

25. Vector Potential - Introduction to Antennas & Radiations

Maxwell's equations are

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}, \quad (1)$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \mathbf{J}, \quad (2)$$

$$\nabla \cdot \mu\mathbf{H} = 0, \quad (3)$$

$$\nabla \cdot \epsilon\mathbf{E} = \rho. \quad (4)$$

Since $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, we can let

$$\mu\mathbf{H} = \nabla \times \mathbf{A}, \quad (5)$$

so that equation (3) is automatically satisfied. Substituting (5) into (1), we have

$$\nabla \times (\mathbf{E} + j\omega\mathbf{A}) = 0. \quad (6)$$

Since $\nabla \times \nabla\phi = 0$, we have

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\phi. \quad (7)$$

Hence, knowing \mathbf{A} and ϕ uniquely determines \mathbf{E} and \mathbf{H} . We shall relate \mathbf{A} and ϕ to the **sources** \mathbf{J} and ρ of Maxwell's equations. Substituting (5) and (7) into (2), we have

$$\nabla \times \nabla \times \mathbf{A} = j\omega\mu\epsilon[-j\omega\mathbf{A} - \nabla\phi] + \mu\mathbf{J}, \quad (8)$$

or

$$\nabla^2 \mathbf{A} + \omega^2 \mu\epsilon \mathbf{A} = -\mu\mathbf{J} + j\omega\mu\epsilon \nabla\phi + \nabla \nabla \cdot \mathbf{A}. \quad (9)$$

Using (7) in (4), we have

$$\nabla \cdot (j\omega\mathbf{A} + \nabla\phi) = -\frac{\rho}{\epsilon}. \quad (10)$$

The above could be simplified for the following observation. Equations (5) and (7) give the same \mathbf{E} and \mathbf{H} fields under the transformation

$$\mathbf{A}' = \mathbf{A} + \nabla\psi, \quad (11)$$

$$\phi' = \phi - j\omega\psi. \quad (12)$$

The above are known as the **Gauge Transformation**. With the new \mathbf{A}' and ϕ' , we can substitute into (5) and (7) and they give the same \mathbf{E} and \mathbf{H} fields, i.e.

$$\nabla \times \mathbf{A}' = \nabla \times \mathbf{A} + \nabla \times \nabla\psi = \nabla \times \mathbf{A} = \mu\mathbf{H}, \quad (13)$$

$$-j\omega\mathbf{A}' - \nabla\phi' = -j\omega\mathbf{A} - j\omega\nabla\psi - \nabla\phi + j\omega\nabla\psi = \mathbf{E}. \quad (14)$$

It implies that \mathbf{A} and ϕ are not unique. The vector field \mathbf{A} is not unique unless we specify both its curl and its divergence. Hence, in order to make \mathbf{A} unique, we have to specify its divergence. If we specify the divergence of \mathbf{A} such that

$$\nabla \cdot \mathbf{A} = -j\omega\mu\epsilon\phi, \quad (15)$$

then (9) and (10) become

$$\nabla^2 \mathbf{A} + \omega^2 \mu \epsilon = -\mu \mathbf{J}, \quad (16)$$

$$\nabla^2 \phi + \omega^2 \mu \epsilon \phi = -\frac{\rho}{\epsilon}. \quad (17)$$

The condition in (15) is also known as the **Lorentz gauge**. Equations (16) and (17) represent a set of four inhomogeneous wave equations driven by the sources of Maxwell's equations. Hence given the sources ρ and \mathbf{J} , we may find \mathbf{A} and ϕ . \mathbf{E} and \mathbf{H} may in turn be found using (5) and (7). However, as a consequence of the Lorentz gauge, we need only to find \mathbf{A} ; ϕ follows directly from equation (15).

Let us consider the relation due to an elemental current that can be described by

$$\mathbf{J} = \hat{z} Il \delta(\mathbf{r}) \quad A/m^2, \quad (18)$$

where Il denotes the strength of this current, and $\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$. Equation (16) becomes

$$\nabla^2 A_z + \omega^2 \mu \epsilon A_z = -\mu Il \delta(\mathbf{r}). \quad (19)$$

Taking advantage of the spherical symmetry of the problem, ∇^2 has only r dependence in spherical coordinates, we have

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} A_z + \beta^2 A_z = -\mu Il \delta(\mathbf{r}), \quad (20)$$

where $\beta^2 = \omega^2 \mu \epsilon$. Equations (19) and (20) are similar in form to Poisson's equation with a point charge Q at the origin,

$$\nabla^2 \phi = -\frac{Q}{\epsilon} \delta(\mathbf{r}). \quad (21)$$

We know that (21) has the solution of the form

$$\phi = \frac{Q}{4\pi\epsilon r}. \quad (22)$$

Hence, we guess that the solution to (20) is of the form

$$A_z = \frac{\mu Il}{4\pi r} C(r). \quad (23)$$

It can be shown that

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} f(r) = \frac{1}{r} \frac{d^2}{dr^2} r f(r). \quad (24)$$

Outside the origin, the RHS of (20) is zero, and after using (23) and (24) in (20), we have

$$\frac{d^2}{dr^2}C(r) + \beta^2 C(r) = 0. \quad (25)$$

This gives

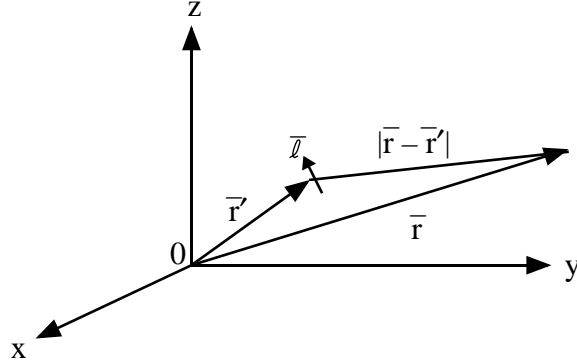
$$C(r) = e^{\pm j\beta r}. \quad (26)$$

Since we are looking for a solution that radiates energy to infinity, we choose an outgoing solution in (26). Hence,

$$A_z(r) = \frac{\mu I l}{4\pi r} e^{-j\beta r}, \quad (27)$$

for a source directed at a \hat{z} -direction. From (16), we note that \mathbf{A} and \mathbf{J} always point in the same direction. Therefore, for a point source directed at \mathbf{l} and located at \mathbf{r}' instead of the origin, the vector potential \mathbf{A} is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu I \mathbf{l}}{4\pi |\mathbf{r} - \mathbf{r}'|} e^{-j\beta |\mathbf{r} - \mathbf{r}'|}. \quad (28)$$



By linear superposition, the vector potential due to an arbitrary source \mathbf{J} is

$$\mathbf{A} = \frac{\mu}{4\pi} \iiint d\mathbf{r}' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} e^{-j\beta |\mathbf{r} - \mathbf{r}'|}. \quad (29)$$

Similarly, we can show that

$$\phi = \frac{1}{4\pi\epsilon} \iiint d\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} e^{-j\beta |\mathbf{r} - \mathbf{r}'|}. \quad (30)$$