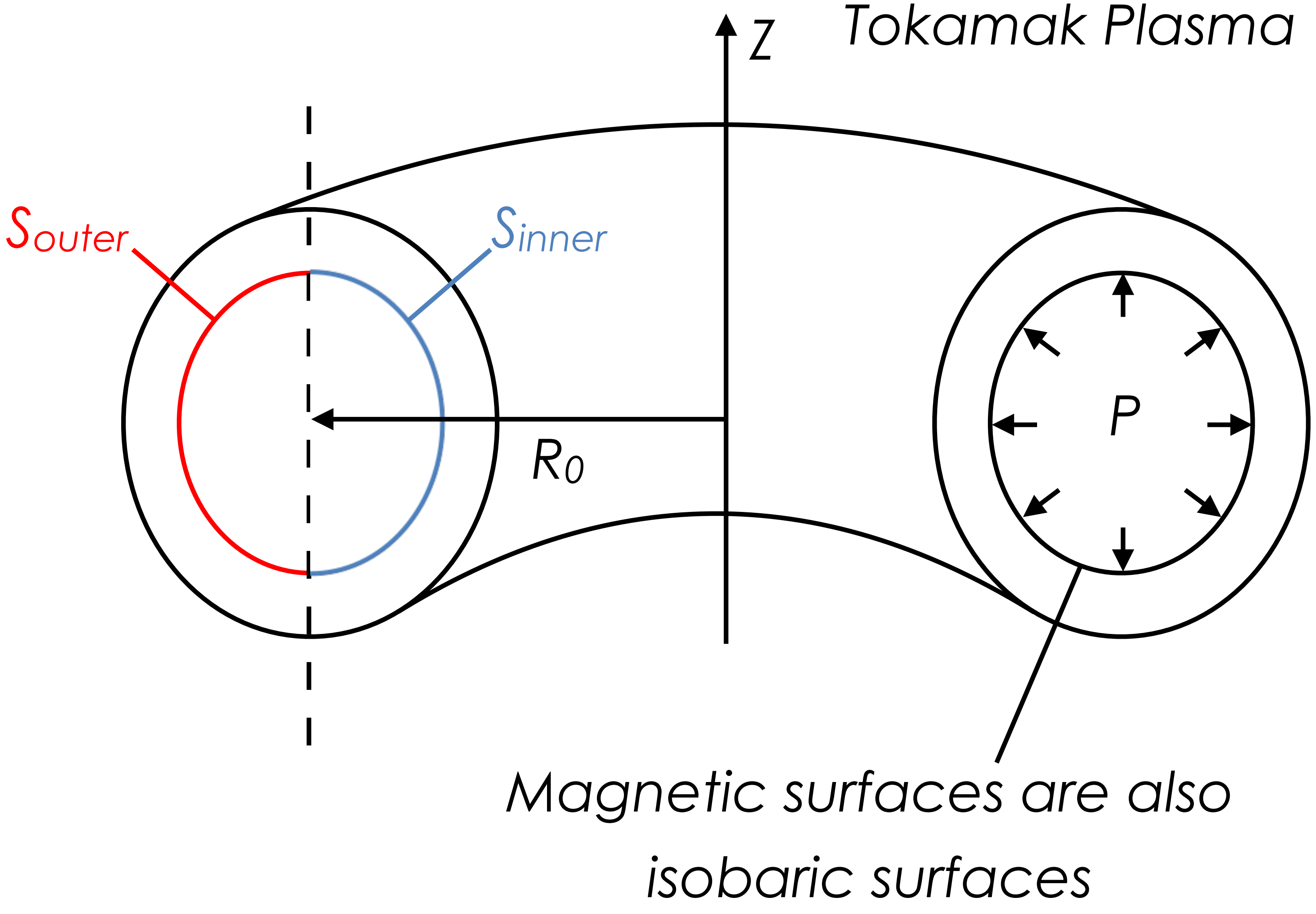
The tire tube force

- **The tire tube force is the result of a constant pressure acting on regions with diference areas**
  - *In a tire tube, as well as in tokamak plasmas, the tire tube force points radially outward and tends to increase the system minor (a) and major (R_0) radii*

$$F_R = F_{outer} - F_{inner} = p_{outer}S_{outer} - p_{inner}S_{inner}$$

$$F_R = p\left(S_{outer} - S_{inner}\right) > 0$$

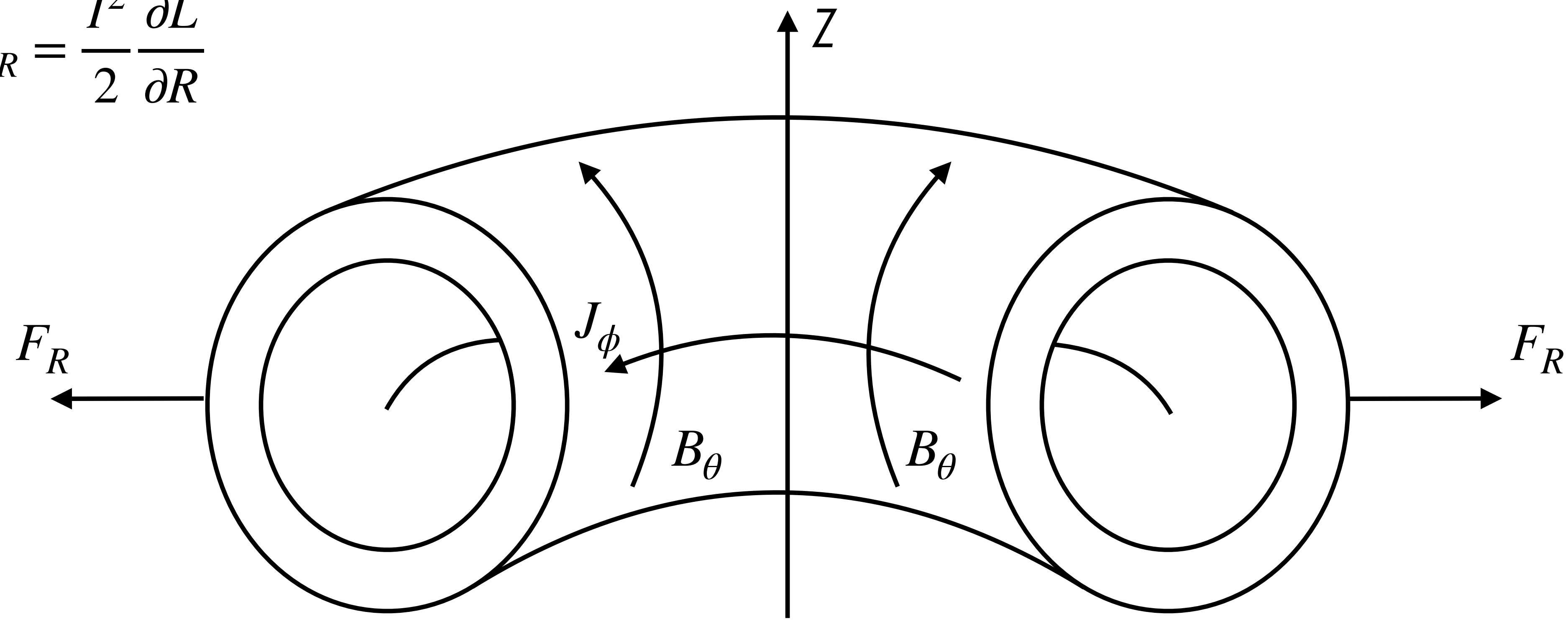
Tire tube



The hoop force

- **The hoop force is the EM equivalent of the tire tube force**
  - *This force tends to decrease the magnetic energy of the system by increasing the major radius of the plasma*

$$F_R = \frac{I^2}{2}\frac{\partial L}{\partial R}$$



The 1/R force

- **A radial force exists due to the interaction between a poloidal current and the toroidal field $B_\phi = R_0 B_0 / R$**

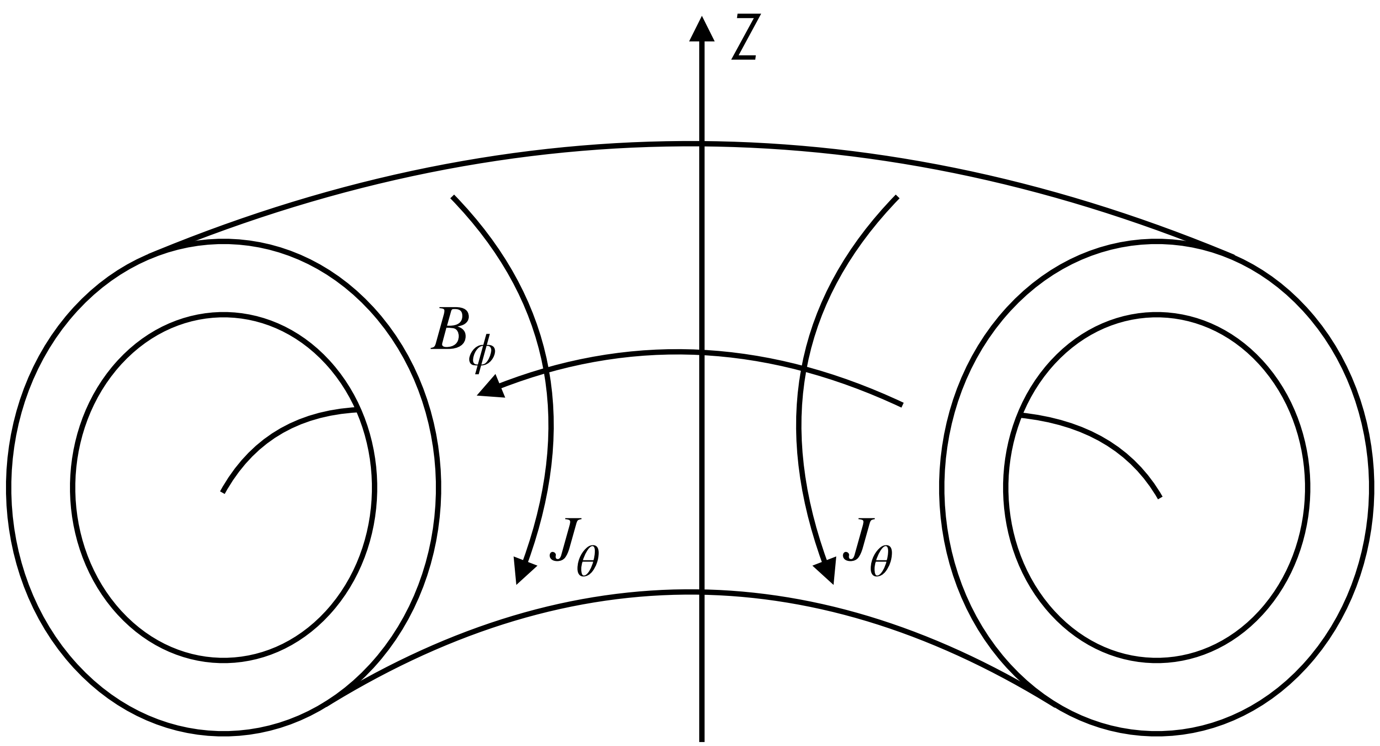
- **Depending on the conditions, the plasma can be diamagnetic or paramagnetic**
  - *The radial force points outward if the plasma is diamagnetic*
  - *The radial force points inward if the plasma is paramagnetic*

$$F_R \propto (J_\theta B_\phi)_{inner} - (J_\theta B_\phi)_{outer}$$

B_ϕ *decreases with* R

J_θ *is larger in the inner parte due to the smaller area*

$$F_R \propto B_0 - B_\phi(r)$$

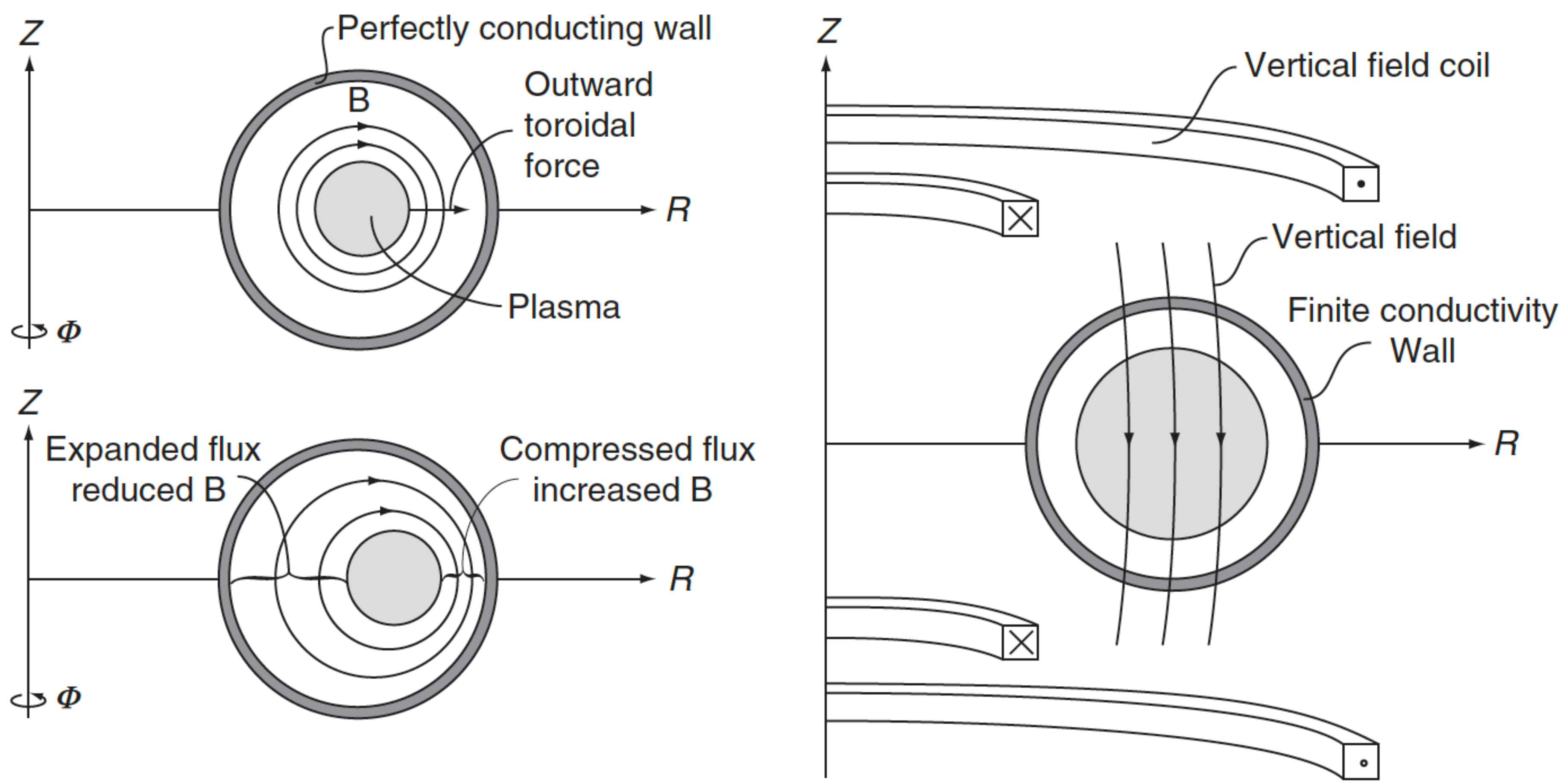


Diamagnetic Plasma

The radial force balance

- **In a configuration maintained by external coil currents, a vertical magnetic field in needed to balance the kinetic and EM forces in the radial direction,**
  - *The Lorentz force (LHS) must balance the hoop force (RHS 1st term), the tire tube force (RHS 2nd term) and the 1/R force (RHS 3rd term)*

$$F_R = \frac{I_P^2}{2}\frac{\partial}{\partial R_0}\left(L_e + L_i\right) + 4\pi^2 \int_0^a \left(p(r) + \frac{B_0}{\mu_0}\left[B_0 - B_\phi(r)\right]\right) r dr$$



The radial force balance

- **Exercício: mostre que**

$$F_R = \frac{\mu_0 I_p^2 \Gamma}{2} \quad \text{with} \quad \Gamma = \ln\left(\frac{8R_0}{a\sqrt{\kappa}}\right) + \beta_p + \frac{\ell_i - 3}{2}$$

  - *Sendo*

$$\ell_i = \frac{2L_i}{\mu_0 R_0} = \frac{1}{\mu_0 I_p^2}\int_{V_p} B_{pol}^2\, dV$$ *(Normalized plasma internal inductance)*

$$\beta_p = \frac{2\mu_0\langle p\rangle}{B_{\text{pol}}^2} = \frac{2\mu_0}{B_{\text{pol}}^2 V_p}\int_{V_p} p\, dV$$ *(Normalized plasma pressure)*

  - *How large is Γ and how strong is the force for a TCABR plasma with $I_p = 150\,\text{kA}$, $R_0 = 0.62\,\text{m}$, $a = 0.16\,\text{m}$, $\kappa = 1.2$, $\beta_p = 0.6$ and $\ell_i = 0.7$?*

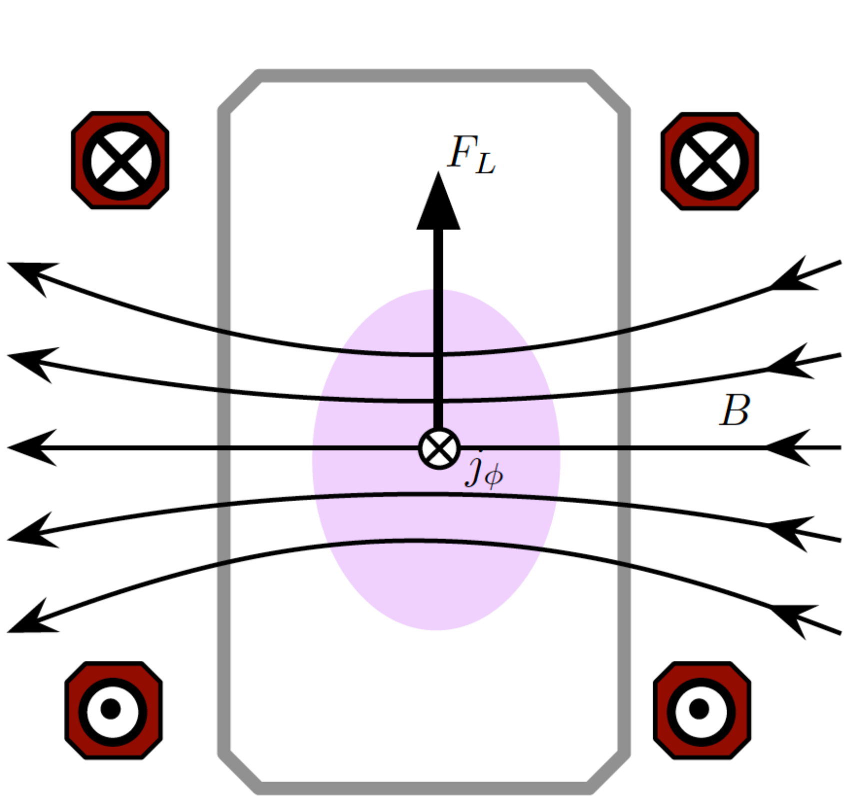
    Resposta: $\Gamma = 2.8$ *and* $F_R = 40\,\text{kN}$

The poloidal field coils are used to control the plasma position

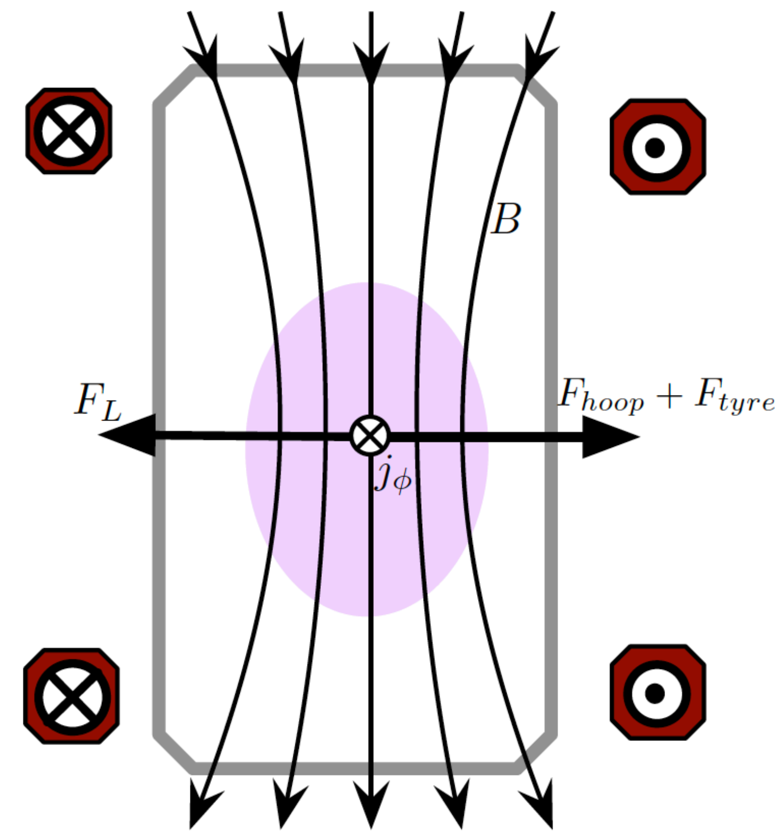
- **There is no natural force acting on the plasma in the vertical direction**
  - *A horizontal field is applied only to correct the plasma position*

- **To balance the naturally occurring radial forces in the plasma, a vertical field must be applied by the PF coils**

Vertical position control



Radial position control



The vertical and radial force balance equations

- **Using Newton's law accounting for the Lorentz force and the radially outward force yields**

$$m_p \frac{d^2\mathbf{R_p}}{dt^2} = \int \mathbf{J}\times\mathbf{B}\,dV + \frac{\mu_0 I_p^2 \Gamma}{2}\hat{\mathbf{e}}_\mathbf{R}$$

  - *Writing* $\mathbf{J} = I_p \delta(R-R_p)\delta(Z-Z_p)\hat{\mathbf{e}}_\phi$ *and* $\mathbf{B} = \mathbf{B_{pl}} + \mathbf{B_{ext}} = \mathbf{B_{pl}} + B_R^{\mathrm{ext}}\hat{\mathbf{e}}_\mathbf{R} + B_\phi^{\mathrm{ext}}\hat{\mathbf{e}}_\phi + B_Z^{\mathrm{ext}}\hat{\mathbf{e}}_\mathbf{Z}$ *leads to*

$$m_p \frac{d^2\mathbf{R_p}}{dt^2} = 2\pi R_p I_p B_Z^{\mathrm{ext}}\hat{\mathbf{e}}_\mathbf{R} - 2\pi R_p I_p B_R^{\mathrm{ext}}\hat{\mathbf{e}}_\mathbf{Z} + \frac{\mu_0 I_p^2 \Gamma}{2}\hat{\mathbf{e}}_\mathbf{R}$$

- **Radial force balance leads to:**

$$m_p \frac{d^2 R_p}{dt^2} = 2\pi R_p I_p B_Z^{\mathrm{ext}} + \frac{\mu_0 I_p^2}{2}\Gamma \quad \rightarrow \quad \text{In the equilibrium}: B_Z^{\mathrm{ext}} = -\frac{\mu_0 I_p \Gamma}{4\pi R_p}$$

  - *For a TCABR plasma with* $I_p = 150\,\mathrm{kA}$, $\Gamma = 2.8$ *and* $R_p = 0.62\,\mathrm{m}$, $B_Z^{\mathrm{ext}} = 68\ \mathrm{mT}$
  - *A vertical field proportional to the plasma current is needed to balance the forces in the radial direction*

The vertical and radial force balance equations

- **Using Newton's law accounting for the Lorentz force and the radially outward force yields**

$$m_p \frac{d^2 \mathbf{R_p}}{dt^2} = \int \mathbf{J} \times \mathbf{B} \, dV + \frac{\mu_0 I_p^2 \Gamma}{2} \hat{\mathbf{e}}_\mathbf{R}$$

  - *Writing* $\mathbf{J} = I_p \delta(R - R_p)\delta(Z - Z_p)\hat{\mathbf{e}}_\phi$ *and* $\mathbf{B} = \mathbf{B_{pl}} + \mathbf{B_{ext}} = \mathbf{B_{pl}} + B_R^{\text{ext}} \hat{\mathbf{e}}_\mathbf{R} + B_\phi^{\text{ext}} \hat{\mathbf{e}}_\phi + B_Z^{\text{ext}} \hat{\mathbf{e}}_\mathbf{Z}$ *leads to*

$$m_p \frac{d^2 \mathbf{R_p}}{dt^2} = 2\pi R_p I_p B_Z^{\text{ext}} \hat{\mathbf{e}}_\mathbf{R} - 2\pi R_p I_p B_R^{\text{ext}} \hat{\mathbf{e}}_\mathbf{Z} + \frac{\mu_0 I_p^2 \Gamma}{2} \hat{\mathbf{e}}_\mathbf{R}$$

- **Vertical force balance leads to:**

$$m_p \frac{d^2 Z_p}{dt^2} = -2\pi R_p I_p B_R^{\text{ext}} \quad \rightarrow \quad \text{In the equilibrium}: B_R^{\text{ext}} = 0$$

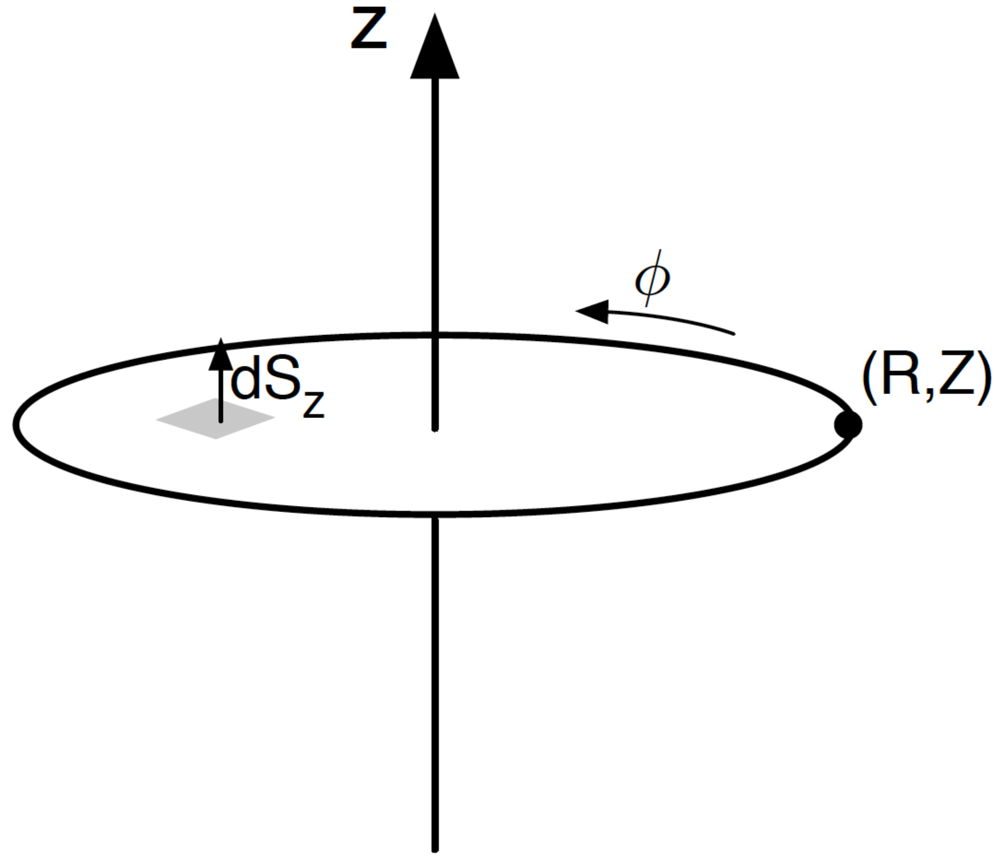
  - *The radial field at the plasma position must be null for the plasma to be in equilibrium)*

Poloidal field coils generate poloidal flux

- **The poloidal flux is defined as**

$$\psi(R,Z) = \int_S \mathbf{B} \cdot d\mathbf{S_Z}$$

$$\psi(R,Z) = 2\pi \int_0^R B_Z(R',Z)\, R'\, dR'$$



  - *Here, S is the surface area defined by a disk, centered on the Z-axis, passing through a point with coordinates (R,Z), and $d\mathbf{S_Z}$ is an element of surface whose normal points in the Z-direction*

Magnetic field from the poloidal flux

- **From the definition of the poloidal flux** $\psi(R,Z) = 2\pi \int_0^R B_Z(R',Z)\, R'\, dR'$

$$\frac{\partial \psi}{\partial R} = 2\pi R B_Z \qquad \frac{\partial \psi}{\partial Z} = 2\pi \int_0^R \frac{\partial B_Z}{\partial Z} R'\, dR'$$

- **Using that**

$$\nabla \cdot \mathbf{B} = \frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{1}{R}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_Z}{\partial Z} = 0$$

combined with axisymmetry, one have that

$$\frac{\partial B_Z}{\partial Z} = -\frac{1}{R}\frac{\partial}{\partial R}(RB_R)$$

and thus

$$\frac{\partial \psi}{\partial Z} = 2\pi \int_0^R \frac{\partial B_Z}{\partial Z} R'\, dR' = -2\pi \int_0^R \frac{1}{R'}\frac{\partial}{\partial R'}(R'B_R)\, R'\, dR' = -2\pi R B_R$$

- **Therefore, the magnetic field can be calculated from the poloidal flux** $\psi(R,Z)$

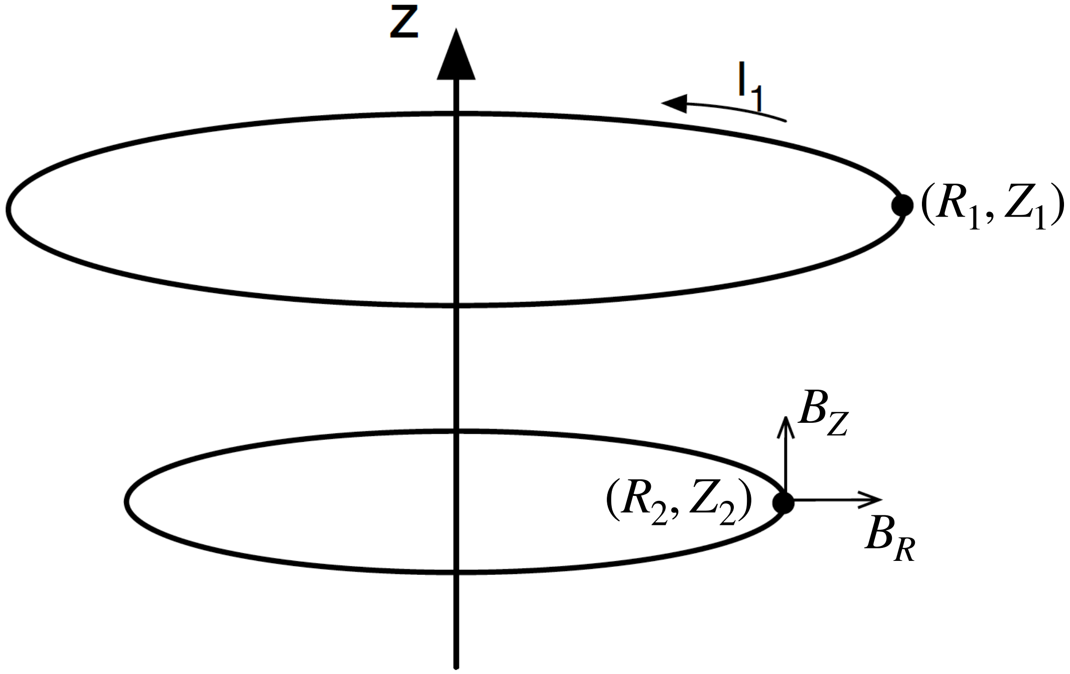
$$B_R = -\frac{1}{2\pi R}\frac{\partial \psi}{\partial Z} \qquad B_Z = \frac{1}{2\pi R}\frac{\partial \psi}{\partial R}$$

Magnetic field from mutual inductances

- **From the definition of mutual inductance**

$$\psi(R_2, Z_2) = M(R_2, Z_2, R_1, Z_1)\, I_1$$

$$\psi_2 = M_{21}\, I_1$$



- **We have already saw that the mutual inductance between two coaxial circular current filament loops is**

$$M_{12} = M_{21} = \mu_0 \sqrt{R_1 R_2} \left[\left(\frac{2}{k} - k \right) K(k^2) - \frac{2}{k} E(k^2) \right]$$

Therefore, the magnetic field at (R_2, Z_2), created by the current I_1, is

$$B_R(R_2, Z_2) = -\frac{I_1}{2\pi R} \frac{\partial M_{21}}{\partial Z} \qquad B_Z(R_2, Z_2) = \frac{I_1}{2\pi R} \frac{\partial M_{21}}{\partial R}$$

Voltage induced by a change in poloidal flux

- **Taking the time derivative of $\psi(t;R,Z)$ through a fixed disk yields**

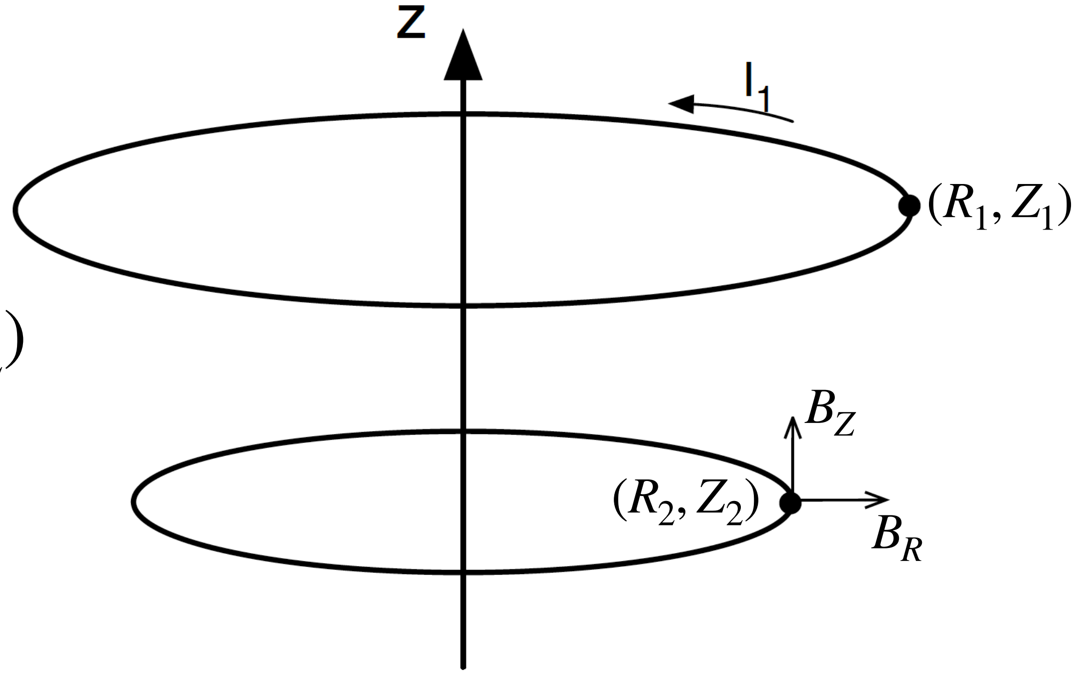
$$\frac{\partial \psi}{\partial t} = \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S_Z}$$

- **From Faraday's induction law, and axisymmetry,**

$$\frac{\partial \psi(t;R,Z)}{\partial t} = -\oint \mathbf{E} \cdot d\mathbf{l} = -2\pi R E_\phi(t;R,Z) = -V_{\mathrm{loop}}(t;R,Z) \;\rightarrow\; E_\phi(t;R,Z) = \frac{V_{\mathrm{loop}}(t;R,Z)}{2\pi R}$$

Therefore, a change in the current I_1 induces a voltage at (R_2, Z_2)

$$\frac{\partial \psi(t;R_2,Z_2)}{\partial t} = M_{21}\frac{dI_1}{dt} = -V_{\mathrm{loop}}(t;R_2,Z_2)$$



Measuring magnetic fields and poloidal fluxes

- **Local measurements of the magnetic field at (R_p, Z_p) can be made by integrating the signal of magnetic probes (of area A_p and N turns)**

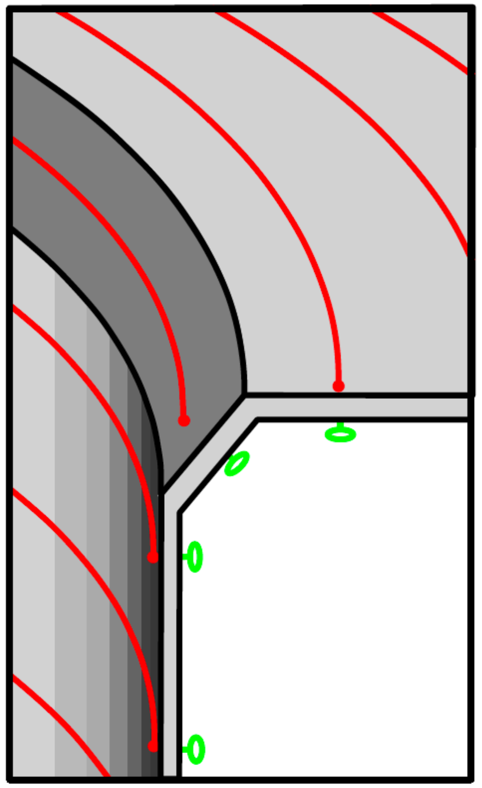
$$V_p = -\frac{d}{dt}\int_{A_p} \mathbf{B}\cdot d\mathbf{S_Z} = -A_pN\frac{dB}{dt} \qquad B_p(t;R_p,Z_p) = -\frac{1}{NA_p}\int_0^t V_p(t')dt'$$

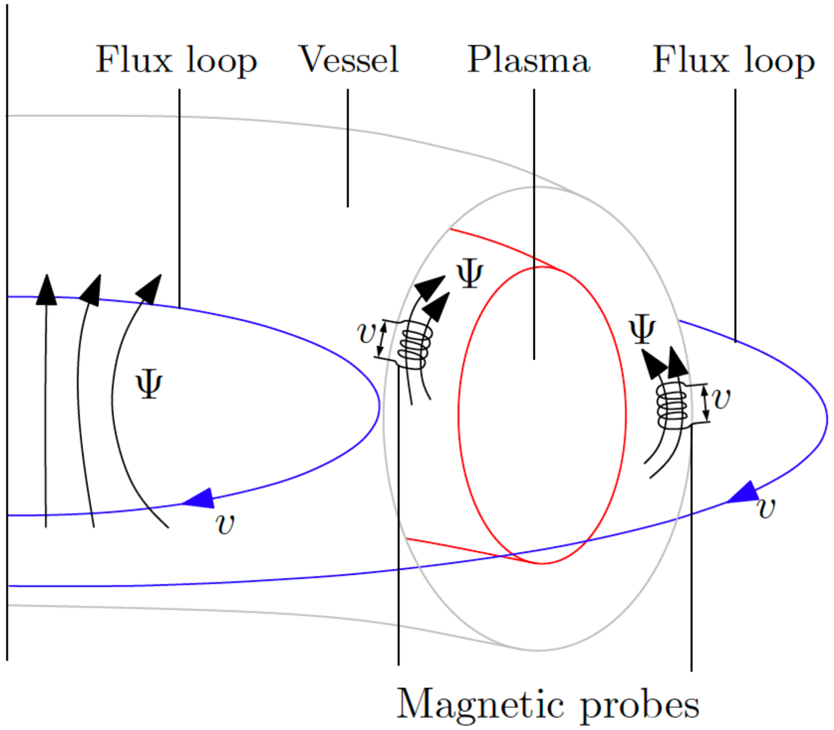
(If A_p is small enough, $\mathbf{B}$ is constant)

- **Measurements of the poloidal flux function at (R_{fl}, Z_{fl}) can be made by integrating the signal from flux loops**

$$V_{fl} = -\frac{\partial\psi}{\partial t}$$

$$\psi(t;R_{fl},Z_{fl}) = -\int_0^t V_{fl}(t')dt'$$





- **Tokamak engineering**
  - *Magnetic forces*
  - *Toroidal field coils*
  - *The central solenoid*
  - *Poloidal field coils*
  - *The vertical plasma instability and the RZIP model*

Elongated plasmas are naturally unstable in the vertical direction

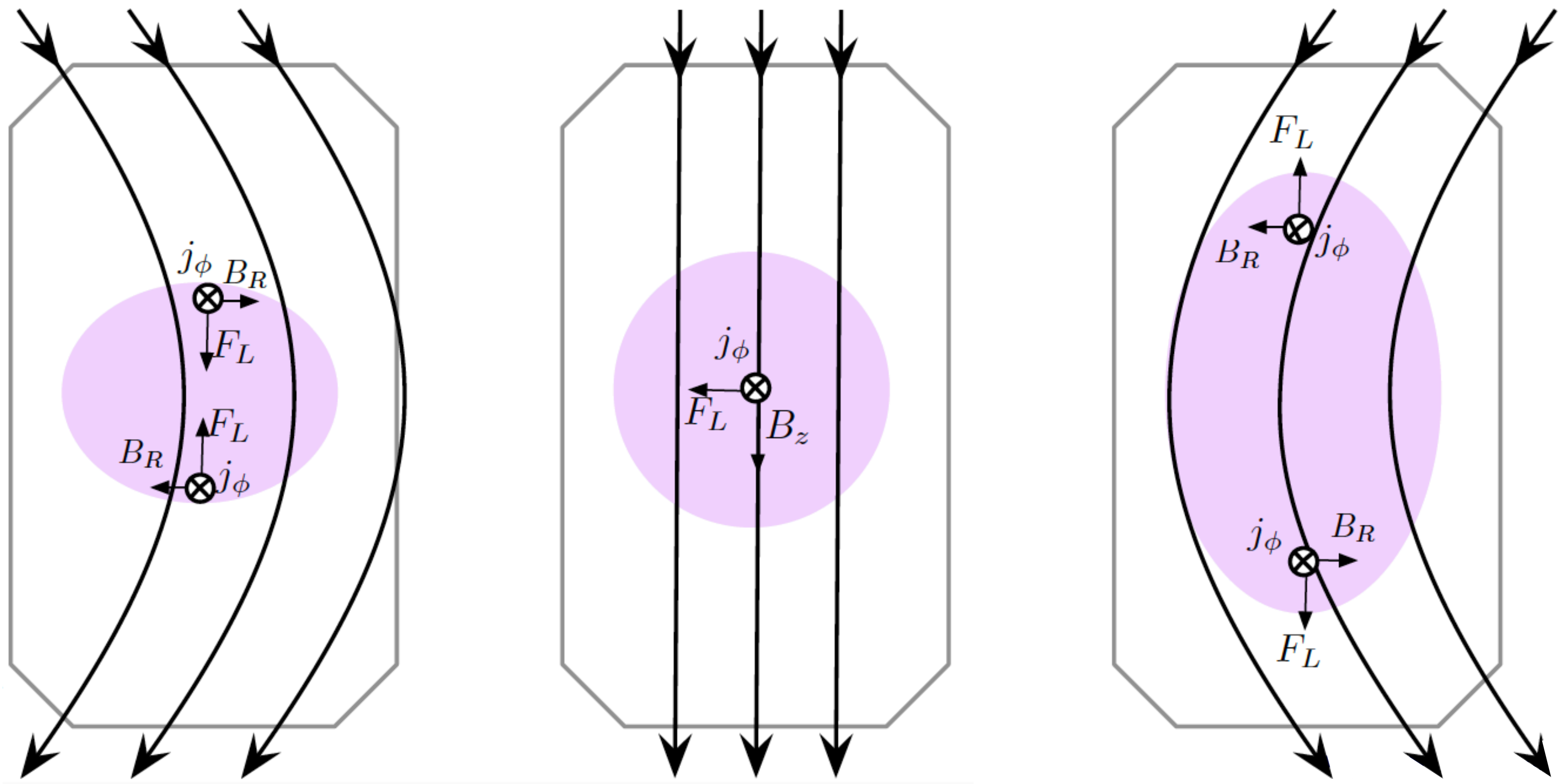
- **A robust control system is needed to control elongated plasma due to the upper and lower competing forces**

Elongation:

$$\kappa = \frac{Z_{\max} - Z_{\min}}{R_{\max} - R_{\min}}$$

Vertical stability index:

$$n = -\left(\frac{Z}{B_Z^{\mathrm{ext}}} \frac{\partial B_R^{\mathrm{ext}}}{\partial Z} \right) \Bigg|_{\mathrm{R_p, Z_p}}$$



$\kappa < 1 \,, n > 0$	$\kappa = 1 \,, n = 0$	$\kappa > 1 \,, n < 0$
(Stable)	*(Metastable)*	*(Unstable)*

- **Modeling of the entire machine, including all the inductive coupling is required to control the plasma**

The vertical plasma position stability

- **The vertical force balance is**

$$m_p \frac{d^2 Z_p}{dt^2} = -2\pi R_p I_p B_R^{\text{ext}}(R_p, Z_p, \mathbf{I_a}, \mathbf{I_v})$$

  - *Let's now separate the contributions to the radial field from the active coils and from the vacuum vessel currents*

$$B_R^{\text{ext}}(R_p, Z_p, \mathbf{I_a}, \mathbf{I_v}) = B_R^a + B_R^v = B_R^a - \frac{1}{2\pi R}\frac{\partial \mathbf{M_{pv}}}{\partial Z}\mathbf{I_v}$$

  - *Let's now allow the plasma to move vertically an to induce currents in the vacuum vessel. Assuming that all other parameters stay constant, one can perturb the equation of motion around an equilibrium defined by*

$$R_p = R_0 \qquad I_p = I_{p0} \qquad Z_p(t) = Z_0 + \delta Z(t)$$

$$\mathbf{I_a} = \mathbf{I_{a0}} \qquad \Gamma = \Gamma_0 \qquad \mathbf{I_v}(t) = 0 + \delta \mathbf{I_v}(t)$$

  - *This leads to*

$$m_p \delta\ddot{Z} + 2\pi R_0 I_{p0} \frac{\partial B_R^a}{\partial Z}\delta Z = I_{p0}\frac{\partial \mathbf{M_{pv}}}{\partial Z}\delta \mathbf{I_v}$$

(Note that $B_R^a(R_0, Z_0, \mathbf{I_{a0}}, \mathbf{I_{v0}} = 0) = 0$)

The vertical plasma position stability

- **Since the coils that produce the poloidal field are outside the plasma,**

$$\nabla \times \mathbf{B^a} = 0 \quad \rightarrow \quad \frac{\partial B_R^a}{\partial Z} = \frac{\partial B_Z^a}{\partial R} = -\frac{n B_Z^a}{R_0} = \frac{\mu_0 I_{p0} \Gamma_0}{4\pi R_0^2} n$$

- **Therefore, one can write**

$$\delta \ddot{Z} + n \frac{\mu_0 I_{p0}^2 \Gamma_0}{2 m_p R_0} \delta Z = \frac{I_{p0}}{m_p} \frac{\partial \mathbf{M_{pv}}}{\partial Z} \delta \mathbf{I_v}$$

- **Defining** $\omega_1^2 = \frac{\mu_0 I_{p0}^2 \Gamma_0}{2 m_p R_0}$ **allows us to write** $\delta \ddot{Z} + n \omega_1^2 \delta Z = \frac{I_{p0}}{m_p} \frac{\partial \mathbf{M_{pv}}}{\partial Z} \delta \mathbf{I_v}$
  - *For a TCABR plasma with* $I_{p0} = 150\,\mathrm{kA}$*,* $\Gamma_0 = 2.8$*,* $R_0 = 0.62\,\mathrm{m}$*,* $a = 0.16\,\mathrm{m}$*,* $m_p = m_i n_e V_p$*,* $n_e = 5 \times 10^{19}\,\mathrm{m}^{-3}$*,* $m_i = 1.67 \times 10^{-27}\,\mathrm{kg}$ *and* $V_p = 2\pi^2 a^2 R_0 = 0.3\,\mathrm{m}^3$*, one has that* $\omega_1 = 1.5 \times 10^6\,\mathrm{s}^{-1}$

- **If no currents are allowed to flow in the vessel (**$\delta \mathbf{I_v} = 0$**), one has (for** $n = -1$**),**

$$\delta \ddot{Z} + n \omega_1^2 \delta Z = 0 \quad \rightarrow \quad \delta Z(t) = Z_0 e^{-t/\tau} \quad \rightarrow \quad \tau = \pm 1 / \left(\omega_1 \sqrt{-n} \right) \approx \pm 0.6\,\mu\mathrm{s}$$

(This mode is to fast for a plasma control system alone to stabilize)

The vertical plasma position stability

- **Let's now include the effect of the vacuum vessel induced current**

$$0 = \mathbf{R_{vv}}\,\mathbf{I_v} + \mathbf{M_{vv}}\,\dot{\mathbf{I}}_\mathbf{v} + \mathbf{M_{vp}}\dot{I}_p + \dot{\mathbf{M}}_\mathbf{vp} I_p + \mathbf{M_{va}}\dot{\mathbf{I}}_\mathbf{a} + \dot{\mathbf{M}}_\mathbf{va}\mathbf{I_a}$$

- **Using the chain rule for $\mathbf{M_{pv}}$ and $\mathbf{M_{av}}$, this equation can be written as**

$$0 = \mathbf{R_{vv}}\,\mathbf{I_v} + \mathbf{M_{vv}}\,\dot{\mathbf{I}}_\mathbf{v} + \frac{\partial \mathbf{M_{vp}}}{\partial Z} I_{p0}\delta\dot{Z} + \frac{\partial \mathbf{M_{va}}}{\partial Z}\mathbf{I_{a0}}\delta\dot{Z}$$

- **One can now consider just the first current eigenmode of the vessel,** $\delta\mathbf{I_v} = \mathbf{v_e}\delta I_e$

$$0 = \left(\mathbf{v_e}^{-1}\mathbf{R_{vv}}\mathbf{v_e}\right)\delta I_e + \left(\mathbf{v_e}^{-1}\mathbf{M_{vv}}\mathbf{v_e}\right)\delta\dot{I}_e + \left(\mathbf{v_e}^{-1}\frac{\partial \mathbf{M_{vp}}}{\partial Z}\right) I_{p0}\delta\dot{Z} + \left(\mathbf{v_e}^{-1}\frac{\partial \mathbf{M_{va}}}{\partial Z}\right)\mathbf{I_{a0}}\delta\dot{Z}$$

- **Therefore, the set of equations that determines the vertical plasma stability is**

$$\delta\ddot{Z} + n\omega_1^2\delta Z = \frac{I_{p0}}{m_p}\frac{\partial M_{pe}}{\partial Z}\delta I_e$$

$$\delta\dot{I}_e + \frac{R_e}{L_e}\delta I_e = -\left[\frac{\partial M_{ep}}{\partial Z}\frac{I_{p0}}{L_e} + \frac{\partial \mathbf{M_{ea}}}{\partial Z}\frac{\mathbf{I_{a0}}}{L_e}\right]\delta\dot{Z}$$

$$R_e = \mathbf{v_e}^{-1}\mathbf{R_{vv}}\mathbf{v_e} \qquad \frac{\partial M_{ep}}{\partial Z} = \mathbf{v_e}^{-1}\frac{\partial \mathbf{M_{vp}}}{\partial Z}$$

$$L_e = \mathbf{v_e}^{-1}\mathbf{M_{vv}}\mathbf{v_e} \qquad \frac{\partial \mathbf{M_{ea}}}{\partial Z} = \mathbf{v_e}^{-1}\frac{\partial \mathbf{M_{va}}}{\partial Z}$$

The vertical plasma position stability

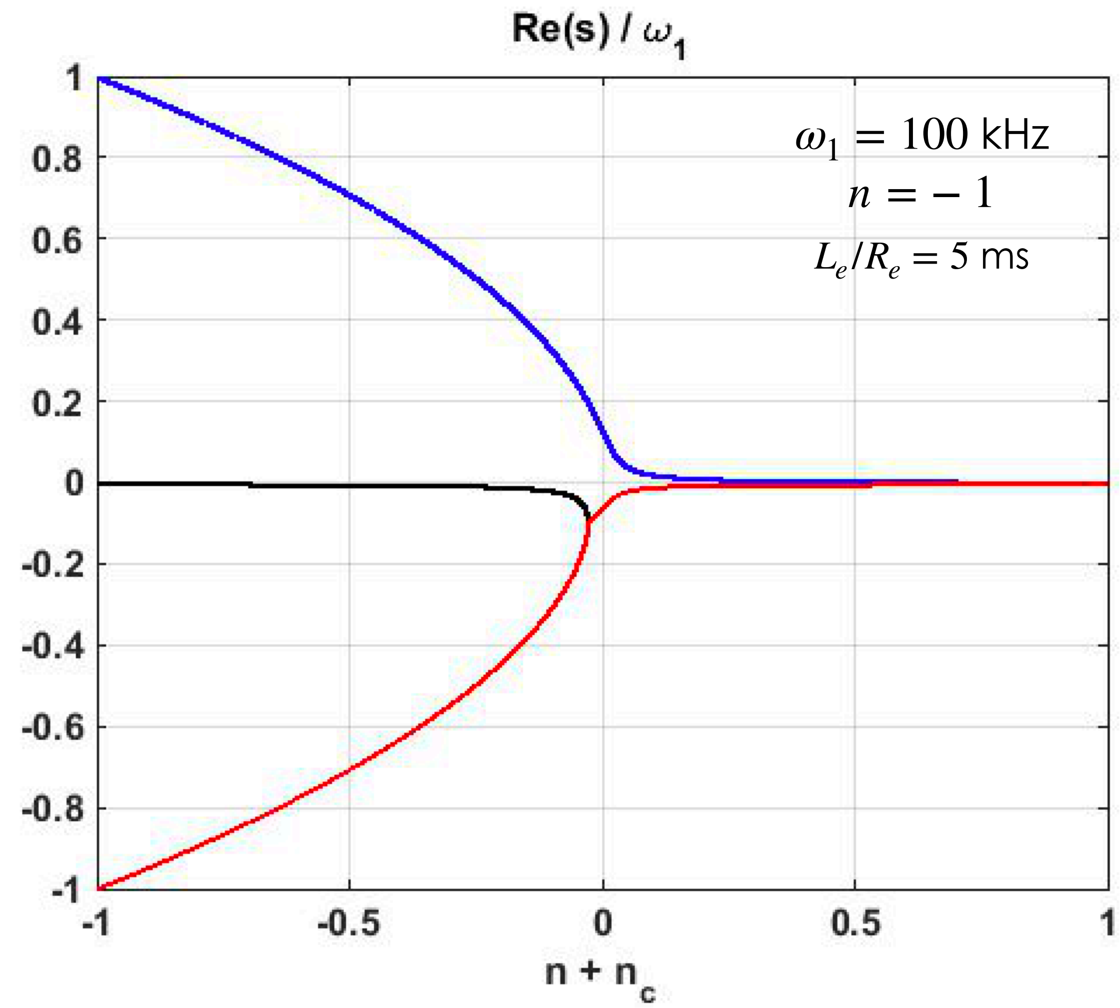
- **In the Laplace domain, with $Z_0 = 0$, these equations become**

$$\left(s^2 + n\omega_1^2\right)\delta\tilde{Z} = \frac{I_{p0}}{m_p}\frac{\partial M_{pe}}{\partial Z}\delta\tilde{I}_e \qquad \left(s + \frac{R_e}{L_e}\right)\delta\tilde{I}_e = -\,s\left[\frac{\partial M_{ep}}{\partial Z}\frac{I_{p0}}{L_e} + \frac{\partial \mathbf{M_{ea}}}{\partial Z}\frac{\mathbf{I_{a0}}}{L_e}\right]\delta\tilde{Z}$$

- **Defining** $n_c = \frac{2R_0}{\mu_0\Gamma_0 L_e}\left(\frac{\partial M_{ep}}{\partial Z} + \frac{\partial \mathbf{M_{ea}}}{\partial Z}\frac{\mathbf{I_{a0}}}{I_{p0}}\right)^2$**, the equations above combine into**

$$s^3 + \frac{R_e}{L_e}s^2 + \omega_1^2(n + n_c)s + \frac{R_e}{L_e}\omega_1^2 n = 0$$

- **The solutions of this equation can be plotted as a function of $n + n_c$**
  - *For $n_c < -n$, the fastest unstable eigenmode is of order ω_1. No hope for stabilization*
  - *For $n_c > -n$, the vessel is able to generate a current to counteract the instability. It can be stabilized by PF coils*



The vertical plasma position stability

- **A simple approximation can be used to estimate the fastest growing eigenmode for $n_c > -n$. Let's suppose that $\omega_1 \gg R_e/L_e$. With this assumption, the equation becomes**

$$\omega_1^2(n+n_c)s + \frac{R_e}{L_e}\omega_1^2 n = 0$$

  - *The solution is* $\mathrm{Real}(s) = \gamma = -\dfrac{nR_e}{L_e\left(n+n_c\right)}$

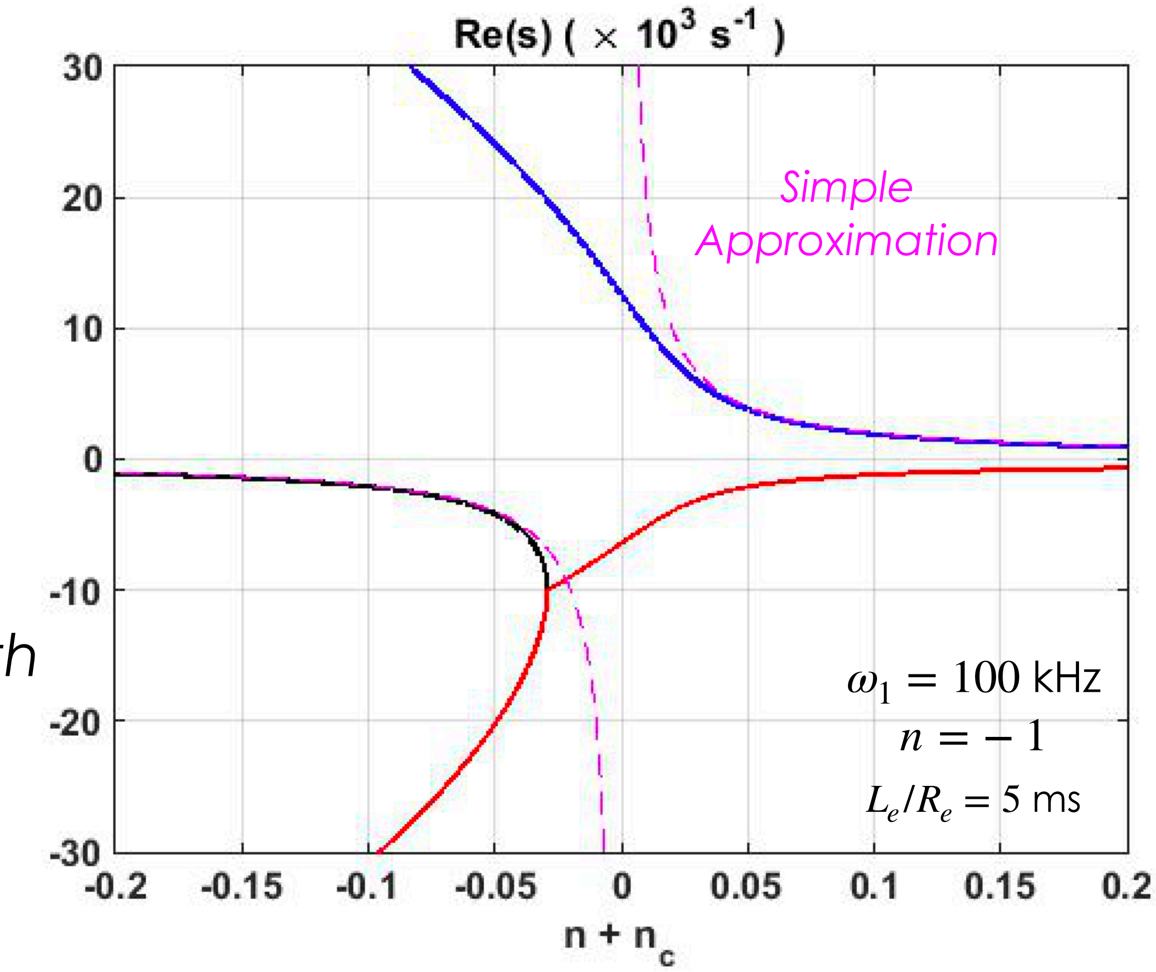
- **For $L_e/R_e = 5\,\mathrm{ms}$ and $n = -1$**

$$\gamma = \frac{200}{n_c - 1}\ \mathrm{s}^{-1}$$

  - *For $n_c = 2$, the fastest eigenmode grows with*

$$\gamma = 200\ \mathrm{s}^{-1}$$

(This mode can be easily stabilized by a plasma control system)

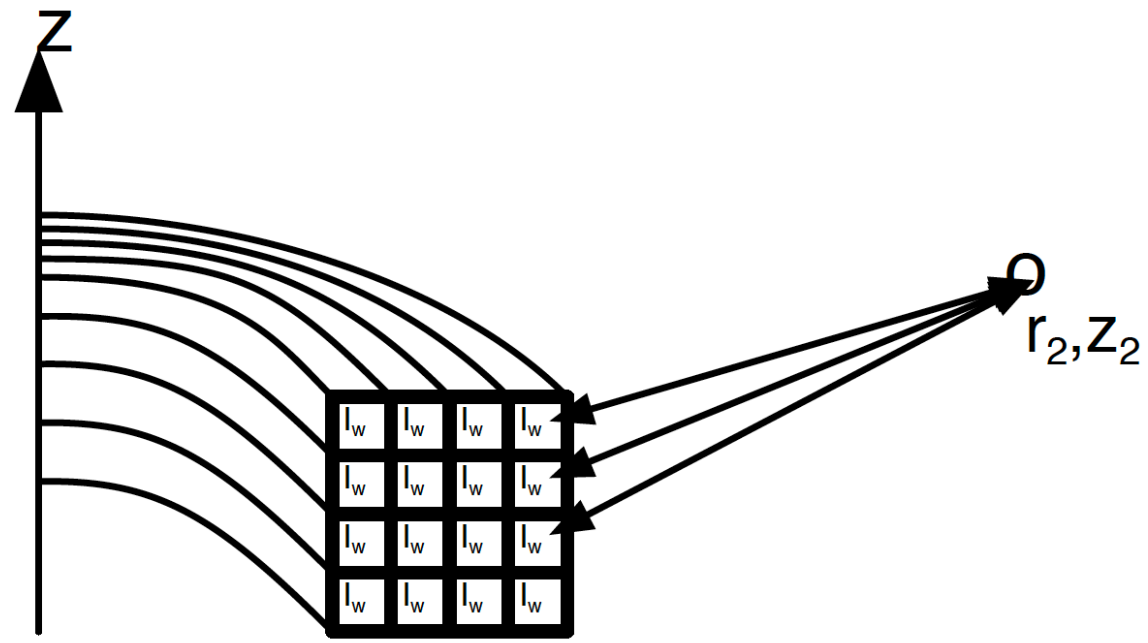


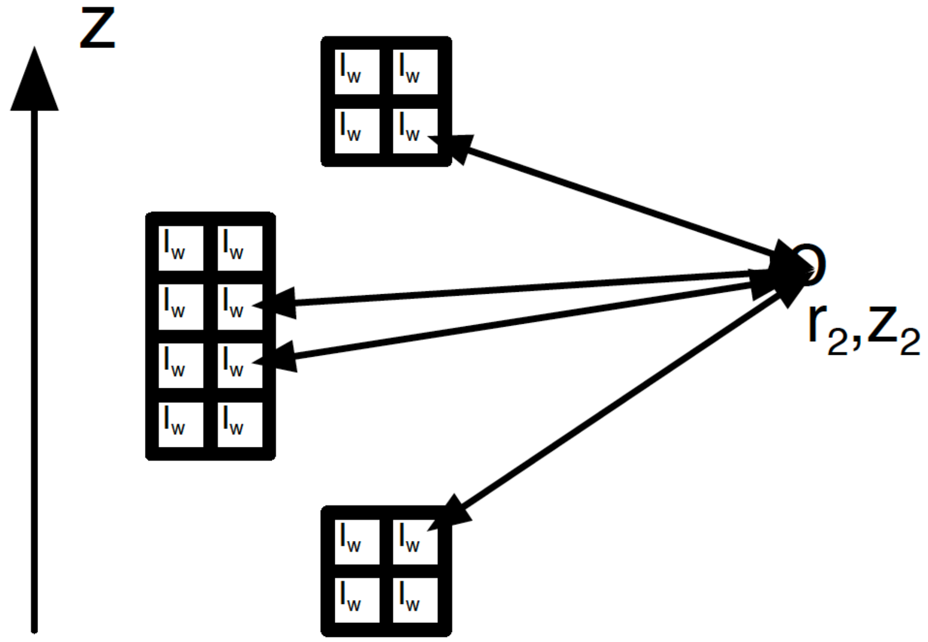
Modeling coils with multiple windings

- **Let's model a magnetic coils with N_w windings as**

$$\psi(R_2, Z_2) = \sum_{i=1}^{N_w} M(R_2, Z_2; R_i, Z_i)\, I_i = I_w \sum_{i=1}^{N_w} M(R_2, Z_2; R_i, Z_i) = M_c(R_2, Z_2)\, I_w$$

- **Filaments connected in series can be treated as a single coil**





The coupling of coils with multiple windings

- **The circuit equation for coil a (with current I_a and N_{wa} windings) with mutual coupling with coil b (with N_{wb})**

$$V_a = \sum_{i=1}^{N_{wa}} \left(R_i I_a + L_i \frac{dI_a}{dt} + M_{ib} \frac{dI_b}{dt} \right) = R_a I_a + L_a \frac{dI_a}{dt} + M_{ab} \frac{dI_b}{dt}$$

with

- $M_{ab} = \sum_{i=1}^{N_{wa}} M_{ib}$ *and* M_{ib} *is the mutual inductance between coil* b *and the i^th^ filament of coil* a*, which is located at* (R_i, Z_i)*. Therefore,* $M_{ab} = \sum_{i=1}^{N_{wa}} \sum_{j=1}^{N_{wb}} M_{ij}$
- $L_a = \sum_{i=1}^{N_{wa}} L_i$ *and* L_i *is the self-inductance of the* i^{th} *filament of coil* a
- $R_a = \sum_{i=1}^{N_{wa}} R_i$ *and* R_i *is the resistance of the* i^{th} *filament of coil* a

The coupling of coils with multiple windings

- **The circuit equations of a set of magnetic coils can be combined as**

$$V_1 = \sum_{i=2}^{N_c} R_1 I_1 + L_1 \frac{dI_1}{dt} + M_{1i}\frac{dI_i}{dt} \qquad \ldots \qquad V_{N_c} = \sum_{i=1}^{N_c-1} R_{N_c} I_{N_c} + L_{N_c}\frac{dI_{N_c}}{dt} + M_{N_c i}\frac{dI_i}{dt}$$

or in matrix form as

$$\mathbf{V_a} = \mathbf{R_a I_a} + \mathbf{M_{aa}\dot{I}_a}$$

where $\mathbf{V_a} = [V_1 \quad V_2 \quad V_3 \ldots \quad V_{N_c}]^T$ **and** $\mathbf{I_a} = [I_1 \quad I_2 \quad I_3 \ldots \quad I_{N_c}]^T$

$$\mathbf{M_{aa}} = \begin{pmatrix} L_1 & M_{12} & \ldots & M_{1N_c} \\ M_{21} & L_2 & \ldots & M_{2N_c} \\ \vdots & \vdots & \ddots & \vdots \\ M_{N_c 1} & M_{N_c 2} & \ldots & L_{N_c} \end{pmatrix} \qquad \mathbf{R_a} = \begin{pmatrix} R_1 & 0 & \ldots & 0 \\ 0 & R_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & R_{Nc} \end{pmatrix}$$

- *Here, the subindex a stands for* ***a****ctive coils with actively controlled voltage V_a*

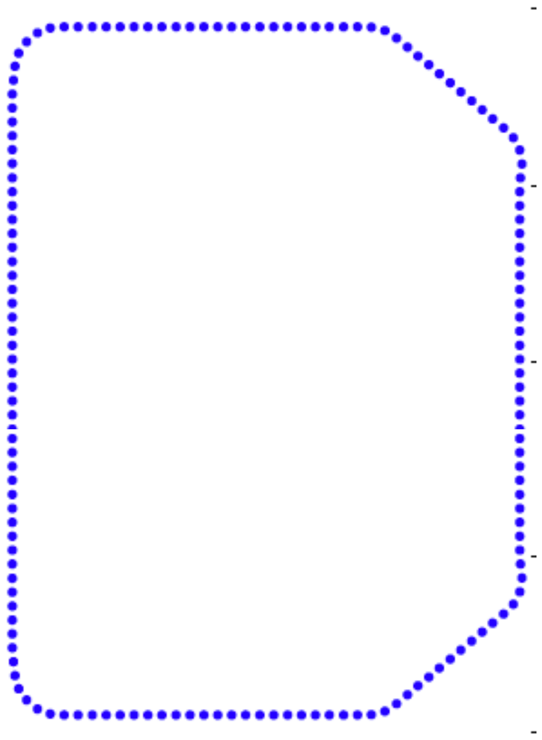
Modeling the vacuum vessel

- **The vacuum vessel is usually a complex 3D structure due to access ports, etc.**
  - *To simplify the model, the vacuum vessel is assumed to be axisymmetric*
  - *The vacuum vessel walls are discretized into toroidal filaments*

- **Since the vacuum vessel filaments are short-circuited the total voltage around the torus must be zero. Therefore, the circuit equation can also be modeled as**

$$0 = \mathbf{R_v I_v} + \mathbf{M_{vv}\dot{I}_v} + \mathbf{M_{va}\dot{I}_a}$$

TCABR Vacuum Vessel



  - *Here,* $\mathbf{I_v} = [I_{v1} \quad I_{v2} \quad I_{v3} \ldots \quad I_{vN_v}]^T$
  - $\mathbf{R_v} \in \mathbb{R}^{N_v \times N_v}$ *contains the resistances of each filament*
  - $\mathbf{M_{vv}} \in \mathbb{R}^{N_v \times N_v}$ *contains the self-inductance of each filament and the mutual inductances between filaments*
  - $\mathbf{M_{va}} \in \mathbb{R}^{N_v \times N_a}$ *contains the mutual inductances between filaments and between filaments and active coils*

Combined model for vessel + active coils

- The circuit equation for the active coils is now also modified to include the voltage induced by changing vessel currents

$$\mathbf{V_a = R_a I_a + M_{aa}\dot{I}_a + M_{av}\dot{I}_v}$$

- Combining these two sets of equations (active coils and vacuum vessel) yields

$$\begin{pmatrix} \mathbf{V_a} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{M_{aa}} & \mathbf{M_{av}} \\ \mathbf{M_{va}} & \mathbf{M_{vv}} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{I}}_\mathbf{a} \\ \dot{\mathbf{I}}_\mathbf{v} \end{pmatrix} + \begin{pmatrix} \mathbf{R_a} & 0 \\ 0 & \mathbf{R_v} \end{pmatrix} \begin{pmatrix} \mathbf{I_a} \\ \mathbf{I_v} \end{pmatrix}$$

or more compactly, just as

$$\mathbf{V = RI + M\dot{I}}$$

Modeling the plasma as made by several filaments

- **To model the plasma, let's discretize the plasma current density into several current filaments**
  - *As we have done for the vacuum vessel, the total voltage around the torus must be zero (note that the loop voltage is driven by* $\dot{\mathbf{I}}_{\mathbf{a}}$*)*

$$0 = R_P I_P + L_P \frac{dI_P}{dt} + \mathbf{M_{pa}} \dot{\mathbf{I}}_{\mathbf{a}} + \mathbf{M_{pv}} \dot{\mathbf{I}}_{\mathbf{v}}$$

- **Combining this equation with the other two sets of equations yields a more complete model**

$$\begin{pmatrix} \mathbf{V_a} \\ \mathbf{0} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{M_{aa}} & \mathbf{M_{av}} & \mathbf{M_{ap}} \\ \mathbf{M_{va}} & \mathbf{M_{vv}} & \mathbf{M_{vp}} \\ \mathbf{M_{pa}} & \mathbf{M_{pv}} & L_p \end{pmatrix} \begin{pmatrix} \dot{\mathbf{I}}_{\mathbf{a}} \\ \dot{\mathbf{I}}_{\mathbf{v}} \\ \dot{I}_p \end{pmatrix} + \begin{pmatrix} \mathbf{R_a} & 0 & 0 \\ 0 & \mathbf{R_v} & 0 \\ 0 & 0 & R_p \end{pmatrix} \begin{pmatrix} \mathbf{I_a} \\ \mathbf{I_v} \\ I_p \end{pmatrix}$$

and, again, this system of equations can be written in a more compact form

$$\mathbf{V} = \mathbf{R}\mathbf{I} + \mathbf{M}\dot{\mathbf{I}}$$

- **When the linearized momentum equations (radial and vertical components) of the plasma are inserted into these matrices, this model is called the RZIP model**
  - *The RZIP model is widely used for tuning plasma control systems*

References

- **Tokamak engineering**
  - *Magnetic forces and the central solenoid*
    - *Magnetic Fields: Ch. 6, section 2*
  - *Toroidal field coils*
    - *Fundamentals of Magnetic Thermonuclear Reactor Design: Ch. 12, section 2*
    - *Magnetic Fields: Ch. 7, section 3*
  - *Poloidal field coils, vertical plasma instability and the RZIP model*
    - *Magnetic modeling and control of tokamaks, by F. Felici, Eindhoven University of Technology*