A Coupled-Mode Theory for Multiwaveguide Systems Satisfying the Reciprocity Theorem and Power Conservation

SHUN-LIEN CHUANG

Abstract—Two sets of coupled-mode equations for multiwaveguide systems are derived using a generalized reciprocity relation; one set for a lossless system and the other for a general lossy or lossless system. The second set of equations also reduces to those of the first set in the lossless case under the condition that the transverse field components are chosen to be real.

Analytical relations between the coupling coefficients are shown and applied to the coupling of mode equations. It is shown analytically that our results satisfy exactly both the reciprocity theorem and power conservation. New orthogonal relations between the supermodes are derived in matrix form with the overlap integrals taken into account.

I. Introduction

THE COUPLING of mode theory in parallel waveguide systems has been of great interest in applications to directional couplers, laser arrays, waveguide switches, etc. [1], [2]. Although it has long been recognized that the previous coupled-mode theory is only applicable to very weakly coupled systems [3]–[8], significant improvements for strongly coupled waveguides have only been presented recently in series of papers [5], [8]–[11].

The major improvement is probably the inclusion of the overlap integrals C_{pq} defined in [8], when evaluating the power, and its resultant corrections to the various parameters such as the propagation constants and the coupling coefficients in the coupled-mode equations. Using two different methods, one based on a generalized reciprocity theorem and the other based on the variational principle, a new set of coupled-mode equations has been derived for a general (lossy or lossless) system [12]. Both methods give the same results.

In this paper, we apply the generalized reciprocity theorem [12] to a multiwaveguide system. The lossless case is treated here separately from the general lossy case, since in a lossless system, one may prefer to deal directly with powers for which the complex conjugates of the fields are needed, while for the general lossy case, one may not require any complex conjugate operations in the formulation [8]–[12]. Thus, the definitions for the overlap integrals and the coupling coefficients presented in Section III will be different from those for the general lossy case presented in Section IV. As will be shown in this paper, only when one chooses the *transverse* electric and magnetic field components to be real functions, the two formulations will be identical in the lossless limit. New properties of our coupled-mode equations are also presented analytically with the overlap integrals properly included.

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The author is with the Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801.

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II. Generalized Reciprocity Relation

Assuming that the electric and the magnetic fields $\boldsymbol{E}^{(1)}$, $\boldsymbol{H}^{(1)}$ satisfy the Maxwell equations in a medium $\epsilon^{(1)}(x, y)$ (for the whole space) and the corresponding boundary conditions and that $\boldsymbol{E}^{(2)}$ and $\boldsymbol{H}^{(2)}$ satisfy the Maxwell equations in another medium $\epsilon^{(2)}(x, y)$ and the corresponding boundary conditions, it is straightforward to show that [12], [13]

$$\nabla \cdot (\boldsymbol{E}^{(1)} \times \boldsymbol{H}^{(2)} - \boldsymbol{E}^{(2)} \times \boldsymbol{H}^{(1)}) = i\omega(\epsilon^{(2)} - \epsilon^{(1)})\boldsymbol{E}^{(1)} \cdot \boldsymbol{E}^{(2)} \tag{1}$$

with the same procedure used for deriving the Lorentz reciprocity relation [14]. When applied to a cylindrical geometry with an infinitesimal distance in the z-direction, (1) reduces to

$$\frac{\partial}{\partial z} \iint (\boldsymbol{E}^{(1)} \times \boldsymbol{H}^{(2)} - \boldsymbol{E}^{(2)} \times \boldsymbol{H}^{(1)}) \cdot \hat{z}\, dx\, dy = i\omega \iint \left(\epsilon^{(2)}(x, y) - \epsilon^{(1)}(x, y)\right) \boldsymbol{E}^{(1)} \cdot \boldsymbol{E}^{(2)}\, dx\, dy. \tag{2}$$

Here $\epsilon^{(1)}(x, y)$ and $\epsilon^{(2)}(x, y)$ can be general media such as a single waveguide or a multiple waveguide system as long as they are translational invariant in the z-direction. The time convention $\exp(-i\omega t)$ will be used in this paper. One notes that the two reciprocal relations (1) and (2) are exact as long as the two sets of field expressions $(\boldsymbol{E}^{(1)}, \boldsymbol{H}^{(1)})$ and $(\boldsymbol{E}^{(2)}, \boldsymbol{H}^{(2)})$ are exact solutions to the Maxwell equations in medium $\epsilon^{(1)}(x, y)$ and $\epsilon^{(2)}(x, y)$ respectively.

III. Coupled-Mode Theory for a Lossless Multiwaveguide System

In this section, we derive the coupled-mode equations for a lossless multiwaveguide system.

A. General Properties of the Fields of the Guided Modes

When the medium $\epsilon(x, y)$ is lossless and translational invariant in the z-direction, one knows that the field solutions of the form exist

$$\text{a)} \quad (\boldsymbol{E}_t + \boldsymbol{E}_z)\, e^{i\beta z} \qquad (\boldsymbol{H}_t + \boldsymbol{H}_z)\, e^{i\beta z}$$

which correspond to the fields propagating in the $+z$ direction. Here, we assume the above set of solutions to be the guided mode of the system. Based on inversion symmetry in the $-z$-direction, the following set of fields will also be solutions to the Maxwell equations [8], [13]–[15]

$$\text{b)} \quad (\boldsymbol{E}_t - \boldsymbol{E}_z)\, e^{-i\beta z} \qquad (-\boldsymbol{H}_t + \boldsymbol{H}_z)\, e^{-i\beta z}$$

which correspond to the fields propagating in the $-z$-direction. If the medium is lossless $\epsilon^*(x, y) = \epsilon(x, y)$, by taking the complex conjugate of the Maxwell equations or applying the time-reversal concept, it is easy to show that the following two sets of solutions also exist:

$$\text{c)} \quad (\boldsymbol{E}_t^* + \boldsymbol{E}_z^*)\, e^{-i\beta^* z} \qquad (-\boldsymbol{H}_t^* - \boldsymbol{H}_z^*)\, e^{-i\beta^* z}$$

$$\text{d)} \quad (\boldsymbol{E}_t^* - \boldsymbol{E}_z^*)\, e^{i\beta^* z} \qquad (\boldsymbol{H}_t^* - \boldsymbol{H}_z^*)\, e^{i\beta^* z}$$

where the * sign means complex conjugate. Since we consider the guided modes of the lossless system (excluding the leaky modes, cutoff modes, etc.), the propagation constant β^* is real. It is thus clear from a) and d) that one can choose the transverse field $\boldsymbol{E}_t$ to be real, and find immediately that $\boldsymbol{H}_t$ is real; $\boldsymbol{E}_z$ and $\boldsymbol{H}_z$ are purely imaginary. However, if one uses complex $\boldsymbol{E}_t$ (e.g., in an optical fiber with a circular cross section, $\boldsymbol{E}_t(\rho, \phi)$ can be of the form $J_m(k_\rho \rho)\, e^{im\phi}$), one finds that $\boldsymbol{H}_t$, $\boldsymbol{E}_z$ and $\boldsymbol{H}_z$ will also be complex. From these general properties of the field solutions, we next derive the coupled-mode equations for a lossless system and some analytical relations between the coupling coefficients and the overlap integrals.

B. The Derivation of the Coupled-Mode Equations

CASE (1): Suppose we choose

$$\epsilon^{\langle 1 \rangle}(x, y) = \epsilon^{(q)}(x, y) \tag{3}$$

$$\boldsymbol{E}^{\langle 1 \rangle} = (\boldsymbol{E}_t^{(q)} + \boldsymbol{E}_z^{(q)})\, e^{i\beta_q z} \tag{4a}$$

$$\boldsymbol{H}^{\langle 1 \rangle} = (\boldsymbol{H}_t^{(q)} + \boldsymbol{H}_z^{(q)})\, e^{i\beta_q z} \tag{4b}$$

to be the guided mode propagating in the $+z$-direction in a medium $\epsilon^{(q)}(x, y)$ with a single waveguide q. We also choose for the second set

$$\epsilon^{\langle 2 \rangle}(x, y) = \epsilon^{(p)*}(x, y) = \epsilon^{(p)}(x, y) \tag{5}$$

and

$$\boldsymbol{E}^{\langle 2 \rangle} = (\boldsymbol{E}_t^{(p)*} + \boldsymbol{E}_z^{(p)*})\, e^{-i\beta_p z} \tag{6a}$$

$$\boldsymbol{H}^{\langle 2 \rangle} = (-\boldsymbol{H}_t^{(p)*} - \boldsymbol{H}_z^{(p)*})\, e^{-i\beta_p z} \tag{6b}$$

which are also solutions as discussed before. They correspond to the fields propagating in the $-z$-direction. Substituting the two sets of expressions into the generalized reciprocity relation (2), we obtain

$$\overline{\overline{K}}_{pq} - \overline{\overline{K}}_{pq}^* = (\beta_p - \beta_q)\left(\frac{\overline{\overline{C}}_{pq} + \overline{\overline{C}}_{qp}^*}{2}\right) \tag{7}$$

where

$$\overline{\overline{K}}_{pq} = \frac{\omega}{4} \iint_{-\infty}^{\infty} \Delta\epsilon^{(q)}(\boldsymbol{E}_t^{(p)*} \cdot \boldsymbol{E}_t^{(q)} + E_z^{(p)*} E_z^{(q)})\, dx\, dy \tag{8}$$

and

$$\overline{\overline{C}}_{pq} = \frac{1}{2} \iint_{-\infty}^{\infty} \boldsymbol{E}_t^{(q)} \times \boldsymbol{H}_t^{(p)*} \cdot \hat{z}\, dx\, dy. \tag{9}$$

We note that (7) is an exact relation since the fields $(\boldsymbol{E}^{\langle 1 \rangle}, \boldsymbol{H}^{\langle 1 \rangle})$ and $(\boldsymbol{E}^{\langle 2 \rangle}, \boldsymbol{H}^{\langle 2 \rangle})$ are exact solutions to Maxwell's equations in $\epsilon^{\langle 1 \rangle}(x, y)$ and $\epsilon^{\langle 2 \rangle}(x, y)$, respectively.

CASE (2): In this case, we choose $\epsilon^{\langle 1 \rangle}(x, y)$ to be the medium of the multiwaveguide system $\epsilon(x, y)$

$$\epsilon^{\langle 1 \rangle}(x, y) = \epsilon(x, y). \tag{10}$$

The solutions to the system are given approximately by

$$\boldsymbol{E}_t^{\langle 1 \rangle} \simeq \sum_{p=1}^{N} a_q(z)\, \boldsymbol{E}_t^{(q)}(x, y) \tag{11a}$$

$$\boldsymbol{H}_t^{\langle 1 \rangle} \simeq \sum_{p=1}^{N} a_q(z)\, \boldsymbol{H}_t^{(q)}(x, y) \tag{11b}$$

for the transverse field components. The z-components are given by

$$\boldsymbol{E}_z^{\langle 1 \rangle} \simeq \sum_{p=1}^{N} \frac{\epsilon^{(q)}(x, y)}{\epsilon(x, y)}\, a_p(z)\, \boldsymbol{E}_z^{(q)}(x, y) \tag{11c}$$

$$\boldsymbol{H}_z^{\langle 1 \rangle} \simeq \sum_{p=1}^{N} a_q(z)\, \boldsymbol{H}_z^{(q)}(x, y). \tag{11d}$$

A similar derivation for the above relations has been given in [13] for the polarization vector or in [12]. One notes that $\boldsymbol{E}_t^{(q)}(x, y)$, $q = 1, \cdots, N$ are not orthogonal functions, and the overlap integrals $C_{pq} \neq 0$. The second set of solutions is chosen as

$$\epsilon^{\langle 2 \rangle}(x, y) = \epsilon^{(p)*}(x, y) = \epsilon^{(p)}(x, y) \tag{12}$$

$$\boldsymbol{E}^{\langle 2 \rangle} = (\boldsymbol{E}_t^{(p)*} + \boldsymbol{E}_z^{(p)*})\, e^{-i\beta_p z} \tag{13a}$$

$$\boldsymbol{H}^{\langle 2 \rangle} = (-\boldsymbol{H}_t^{(p)*} - \boldsymbol{H}_z^{(p)*})\, e^{-i\beta_p z}. \tag{13b}$$

Substituting the two sets of expressions (10)–(13) into the generalized reciprocity relation (2), we obtain

$$\sum_q \tilde{\tilde{C}}_{pq} \frac{d}{dz} a_q(z) = i \sum_q (\tilde{\tilde{K}}^*_{pq} + \beta_p \tilde{\tilde{C}}_{pq}) a_q(z) \quad (14)$$

where

$$\tilde{\tilde{K}}_{pq} = \frac{\omega}{4} \iint_{-\infty}^{\infty} \Delta\epsilon^{(q)} \left(\boldsymbol{E}_t^{(p)*} \cdot \boldsymbol{E}_t^{(q)} + \frac{\epsilon^{(p)}}{\epsilon} E_z^{(p)*} E_z^{(q)} \right) dx\, dy \quad (15)$$

and

$$\tilde{\tilde{C}}_{pq} = \frac{1}{2} (\overline{\overline{C}}_{pq} + \overline{\overline{C}}^*_{qp}). \quad (16)$$

One notes that $\tilde{\tilde{C}}_{pq}$ is a hermitian matrix, $\tilde{\tilde{C}}_{pq} = \tilde{\tilde{C}}^*_{qp}$. It is straightforward to show that $\tilde{\tilde{K}}_{pq}$ satisfies the same relation (7) as $\overline{\overline{K}}_{pq}$ because

$$\tilde{\tilde{K}}_{pq} = \overline{\overline{K}}_{pq} + \frac{\omega}{4} \iint \Delta\epsilon^{(q)} \Delta\epsilon^{(p)} E_z^{(p)*} E_z^{(q)}\, dx\, dy \quad (17)$$

where the second term is equal to its complex conjugate quantity if one exchanges p and q where both $\Delta\epsilon^{(p)}$ and $\Delta\epsilon^{(q)}$ are real (lossless). Thus

$$\tilde{\tilde{K}}_{pq} - \tilde{\tilde{K}}^*_{qp} = \overline{\overline{K}}_{pq} - \overline{\overline{K}}^*_{qp} = (\beta_p - \beta_q) \left(\frac{\overline{\overline{C}}_{pq} + \overline{\overline{C}}^*_{qp}}{2} \right) = (\beta_p - \beta_q) \tilde{\tilde{C}}_{pq} \quad (18)$$

which is an exact relation. It is seen clearly that only if $\beta_p = \beta_q$, one has $\tilde{\tilde{K}}_{pq} = \tilde{\tilde{K}}^*_{qp}$ (or if the overlap integrals are very small in the extremely weak coupling case, $\tilde{\tilde{K}}_{pq} \simeq \tilde{\tilde{K}}^*_{qp}$). Otherwise one should treat $\tilde{\tilde{K}}_{pq}$ and $\tilde{\tilde{K}}^*_{qp}$ as different quantities in general. One defines the matrix elements:

$$\tilde{\tilde{Q}}_{pq} = \tilde{\tilde{K}}^*_{qp} + \beta_p \tilde{\tilde{C}}_{pq} = \tilde{\tilde{K}}_{pq} + \tilde{\tilde{C}}_{pq} \beta_q. \quad (19)$$

Thus, the coupling of mode equations can be written as

$$\tilde{\tilde{\boldsymbol{C}}} \frac{d\boldsymbol{a}}{dz} = i \tilde{\tilde{\boldsymbol{Q}}} \boldsymbol{a} \quad (20)$$

where $\tilde{\tilde{\boldsymbol{Q}}}$ is clearly hermitian since

$$\tilde{\tilde{Q}}_{pq} = \tilde{\tilde{Q}}^*_{pq} \quad (21)$$

which can be shown from (19) and $\boldsymbol{a}$ is a vector with its elements given by $a_q(z)$, $q = 1, 2, \cdots, N$. Another way to write the above equation is either

$$\tilde{\tilde{\boldsymbol{C}}} \frac{d\boldsymbol{a}}{dz} = i(\tilde{\tilde{\boldsymbol{K}}}^+ + \boldsymbol{B}\tilde{\tilde{\boldsymbol{C}}}) \boldsymbol{a} \quad (22)$$

or

$$\tilde{\tilde{\boldsymbol{C}}} \frac{d\boldsymbol{a}}{dz} = i(\tilde{\tilde{\boldsymbol{K}}} + \tilde{\tilde{\boldsymbol{C}}}\boldsymbol{B}) \boldsymbol{a} \quad (23)$$

where $\boldsymbol{B}$ is a diagonal matrix with the elements given by the propagation constants of individual waveguides β_p. Here the superscript $^+$ means complex conjugate and transpose of the matrix. The second form (23) is useful since

$$\frac{d\boldsymbol{a}}{dz} = i(\boldsymbol{B} + \tilde{\tilde{\boldsymbol{C}}}^{-1} \tilde{\tilde{\boldsymbol{K}}}) \boldsymbol{a} \quad (24)$$

while the first form (22), which is similar (but not identical) to that of [10], [11], requires more algebraic manipulations in evaluating $(\tilde{\tilde{\boldsymbol{C}}}^{-1} \boldsymbol{B} \tilde{\tilde{\boldsymbol{C}}} + \tilde{\tilde{\boldsymbol{C}}}^{-1} \tilde{\tilde{\boldsymbol{K}}}^+)$.

C. Power Conservation

In Section III-B, we derived the coupled-mode equations in matrix form (20), where $\tilde{\tilde{\boldsymbol{C}}}$ is related to the overlap integrals C_{pq} and C^*_{qp}, and $\tilde{\tilde{\boldsymbol{Q}}}$ is defined in (19). Both $\tilde{\tilde{\boldsymbol{C}}}$ and $\tilde{\tilde{\boldsymbol{Q}}}$ are proved to be hermitian without any approximation in the matrix elements. Let us look at the power guided along the multiwaveguide system

$$P(z) = \frac{1}{2} \mathrm{Re} \iint \boldsymbol{E}_t \times \boldsymbol{H}^*_t \cdot \hat{z}\, dx\, dy = \mathrm{Re} \left[\sum_{p,q} a^*_p(z)\, \overline{\overline{C}}_{pq} a_q(z) \right] = \sum_{p,q} a^*_p(z) \left(\frac{\overline{\overline{C}}_{pq} + \overline{\overline{C}}^*_{qp}}{2} \right) a_q(z) = \boldsymbol{a}^+(z)\, \tilde{\tilde{\boldsymbol{C}}} \boldsymbol{a}(z) \quad (25)$$

where $\tilde{\tilde{\boldsymbol{C}}}$ is defined in (16). If the medium is lossless, the power of the guided mode must be independent of the position z, i.e., $dP/dz = 0$. We have the lossless condition

$$\sum_{p,q} \left(\frac{da^*_p(z)}{dz} \right) \tilde{\tilde{C}}_{pq} a_q(z) + \sum_{i,j} a^*_p(z)\, \tilde{\tilde{C}}_{pq} \frac{da_q(z)}{dz} = 0. \quad (26)$$

Using the coupled-mode equation (20), one finds immediately that the *lossless condition* is equivalent to

$$i \sum_{p,q} a^*_p (\tilde{\tilde{Q}}_{pq} - \tilde{\tilde{Q}}^*_{qp}) a_q = 0 \quad (27)$$

where we have used the fact that $\tilde{\tilde{C}}_{pq}$ is hermitian. Since a^*_p and a_q can be arbitrary values, we obtain

$$\tilde{\tilde{Q}}_{pq} - \tilde{\tilde{Q}}^*_{qp} = 0. \quad (28)$$

That is, $\tilde{\tilde{Q}}_{pq}$ must be hermitian, which is true, since we have shown that $\tilde{\tilde{Q}}_{pq}$ is indeed hermitian in (21) from the definition (19). Thus our formulation satisfies exactly the power conservation. An example will be shown later which illustrates this power conservation criterion.

D. Power Orthogonality of the Supermodes

Let us choose two sets of solutions to be two distinct supermodes in the multiwaveguide system $\epsilon(x, y)$

$$\epsilon^{(1)}(x, y) = \epsilon(x, y) \quad (29)$$

$$E_t^{\langle 1 \rangle} = \left(\sum_{q=1}^{N} a_q^{(i)} E_t^{(q)} \right) e^{i\gamma_i z} \qquad (30a)$$

$$H_t^{\langle 1 \rangle} = \left(\sum_{q=1}^{N} a_q^{(i)} H_t^{(q)} \right) e^{i\gamma_i z} \qquad (30b)$$

where $\boldsymbol{a}^{(i)}$ with elements $a_q^{(i)}$, $q = 1, 2, \cdots, N$, is the eigenvector for the first supermode with a propagation constant γ_i

$$\epsilon^{\langle 2 \rangle}(x, y) = \epsilon(x, y) = \epsilon^*(x, y) \qquad (31)$$

$$E_t^{\langle 2 \rangle} = \left(\sum_{p=1}^{N} a_p^{(j)*} E_t^{(p)*} \right) e^{-i\gamma_j z} \qquad (32a)$$

$$H_t^{\langle 2 \rangle} = \left(- \sum_{p=1}^{N} a_p^{(j)*} H_t^{(p)*} \right) e^{-i\gamma_j z} \qquad (32b)$$

and $\boldsymbol{a}^{(j)}$ with elements $a_p^{(j)}$, $p = 1, 2, \cdots, N$, is the eigenvector for the second supermode with a propagation constant γ_j. The reciprocity relation (2) gives

$$(\gamma_i - \gamma_j) \sum_{p,q}\sum a_q^{(i)} a_p^{(j)*} (\overline{\overline{C}}_{pq} + \overline{\overline{C}}_{qp}^*) = 0. \qquad (33)$$

Since $\gamma_i \neq \gamma_j$ we obtain the general orthogonality condition:

$$\boldsymbol{a}^{(j)+} \tilde{\tilde{C}} \boldsymbol{a}^{(i)} = 0, \qquad i \neq j. \qquad (34)$$

That is, any two eigenvectors corresponding to different propagation constants are orthogonal to each other with a weighting matrix given by $\tilde{\tilde{C}}$.

An alternative way of deriving (34) is simply by looking at the coupled-mode equation (20). The supermode solution $\boldsymbol{a}(z)$ is given by the form

$$\boldsymbol{a}(z) = \boldsymbol{a} e^{i\gamma z}. \qquad (35)$$

Thus, the matrix equation (20) for the coupled-mode equations reduces to the eigenequation

$$\gamma \tilde{\tilde{C}} \boldsymbol{a} = \tilde{\tilde{Q}} \boldsymbol{a}. \qquad (36)$$

The eigenvalue γ satisfies

$$\det \left| \tilde{\tilde{Q}} - \gamma \overline{\overline{C}} \right| = 0. \qquad (37)$$

Since both $\tilde{\tilde{C}}$ and $\tilde{\tilde{Q}}$ are hermitian, the eigenvalues for (36) must be real, that can be shown from elementary matrix theory [16]. It is also obviously true from the fact that the medium is lossless. Another property of the matrix equation (36) is that two distinct eigenvectors $\boldsymbol{a}^{(i)}$ and $\boldsymbol{a}^{(j)}$ are orthogonal to each other with the "weighting matrix" $\tilde{\tilde{C}}$

$$\boldsymbol{a}^{(j)+} \tilde{\tilde{C}} \boldsymbol{a}^{(i)} = 0, \qquad i \neq j. \qquad (38)$$

One notes that in the extremely weak coupling case, the coupling of mode equations have the same form as (20) except that $\tilde{\tilde{C}}$ should be replaced by I, the identity matrix. Thus, the orthogonality relation (38) reduces to the well-known results: $\boldsymbol{a}^{(j)+}\boldsymbol{a}^{(i)} = 0$ for $i \neq j$ in conventional theory.

IV. Coupled-Mode Theory for a General (Lossy or Lossless) Multiwaveguide System

In general, a multiwaveguide system can be lossy. The previous formulation will not be applicable anymore. Actually, the formulation for a lossy medium is very similar to the formulation in the previous section, except one does not have any complex conjugate operation, and special care is taken for the z-components of the fields as can be seen from Section III-A. A derivation has been presented in [12] which is similar to that in the previous section. Therefore, we briefly give the results below.

A. The Derivation of the Coupled Mode Equations for a Lossy System

CASE (1): Following the procedure in Case (1) of Section III-B, except that we choose the second set of solutions to be the form b) in Section III-A, it is easy to derive [12]

$$\overline{K}_{pq} - \overline{K}_{qp} = (\beta_p - \beta_q) \frac{C_{pq} + C_{qp}}{2} \qquad (39)$$

where

$$\overline{K}_{pq} = \frac{\omega}{4} \iint_{-\infty}^{\infty} \Delta\epsilon^{(q)} (E_t^{(p)} \cdot E_t^{(q)} - E_z^{(p)} E_z^{(q)}) \, dx \, dy \qquad (40)$$

and

$$C_{pq} = \frac{1}{2} \iint_{-\infty}^{\infty} E_t^{(q)} \times H_t^{(p)} \cdot \hat{z} \, dx \, dy \qquad (41)$$

where no complex conjugate operation is involved, and there is a negative sign in the integrand of (40). The above definitions (40) and (41) are the same as those used in [8] except for the constant factor of 4. The difference is only apparent because once we choose the normalization condition $C_{11} = C_{22} = \cdots C_{NN} = 1$, the factor of 4 is absorbed in E_t and H_t. Thus, numerically, $\overline{K}_{pq}$ is identical to that in [8].

CASE (2): Following the procedures in Case (2) of Section III-B, we choose the first medium and the field solutions to be the same as (10) and (11), and the second medium and the field solutions to be

$$\epsilon^{\langle 2 \rangle}(x, y) = \epsilon^{(p)}(x, y) \qquad (42)$$

$$E^{\langle 2 \rangle} = (E_t^{(p)} - E_z^{(p)}) \, e^{-i\beta_p z} \qquad (43a)$$

$$H^{\langle 2 \rangle} = (-H_t^{(p)} + H_z^{(p)}) \, e^{-i\beta_p z}. \qquad (43b)$$

We obtain again from the generalized reciprocity relation (2)

$$\sum_q \frac{C_{pq} + C_{qp}}{2} \frac{d}{dz} a_q(z) = i \sum_q \left(K_{pq} + \beta_p \frac{C_{pq} + C_{qp}}{2} \right) a_q(z) \qquad (44)$$

where

$$K_{pq} = \frac{\omega}{4} \iint \Delta\epsilon^{(q)} \left(\boldsymbol{E}_t^{(p)} \cdot \boldsymbol{E}_t^{(q)} - \frac{\epsilon^{(p)}}{\epsilon} E_z^{(p)} E_z^{(q)} \right) \quad (45)$$

and C_{pq} has been defined in (41). One can also show that K_{pq} satisfies the same equation as $\overline{K}_{pq}$ in (39) by recognizing that

$$K_{pq} = \overline{K}_{qp} + \frac{\omega}{4} \iint \Delta\epsilon^{(q)} \frac{\Delta\epsilon^{(p)}}{\epsilon} E_z^{(p)} E_z^{(q)} \, dx \, dy \quad (46)$$

where the second term is symmetric with respect to p and q. Thus

$$K_{pq} - K_{qp} = (\beta_p - \beta_q) \frac{C_{pq} + C_{qp}}{2} \quad (47)$$

which is also an exact relation.

The coupled-mode equation can be written in a matrix form

$$\overline{\boldsymbol{C}} \frac{d}{dz} \boldsymbol{a}(z) = i\boldsymbol{Q}\boldsymbol{a} \quad (48)$$

where

$$\overline{C}_{pq} = \frac{C_{pq} + C_{qp}}{2} = \overline{C}_{qp} \quad (49)$$

is symmetric, and

$$\begin{aligned} Q_{pq} &= K_{qp} + \beta_p \frac{C_{pq} + C_{qp}}{2} \\ &= K_{pq} + \frac{C_{pq} + C_{qp}}{2} \beta_q = Q_{qp} \end{aligned} \quad (50)$$

is also symmetric. The matrix equation can also be written as

$$\frac{d}{dz} \boldsymbol{a}(z) = i\boldsymbol{M}\boldsymbol{a} \quad (51)$$

$$\boldsymbol{M} = \overline{\boldsymbol{C}}^{-1} \boldsymbol{Q} \quad (52)$$

or

$$\boldsymbol{M} = \overline{\boldsymbol{C}}^{-1} \boldsymbol{B} \overline{\boldsymbol{C}} + \overline{\boldsymbol{C}}^{-1} \boldsymbol{K}^T \quad (53)$$

or

$$\boldsymbol{M} = \boldsymbol{B} + \overline{\boldsymbol{C}}^{-1} K \quad (54)$$

where $\boldsymbol{B}$ is again a diagonal matrix with the propagation constants β_p as the elements, and the superscript T means transpose of the matrix. Equation (53) is compared with the form in [10], [11]. One sees that the only difference is that the matrix $\overline{\boldsymbol{C}}$ is used here while the matrix C is used in [8]–[11]. $\boldsymbol{K}^T$ used in this paper is the same as K in [10], [11] from the definition (40) except for the factor of 4. The final form (54) is simpler than (53) since $\boldsymbol{B}$ is used instead of $\overline{\boldsymbol{C}}^{-1}\boldsymbol{B}\overline{\boldsymbol{C}}$. Thus, our coupled-mode equation looks simpler using the form (51) with $\boldsymbol{M}$ given by (54), than that in [10], [11]. The coupling coefficients $\overline{\overline{K}}_{pq}$ defined in (15) for the lossless case or K_{pq} defined in (45) for the general case differ from $\overline{\overline{K}}_{pq}$ (defined in (8)) or $\overline{K}_{pq}$ (defined in (40)) by the factor $\epsilon^{(p)}/\epsilon$ in the second part of the integrand. This factor is also taken as one in [17]. We believe that it should be more self-consistent to keep the factor since it was derived from (11c) making use of Maxwell's equation as shown in [12, appendix A].

B. General Orthogonality Property of the Supermodes

Following a similar procedure to that in Section III-D, one applies the reciprocity relation (2) to any two supermodes

$$\boldsymbol{E}_t^{\langle 1 \rangle} = \left(\sum_{q=1}^{N} a_q^{(i)} \boldsymbol{E}_t^{(q)} \right) e^{i\gamma_i z} \quad (55a)$$

$$\boldsymbol{H}_t^{\langle 1 \rangle} = \left(\sum_{q=1}^{N} a_q^{(i)} \boldsymbol{H}_t^{(q)} \right) e^{i\gamma_i z} \quad (55b)$$

and

$$\boldsymbol{E}_t^{\langle 2 \rangle} = \left(\sum_{p=1}^{N} a_p^{(j)} \boldsymbol{E}_t^{(p)} \right) e^{-i\gamma_j z} \quad (56a)$$

$$\boldsymbol{H}_t^{\langle 2 \rangle} = \left(-\sum_{p=1}^{N} a_p^{(j)} \boldsymbol{H}_t^{(p)} \right) e^{-i\gamma_j z}. \quad (56b)$$

Using the eigenvector $\boldsymbol{a}^{(i)}$ = column $(a_1^{(i)}, a_2^{(i)}, \cdots, a_N^{(i)})$ and a similar form for $\boldsymbol{a}^{(j)}$, one obtains

$$\boldsymbol{a}^{(j)T} \overline{\boldsymbol{C}} \boldsymbol{a}^{(i)} = 0, \qquad \text{for } \gamma_i \neq \gamma_j \quad (57)$$

which is the reciprocity relation that should be satisfied by any two eigenvectors of the matrix equation

$$\gamma \overline{\boldsymbol{C}} \boldsymbol{a} = \boldsymbol{Q}\boldsymbol{a} \quad (58)$$

which follows (48). Alternatively, because both $\overline{\boldsymbol{C}}$ and $\boldsymbol{Q}$ are symmetric matrices, the general orthogonality relation (57) is a well-known property in matrix theory [16].

C. Reciprocity Relation for Two Sets of Solutions with Separate Boundary Conditions

Let us look at the boundary value problem for a set of solutions to the coupled-mode equation (48). The general solution $\boldsymbol{a}(z)$ is given by

$$\boldsymbol{a}(z) = \boldsymbol{A} e^{i\boldsymbol{\Gamma} z} \boldsymbol{A}^{-1} \boldsymbol{a}(0) \quad (59)$$

where

$$\boldsymbol{\Gamma} = \text{diagonal}\,(\gamma_1, \gamma_2, \cdots, \gamma_N) \quad (60)$$

for a given boundary condition $\boldsymbol{a}(0)$ and the wave propagating in the $+z$ direction. Here the matrix $\boldsymbol{A}$ is defined to have the ith column given by the ith eigenvector of the matrix (58), and $\gamma_1, \cdots, \gamma_N$ are the eigenvalues of (58).

Consider a first set of solutions at $z = 0$ given the condition $\boldsymbol{a}^{\langle 1 \rangle}(z = -l)$,

$$\boldsymbol{a}^{\langle 1 \rangle}(0) = \boldsymbol{A} e^{i\boldsymbol{\Gamma} l} \boldsymbol{A}^{-1} \boldsymbol{a}^{\langle 1 \rangle}(-l). \quad (61)$$

Let us look at another set of solutions with the boundary condition given at $z = 0$ and the wave propagating in the $-z$ direction to $z = -l$

$$\boldsymbol{a}^{\langle 2 \rangle}(-l) = \boldsymbol{A} e^{i\boldsymbol{\Gamma} l} \boldsymbol{A}^{-1} \boldsymbol{a}^{\langle 2 \rangle}(0). \quad (62)$$

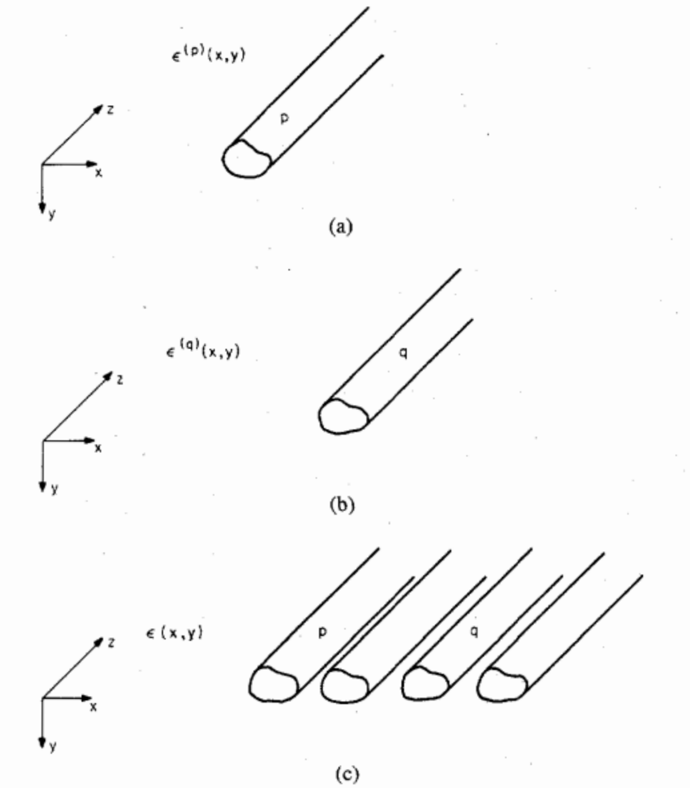

Fig. 1. (a) A single waveguide p described by $\epsilon^{(p)}(x, y)$ in whole space. (b) A single waveguide q described by $\epsilon^{(q)}(x, y)$ in whole space. (c) A multiwaveguide system described by $\epsilon(x, y)$ in whole space.

Applying the reciprocity relation (1) to a cylindrical surface enclosing the planes $z = -l$ and $z = 0$ with a radius going to infinity, one finds [12]

$$\iint\limits_{z=-l} (\boldsymbol{E}^{\langle 1\rangle} \times \boldsymbol{H}^{\langle 2\rangle} - \boldsymbol{E}^{\langle 2\rangle} \times \boldsymbol{H}^{\langle 1\rangle}) \cdot \hat{z}\, dx\, dy$$

$$= \iint\limits_{z=0} (\boldsymbol{E}^{\langle 1\rangle} \times \boldsymbol{H}^{\langle 2\rangle} - \boldsymbol{E}^{\langle 2\rangle} \times \boldsymbol{H}^{\langle 1\rangle}) \cdot \hat{z}\, dx\, dy. \tag{63}$$

From the previous two sets of solutions, we have (Fig. 2)

$$\boldsymbol{E}_t^{\langle 1\rangle} = \sum_{p=1}^{N} a_p^{\langle 1\rangle}(z)\, \boldsymbol{E}_t^{(p)} \tag{64a}$$

$$\boldsymbol{H}_t^{\langle 1\rangle} = \sum_{p=1}^{N} a_p^{\langle 1\rangle}(z)\, \boldsymbol{H}_t^{(p)} \tag{64b}$$

and

$$\boldsymbol{E}_t^{\langle 2\rangle} = \sum_{q=1}^{N} a_q^{\langle 2\rangle}(z)\, \boldsymbol{E}_t^{(q)} \tag{65a}$$

$$\boldsymbol{H}_t^{\langle 2\rangle} = -\sum_{q=1}^{N} a_q^{\langle 2\rangle}(z)\, \boldsymbol{H}_t^{(q)} \tag{65b}$$

where the second set of fields propagates in the $-z$-direction. The vector $\boldsymbol{a}^{\langle 1\rangle}(z = 0)$ is related to $\boldsymbol{a}^{\langle 1\rangle}(z = -l)$ by (61), and $\boldsymbol{a}^{\langle 2\rangle}(z = -l)$ is related to $\boldsymbol{a}^{\langle 2\rangle}(z = 0)$ by

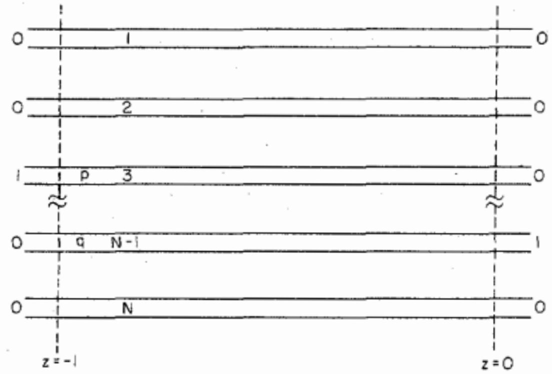

Fig. 2. A multiwaveguide system with possible excitation either at waveguide p at $z = -l$ and the wave propagating in the $+z$ direction, or at waveguide q at $z = 0$ and the wave propagating in the $-z$-direction.

(62). The reciprocity relation (62) reduces to

$$\boldsymbol{a}^{\langle 1\rangle T}(-l)\, \overline{\boldsymbol{C}}\boldsymbol{a}^{\langle 2\rangle}(-l) = \boldsymbol{a}^{\langle 1\rangle T}(0)\, \overline{\boldsymbol{C}}\boldsymbol{a}^{\langle 2\rangle}(0) \tag{66}$$

or equivalently

$$\begin{aligned}\boldsymbol{a}^{\langle 1\rangle T}(-l)\, \overline{\boldsymbol{C}}\boldsymbol{A}e^{i\boldsymbol{\Gamma}l}\boldsymbol{A}^{-1}\boldsymbol{a}^{\langle 2\rangle}(0) \\ = \boldsymbol{a}^{\langle 1\rangle T}(-l)\, (\boldsymbol{A}e^{i\boldsymbol{\Gamma}l}\boldsymbol{A}^{-1})^T\, \overline{\boldsymbol{C}}\boldsymbol{a}^{\langle 2\rangle}(0).\end{aligned} \tag{67}$$

Since the initial conditions for $\boldsymbol{a}^{\langle 1\rangle}(-l)$ and $\boldsymbol{a}^{\langle 2\rangle}$ can be

$$\boldsymbol{a}^{\langle 1\rangle}(-l) = \begin{array}{cl} \left[\begin{matrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right] & \begin{matrix} \\ \\ \rightarrow p\text{th position} \\ \\ \\ \\ \\ \end{matrix} \end{array} \tag{68}$$

$$\boldsymbol{a}^{\langle 2\rangle}(0) = \begin{array}{cl} \left[\begin{matrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{matrix}\right] & \begin{matrix} \\ \\ \\ \\ \\ \rightarrow q\text{th position} \\ \\ \end{matrix} \end{array} \tag{69}$$

where p and q can be arbitrarily set between 1 and N, we find the reciprocity condition:

$$\overline{\boldsymbol{C}}\boldsymbol{A}e^{i\boldsymbol{\Gamma}l}\boldsymbol{A}^{-1} = (\overline{\boldsymbol{C}}\boldsymbol{A}e^{i\boldsymbol{\Gamma}l}\boldsymbol{A}^{-1})^T \tag{70}$$

where we have used the fact that $\overline{\boldsymbol{C}}^T = \overline{\boldsymbol{C}}$. Since the matrix $\boldsymbol{A}$ has each column given by the eigenvector of (58), we have

$$\overline{\boldsymbol{C}}\boldsymbol{A}\boldsymbol{\Gamma} = \boldsymbol{Q}\boldsymbol{A} \tag{71}$$

or

$$\boldsymbol{\Gamma} = \boldsymbol{A}^{-1}\overline{\boldsymbol{C}}^{-1}\boldsymbol{Q}\boldsymbol{A} = \boldsymbol{A}^{-1}\boldsymbol{M}\boldsymbol{A}. \tag{72}$$

Substituting the above relation into (70), we find that the reciprocity condition (70) is the same as

$$\overline{\boldsymbol{C}}\boldsymbol{M} = (\overline{\boldsymbol{C}}\boldsymbol{M})^T \qquad \text{(reciprocity condition)} \tag{73}$$

i.e., the product $\overline{C}M$ must be symmetric. We see clearly that our formulation (48) satisfies this condition because

$$\overline{\boldsymbol{C}}\boldsymbol{M} = \boldsymbol{Q} \tag{74}$$

where $\boldsymbol{Q}$ has been proved to be symmetric in Section IV-A. To illustrate our results, we show in the next section the special cases for two coupled waveguides and three coupled waveguides.

V. Special Cases

In summary, the coupled-mode equation is put in matrix form

$$\overline{\boldsymbol{C}} \frac{d}{dz} \boldsymbol{a} = i\boldsymbol{Q}\boldsymbol{a} \tag{75}$$

where both $\overline{\boldsymbol{C}}$ and $\boldsymbol{Q}$ are symmetric. In another form, it is given by

$$\frac{d}{dz} \boldsymbol{a} = i\boldsymbol{M}\boldsymbol{a} \tag{76}$$

where

$$\boldsymbol{M} = \overline{\boldsymbol{C}}^{-1}\boldsymbol{Q}. \tag{77}$$

The reciprocity condition (74) requires $\overline{\boldsymbol{C}}\boldsymbol{M}$ (or $\boldsymbol{Q}$) to be symmetric. The above formulation is very general and is applicable to both lossy as well as lossless systems.

A. Two-Coupled Waveguides

If $N = 2$, one has

$$\boldsymbol{M} = \begin{bmatrix} \gamma_a & K_{ab} \\ K_{ba} & \gamma_b \end{bmatrix} \tag{78}$$

where

$$\gamma_a = \beta_1 + (K_{11} - \overline{C}_{12}K_{21})/(1 - \overline{C}_{12}^2) \tag{79a}$$

$$\gamma_b = \beta_2 + (K_{22} - \overline{C}_{12}K_{12})/(1 - \overline{C}_{12}^2) \tag{79b}$$

$$K_{ab} = (K_{12} - K_{22}\overline{C}_{12})/(1 - \overline{C}_{12}^2) \tag{79c}$$

$$K_{ba} = (K_{21} - K_{11}\overline{C}_{12})/(1 - \overline{C}_{12}^2). \tag{79d}$$

As has been pointed out in [8], the overlap integrals C_{12} and C_{21} or $\overline{C}_{12}$ are obtained from the integration over whole space in the transverse direction, and can be significantly large even if $\tilde{K}_{12}$ is small. Thus the factor $1 - \overline{C}_{12}^2$ may become very small and K_{ab} is large. The reciprocity condition that $\overline{C}M$ be symmetric gives

$$K_{ab} - K_{ba} = (\gamma_a - \gamma_b)\overline{C}_{12} \tag{80}$$

which has been shown in [12], and can also be proved by substituting (79a)–(79d) into (80). The two eigenvalues γ_1, γ_2, and eigenvectors are well known:

$$\gamma_1 = \phi + \psi \tag{81a}$$

$$\gamma_2 = \phi - \psi \tag{81b}$$

where

$$\phi = \frac{\gamma_b + \gamma_a}{2} \tag{81c}$$

$$\psi = \sqrt{\Delta^2 + K_{ab}K_{ba}} \tag{81d}$$

$$\Delta = \frac{\gamma_b - \gamma_a}{2} \tag{81e}$$

and

$$\boldsymbol{a}^{(1)} = \begin{bmatrix} K_{ab} \\ \Delta + \psi \end{bmatrix} \tag{82a}$$

$$\boldsymbol{a}^{(2)} = \begin{bmatrix} -\Delta - \psi \\ K_{ba} \end{bmatrix} \tag{82b}$$

where the orthogonality relation $\boldsymbol{a}^{(1)T}\overline{\boldsymbol{C}}\boldsymbol{a}^{(2)} = 0$ is indeed satisfied and it is the same as the reciprocity condition (80).

B. Three-Coupled Waveguides

If $N = 3$, we have

$$\boldsymbol{M} = \boldsymbol{B} + \overline{\boldsymbol{C}}^{-1}\boldsymbol{K} \tag{83}$$

which can be calculated easily by inversion of $\overline{C}$, noting that $\overline{C}$ is symmetric:

$$\overline{C}_{12} = \overline{C}_{21} = (C_{12} + C_{21})/2 \tag{84a}$$

$$\overline{C}_{13} = \overline{C}_{31} = (C_{13} + C_{31})/2 \tag{84b}$$

$$\overline{C}_{11} = \overline{C}_{22} = \overline{C}_{33} = 1. \tag{84c}$$

The reciprocity condition that $\overline{C}M$ is symmetric leads to, e.g., $(\overline{C}M)_{12} = (CM)_{21}$

$$m_{12} - m_{21} = \overline{C}_{12}(m_{12} - m_{22}) + \overline{C}_{21}m_{31} - \overline{C}_{13}m_{32} \tag{85}$$

which will be useful later. Let us consider a symmetric case with the two outer waveguides identical [10]:

$$K_{12} = K_{32} \neq K_{21} = K_{23} \tag{86a}$$

$$K_{13} = K_{31} \tag{86b}$$

$$K_{11} = K_{33} \neq K_{22}. \tag{86c}$$

The matrix elements of $\boldsymbol{M}$ are obtained from (83)

$$m_{11} = m_{33} = \beta_1 + \left[K_{11}(1 - \overline{C}_{12}^2) - K_{21}\overline{C}_{12}(1 - \overline{C}_{13}) + K_{13}(\overline{C}_{12}^2 - \overline{C}_{13})\right]/D \tag{87a}$$

$$m_{22} = \beta_2 + \left[K_{22}(1 + \overline{C}_{13}) - 2K_{12}\overline{C}_{12}\right](1 - \overline{C}_{13})/D \tag{87b}$$

$$m_{12} = m_{32} = (K_{12} - K_{22}\overline{C}_{12})(1 - \overline{C}_{13})/D \tag{87c}$$

$$m_{13} = m_{31} = \left[K_{13}(1 - \overline{C}_{12}^2) - K_{21}\overline{C}_{12}(1 - \overline{C}_{13}) + K_{11}(\overline{C}_{12}^2 - \overline{C}_{13})\right]/D \tag{87d}$$

$$m_{21} = m_{23} = \left[K_{21}(1 + \overline{C}_{13}) - (K_{11} + K_{13})\overline{C}_{12}\right](1 - \overline{C}_{13})/D \tag{87e}$$

$$D = (1 - \overline{C}_{13})(1 + \overline{C}_{13} - 2\overline{C}_{12}^2). \tag{87f}$$

The three eigenvalues and eigenvectors have been calculated in [10], [11], and are given here:

$$\gamma_1 = \frac{\phi + \psi}{2} \tag{88a}$$

$$\gamma_2 = m_{11} - m_{13} \tag{88b}$$

$$\gamma_3 = \frac{\phi - \psi}{2} \tag{88c}$$

where

$$\phi = m_{11} + m_{13} + m_{22} \tag{88d}$$

$$\psi = \sqrt{(m_{11} + m_{13} - m_{22})^2 + 8m_{12}m_{21}} \tag{88e}$$

and

$$a^{(1)} = \begin{bmatrix} 1 \\ (2m_{22} - \phi + \psi)/2m_{12} \\ 1 \end{bmatrix} \tag{89a}$$

$$a^{(2)} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{89b}$$

and

$$a^{(3)} = \begin{bmatrix} 1 \\ (2m_{22} - \phi - \psi)/2m_{12} \\ 1 \end{bmatrix}. \tag{89c}$$

It is straightforward to show that these three eigenvectors satisfy the general orthogonality relation (57) by direct substitutions. Notice that the formulation in [8]–[11] does not satisfy this condition since in general $C_{12} \neq C_{21}$.

Finally, let us consider an excitation with the boundary condition at $z = 0$ given by

$$a(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{90}$$

The general solution at z is [10], [11]

$$a(z) = Ae^{i\Gamma z}A^{-1}a(0). \tag{91}$$

Here, the matrix A is given by the three eigenvectors from (89a)–(89c). (Note that our definition of A is the inverse of that in [10] and [11] with some typos corrected.)

$$A = [a^{(1)}, a^{(2)}, a^{(3)}]. \tag{92}$$

The results of $Ae^{i\Gamma z}A^{-1}$ have been calculated in [10] and [11]. The solution at position z is

$$a_1(z) = a_3(z) = i\,\frac{2m_{12}}{\psi} \sin \frac{\psi z}{2}\, e^{i\phi z/2} \tag{93a}$$

$$a_2(z) = \left[\cos \frac{\psi z}{2} + i\,\frac{(2m_{22} - \phi)}{\psi} \sin \frac{\psi z}{2}\right] e^{i\phi z/2}. \tag{93b}$$

The total guided power is given by

$$P(z) = \mathrm{Re}\,[a^{+}(z)\, Ca(z)]$$
$$= a^{+}(z)\, \tilde{\tilde{C}}a(z) = 1 + F \sin^2 \frac{\psi z}{2} \tag{94}$$

where the factor F is given by

$$F = \frac{8m_{12}}{\psi^2}\left[m_{12}(1 + \tilde{\tilde{C}}_{13}) - m_{21} + \tilde{\tilde{C}}_{12}(m_{22} - m_{11} - m_{13})\right]. \tag{95}$$

For a lossless system, the power conservation requires that $P(z)$ be independent of the position z. Thus the factor F provides a check of the energy conservation. Although it has been shown before that our formulation satisfies the energy conservation exactly by using the fact that the two matrices $\tilde{\tilde{C}}$ and $\tilde{\tilde{Q}}$ are hermitian, one can also see from the reciprocity relation (85) substituted into (95) that F is indeed zero provided that we choose $E_t^{(p)}$ and $H_t^{(p)}$ to be real functions for a lossless system. Therefore, $C_{pq} = \bar{\bar{C}}_{pq}$ = real. Numerical results will be given in the next section. The factor F from two previous methods [4], [10] will also be calculated.

VI. A Numerical Example and Discussions

In this section, we illustrate our theory by a numerical example [10] and compare it with those of two previous methods [4], [10]. We consider three coupled waveguides with the two outside waveguides identical and symmetric with respect to the center waveguide (Fig. 3). Using the theoretical results discussed in Section V-B, we calculate K_{pq}, C_{pq}, and β_p. The analytical relation (47) is used to check the numerical accuracies of these quantities. We show in Fig. 4(a) the three eigenvalues γ_1, γ_2, and γ_3 from (88a)–(88c), which are the propagation constants of the three supermodes, versus the separation t between the waveguides. We compare our results (dotted line) and the exact solutions (solid lines) of the multilayered structure in Fig. 4(a) and those of the method in [10] (dashed lines), the method in [4] (crosses) in Fig. 4(b). We see clearly that the results using the method in [10] and our theory agree very well with the exact calculation. There is a slight error for the third eigenvalue γ_3 near cutoff where the separation t is reduced to near 0.2 μm. In our calculation, we choose the same parameters as in [10], $n = 3.4$, $n_1 = 3.6$, $n_2 = 3.63$, $d_1 = d_2 = 0.15$ μm, and t varies. The method of [4] clearly has larger errors in γ_1 and γ_3, especially γ_3 deviates from the exact results over a wide range of t near cutoff. The result of γ_2 using three methods agree with each other very well because

$$\gamma_2 = m_{11} - m_{13} = \beta_1 + (K_{11} - K_{13})/(1 - \bar{C}_{13}) \tag{96}$$

using (87) and (88). Since $\bar{C}_{13}$ and $(K_{11} - K_{13})$ are very small, ($\bar{C}_{13} = 0.136 \sim 0.00436$ and $(K_{11} - K_{13}) = -0.0237 \sim -0.0004$ (1/μm) at $t = 0.2$ μm ~ 0.6 μm),

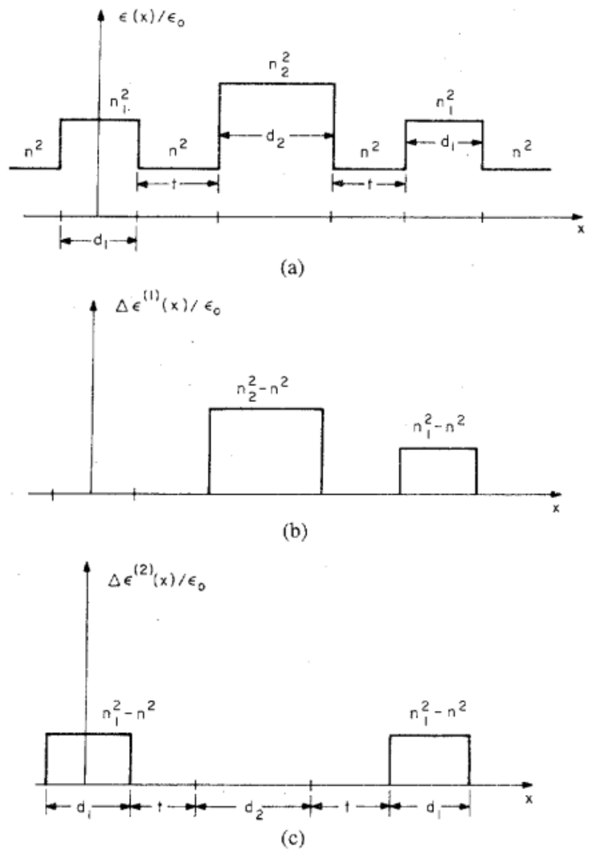

Fig. 3. Three coupled waveguides under investigation. (a) $\epsilon(x)/\epsilon_0$ for the three-waveguide system. (b) $\Delta\epsilon^{(1)}(x)/\epsilon_0$. (c) $\Delta\epsilon^{(2)}(x)/\epsilon_0$.

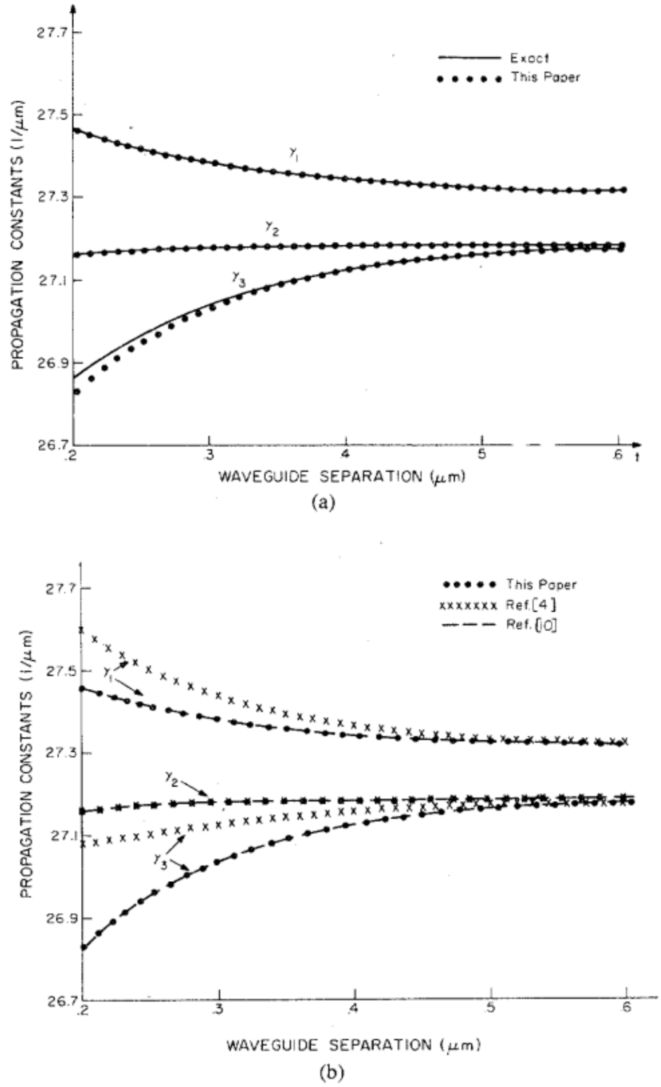

Fig. 4. (a) A comparison of the propagation constants of the three supermodes between the exact results (solid lines) and the results of this paper (dotted lines) for the three coupled waveguides in Fig. 3. (b) A comparison with other two methods: results using [10] (dashed lines), results using [4] (crosses). The results of this paper are given by dotted lines.

if we have $\overline{C}_{13} = 0$ (theory of [4]) the difference is negligible. Since $C_{13} = C_{31} = \overline{C}_{13}$, the theory of [10] gives the same results for γ_2 as the results of this paper.

In Fig. 5, we show the power conservation violation factor F for the initial excitation at the center waveguide, $\boldsymbol{a}(0) =$ column (0, 1, 0). We see clearly that our results indeed satisfy the power conservation very well and F is always zero. The factor F calculated from [10] is always very small (less than 0.08 percent), but F calculated from [4] can be as much as 42 percent.

Numerically speaking, our results are as good as or slightly better than those obtained from [10]. The new features are that our formulation is derived using a simpler approach, and it satisfies both the reciprocity theorem and the power conservation law analytically, while [8]–[11] can only show numerically that their method satisfies the power conservation and the reciprocity theorem approximately. (One should note that power conservation and reciprocity are only satisfied self-consistently and not exactly since the modal expansions (11) of the fields are approximate.) Our formulation also leads to the general orthogonality relations (38) and (57) with the overlap integrals properly taken into account, that cannot be obtained from the formulation in [8]–[11]. By setting the matrix $\overline{C}$ or $\tilde{\overline{C}}$ to be the identity matrix, the coupled-mode equations and the orthogonality relations all reduce to the results of a conventional analysis [4]. Our numerical results also show that ignoring the overlap integrals does

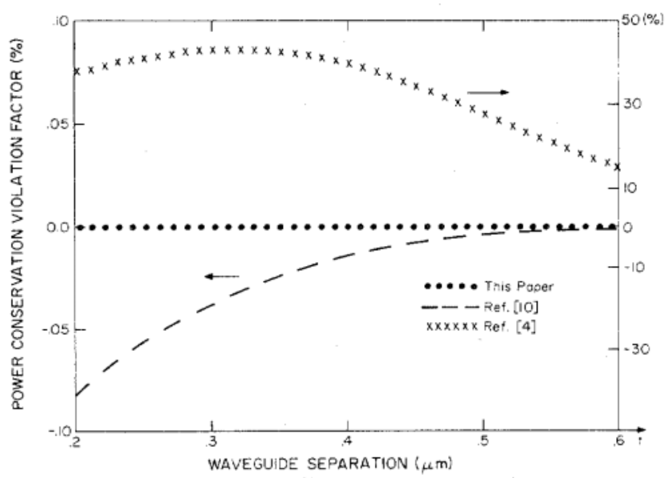

Fig. 5. The power conservation violation factor F for an excitation at the center waveguide of Fig. 3. The results of F using [10] (dashed line), and results of this paper (dotted line) are shown using the left scale, and the results of [4] (crosses) follow the right scale.

lead to erroneous results violating the power conservation significantly as has also been pointed out in [8] and [12].

VII. Conclusions

Two sets of coupled-mode equations for a multiwaveguide system have been derived using a generalized reciprocity relation, one set for a lossless, and the other for a lossy or lossless system. New general orthogonality relations between the eigenvectors of the supermodes have been derived. We have derived the conditions on the matrix elements for the reciprocity theorem and the power conservation laws and have shown that our formulations do indeed satisfy those conditions analytically.

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Shun-Lien Chuang (S'78–M'82) was born in Taiwan in 1954. He received the B.S. degree in electrical engineering from National Taiwan University in 1976, and the M.S., E.E., and Ph.D. degrees in electrical engineering from the Massachusetts Institute of Technology in 1980, 1981, and 1983, respectively.

While in graduate school, he held Research and Teaching Assistantships, and also served as a Recitation Instructor. He conducted research at Schlumberger-Doll Research in Ridgefield, CT, during the summers of 1981 and 1982 and also in 1983 as a Member of the Professional Staff. He is now an Assistant Professor in the Department of Electrical and Computer Engineering at the University of Illinios at Urbana-Champaign. He is conducting research in electromagnetics, integrated optics, and semiconductor devices including superlattice photodetectors and microwave transistors. He teaches courses on Electromagnetics and Solid-State Electronic Devices. He has been cited several times in the list for Excellence in Teaching. He recently developed a new graduate course on Integrated Optics and Optoelectronics.

Dr. Chuang is a member of the Optical Society of America, Sigma Xi, and the American Physical Society.