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ECE 350 Lecture Notes

2. Review of Vector Analysis

A vector $\mathbf{A}$ can be written as

$$\mathbf{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z. \tag{1}$$

Similarly, a vector $\mathbf{B}$ can be written as

$$\mathbf{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z. \tag{2}$$

In the above, $\hat{x}, \hat{y}, \hat{z}$ are unit vectors pointing in the x, y, z directions respectively. A_x, A_y and A_z are the components of the vector $\mathbf{A}$ in the x, y, z directions respectively. The same statement applies to B_x, B_y, and B_z.

Addition

$$\mathbf{A} + \mathbf{B} = \hat{x}(A_x + B_x) + \hat{y}(A_y + B_y) + \hat{z}(A_z + B_z). \tag{3}$$

Multiplication

(a) Dot Product (scalar product)

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z, \tag{4}$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}, \qquad \text{commutative property} \tag{5}$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}, \qquad \text{distributive property} \tag{6}$$

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos\theta. \tag{7}$$

In (7), θ is the angle between vectors $\mathbf{A}$ and $\mathbf{B}$.

(b) Cross Product (vector product)

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}(A_yB_z - A_zB_y) + \hat{y}(A_zB_x - A_xB_z) + \hat{z}(A_xB_y - A_yB_x), \tag{8}$$

$$\mathbf{A} \times \mathbf{B} = \hat{u}\, |\mathbf{A}|\, |\mathbf{B}| \sin\theta, \tag{9}$$

where $\hat{u}$ is a unit vector obtained from $\mathbf{A}$ and $\mathbf{B}$ via the right hand rule.

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}, \qquad \text{distributive property} \tag{10}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}, \quad \text{non-associative property} \tag{11}$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}, \qquad \text{anti-commutative property} \tag{12}$$

Vector Derivatives

$$\text{Del} \quad \nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}, \tag{13}$$

$$\text{Gradient} \quad \nabla\phi = \hat{x}\frac{\partial}{\partial x}\phi + \hat{y}\frac{\partial}{\partial y}\phi + \hat{z}\frac{\partial}{\partial z}\phi, \tag{14}$$

$$\text{Divergent} \quad \nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z, \tag{15}$$

$$\begin{aligned} \text{Curl} \quad \nabla \times \mathbf{A} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\ &= \hat{x}\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right) + \hat{y}\left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right) \\ &\quad + \hat{z}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_z\right). \end{aligned} \tag{16}$$

Divergence Theorem

$$\oint_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot \hat{n} dS. \tag{17}$$

Stokes Theorem

$$\oint_S (\nabla \times \mathbf{A}) \cdot \hat{n} dS = \oint_C \mathbf{A} \cdot d\mathbf{l}. \tag{18}$$

Some Useful Vector Identities

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}), \tag{19}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b}), \tag{20}$$

$$\mathbf{a} \times \mathbf{a} = 0, \tag{21}$$

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0, \tag{22}$$

$$\nabla \times (\nabla \phi) = 0, \tag{23}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0, \tag{24}$$

$$\nabla \cdot (\psi \mathbf{A}) = \mathbf{A} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{A}, \tag{25}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B}, \tag{26}$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \cdot \nabla \mathbf{A}, \tag{27}$$

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{28}$$