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ECE 350X Lecture Notes

Supplementary Notes.

The Q of a Cavity.

The Q of a cavity is a **quality factor** that measures how ideal a cavity resonator is. An ideal cavity resonator will sustain free oscillations forever, while most resonators sustain free oscillations for a finite time due to losses and dissipations. Because of losses, the free oscillation in a cavity will give rise to electromagnetic field with time dependence as follows:

$$\mathbf{E}\alpha e^{-\alpha t}\cos(\omega t+\phi_1), \quad \mathbf{H}\alpha E^{-\alpha t}\cos(\omega t+\phi_2). \tag{1}$$

The total stored energy, which is proportional to $\frac{1}{4}\epsilon\left|\mathbf{E}\right|^2+\frac{1}{4}\mu\left|\mathbf{H}\right|^2$ is of the form

$$< W_T >=< W_E > + < W_H >= W_0 e^{-2\alpha t}. \tag{2}$$

If there is no loss, $< W_T >$ will remain constant. However, with loss, the average stored energy will decrease to $\frac{1}{e}$ of its original value at $t=\frac{1}{2}\alpha$. The Q of a cavity is a measure of the number of free oscillations the field would have before the energy stored decreases to $\frac{1}{e}$ of its original value. In a time $t=\frac{1}{2}\alpha$, the number of free oscillations in radians is $\frac{\omega}{2\alpha}$, hence, the Q is defined to be

$$Q=\frac{\omega}{2\alpha}. \tag{3}$$

Furthermore, by energy conservation, the decrease in stored energy per unit time must be equal to the total power dissipated in the losses of a cavity, in other words,

$$< P_d >= -\frac{d < W_T >}{dt} = 2\alpha W_0 e^{-2\alpha t} = 2\alpha W_t. \tag{4}$$

Hence, we can write equation (3) as

$$Q=\frac{\omega < W_T >}{< P_d >} = 2\pi\frac{< W_T >}{< P_d > T} = 2\pi\frac{\text{total energy stored}}{\text{Energy dissipated/cycle}}. \tag{5}$$

In a cavity, the energy can dissipate in either the dielectric loss or the wall of the cavity due to the finiteness of the conductivity. If the cavity is filled with air, then, the loss comes mainly from the copper-loss from the cavity wall. In this case, the power dissipated on the wall is given by

$$< P_d >= \frac{1}{2}\Re e\oint_S (\mathbf{E}\times\mathbf{H}^*)\cdot\hat{n}\,ds = \frac{1}{2}\Re e\oint_S(\hat{n}\times\mathbf{E})\cdot\mathbf{H}^*\,dS, \tag{6}$$

where S is the surface of the cavity wall. $(\hat{n} \times \mathbf{E})$ is the tangential component of the electric field which would have been zero if the cavity is ideal. However, for metallic walls, $\hat{n} \times \mathbf{E} = \mathbf{H}_t Z_m$ where Z_m is the intrinsic impedance for the metallic conductor, $Z_m = \sqrt{\frac{\mu}{\epsilon_m}} = \sqrt{\frac{\mu}{-j\frac{\sigma}{\omega}}} = \sqrt{\frac{\omega\mu}{2\sigma}}(1+j)$, and H_t is the tangential magnetic field. Hence,

$$< P_d >= \frac{1}{2}\Re e \oint_S \sqrt{\frac{\omega\mu}{2\sigma}}(1+j)\left|\mathbf{H}_t\right|^2 dS = \frac{1}{2}\sqrt{\frac{\omega\mu}{2\sigma}} \oint_S \left|\mathbf{H}_t\right|^2 dS. \tag{7}$$

Written explicitly, the Q becomes

$$Q = \sqrt{4\pi f \mu\sigma}\frac{\oint_V |\mathbf{H}|^2 \, dV}{\oint_S |\mathbf{H}_t|^2 \, dS}. \tag{8}$$

Hence, the more energy stored we can have with respect to the power dissipated, the higher the Q of a resonating system.

Example: The Q of TM_{110} mode

For the TM_{110} mode,

$$E_Z = E_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right), \tag{9}$$

and

$$H_x = \frac{+j\omega\epsilon}{\omega^2\mu\epsilon}\frac{\partial}{\partial y}E_Z = \frac{+j\left(\frac{\pi}{b}\right)}{\omega\mu}E_0 \sin\left(\frac{\pi x}{a}\cos\right)\left(\frac{\pi y}{b}\right), \tag{10}$$

$$H_y = \frac{-j\omega\epsilon}{\omega^2\mu\epsilon}\frac{\partial}{\partial x}E_Z = -\frac{j\left(\frac{\pi}{a}\right)}{\omega\mu}E_0 \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right). \tag{11}$$

Therefore

$$\oint_V |\mathbf{H}|^2 \, dV = \int_{-d}^{0}\int_0^b\int_0^a dx\, dy\, dv \left[|H_x|^2 + |H_y|^2\right]$$

$$= \frac{|E_0|^2}{\omega^2\mu^2}\int_{-d}^{0}\int_0^b\int_0^a dx\, dy\, dz \times,$$

$$\left[\left(\frac{\pi}{b}\right)^2 \sin^2\left(\frac{\pi x}{a}\right)\cos^2\left(\frac{\pi y}{b}\right) + \left(\frac{\pi}{a}\right)^2 \cos^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{\pi y}{b}\right)\right]$$

$$= \frac{|E_0|^2}{\omega^2\mu^2}\frac{\pi^2}{4}\left[\frac{ad}{b} + \frac{bd}{a}\right]. \tag{12}$$

A cavity has six faces, finding the tangential exponent at each face and integrate

$$\oint_S |\mathbf{H}_t| \, dS$$

$$= 2\int_0^b \int_0^a dx\, dy \left[|H_x|^2 + |H_y|^2\right]$$

$$+2\int_{-d}^0 \int_0^a dx\, dz\, |H_x(y=0)|^2 + 2\int_{-d}^0 \int_0^b dy\, dz\, |H_y(x=0)|^2$$

$$= \frac{2|E_0|^2}{\omega^2\mu^2}\frac{\pi^2 ab}{4}\left[\frac{1}{a^2}+\frac{1}{b^2}\right] + \frac{2\left(\frac{\pi}{b}\right)^2}{\omega^2\mu^2}|E_0|^2\frac{ad}{2} + \frac{2\left(\frac{\pi}{a}\right)^2}{\omega^2\mu^2}|E_0|^2\frac{bd}{2}$$

$$= \frac{\pi^2|E_0|^2}{\omega^2\mu^2}\left[\frac{b}{2a}+\frac{a}{2b}+\frac{ad}{b^2}+\frac{bd}{a^2}\right]. \tag{13}$$

Hence the Q is

$$Q = \sqrt{4\pi f\mu\sigma}\frac{1}{4}\frac{\left(\frac{ad}{b}+\frac{bd}{a}\right)}{\left(\frac{b}{2a}+\frac{a}{2b}+\frac{ad}{b^2}+\frac{bd}{a^2}\right)}. \tag{14}$$