

16. Real Poynting Theorem.

Since $\mathbf{E} \times \mathbf{H}$ has the dimension of watts/ m^2 , we can study its divergence property and its conservative property. Using the vector identity in (1.26), we have,

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}. \quad (1)$$

From Maxwell's equations, we can replace $\nabla \times \mathbf{E}$ by $-\frac{\partial \mathbf{B}}{\partial t}$ and $\nabla \times \mathbf{H}$ by $\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$. Hence,

$$\begin{aligned} \nabla \cdot (\mathbf{E} \times \mathbf{H}) &= -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \cdot \mathbf{J} \\ &= -\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} - \epsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} - \mathbf{E} \cdot \mathbf{J}. \end{aligned} \quad (2)$$

We can show that

$$\frac{1}{2} \frac{\partial |\mathbf{H}|^2}{\partial t} = \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t}. \quad (3)$$

Hence,

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t} \left(\frac{1}{2} \mu |\mathbf{H}|^2 + \frac{1}{2} \epsilon |\mathbf{E}|^2 \right) - \mathbf{E} \cdot \mathbf{J}. \quad (4)$$

We can define

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \text{ Poynting vector (Power Flow Density } \textit{watt m}^{-2}\text{)}, \quad (5)$$

$$U_H = \frac{1}{2} \mu |\mathbf{H}|^2 \text{ Magnetic Energy Density (} \textit{joule m}^{-3}\text{)}, \quad (6)$$

$$U_E = \frac{1}{2} \epsilon |\mathbf{E}|^2 \text{ Electric Energy Density (} \textit{joule m}^{-3}\text{)}, \quad (7)$$

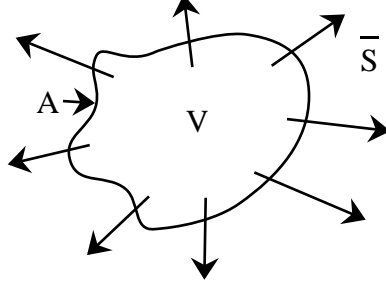
$$\mathbf{E} \cdot \mathbf{J} = \text{Energy Dissipation Density (} \textit{watt m}^{-3}\text{)}. \quad (8)$$

U_H and U_E represent the energy stored in the magnetic field and electric field respectively. Alternatively, (4) becomes

$$\nabla \cdot \mathbf{S} = -\frac{\partial}{\partial t} (U_H + U_E) - \mathbf{E} \cdot \mathbf{J}. \quad (9)$$

Using the divergence theorem, (9) can be written in integral form,

$$\oint_A \mathbf{S} \cdot \hat{n} dA = -\frac{\partial}{\partial t} \int_V (U_H + U_E) dV - \int_V \mathbf{E} \cdot \mathbf{J} dV. \quad (10)$$



The equation says that the LHS will be positive only if there is a net outflow of the flux due to the vector field \mathbf{S} . If there is no current inside V so that $\mathbf{E} \cdot \mathbf{J} = 0$, then this is only possible if the stored energy $U_H + U_E$ inside V decreases with time.

If $\mathbf{J} = \sigma \mathbf{E}$, then the last term is $-\int \sigma |\mathbf{E}|^2 dV$ is always negative. Hence, the last term tends to make $\oint_S \mathbf{S} \cdot \hat{n} dA$ negative, because energy dissipation has to be compensated by power flux flowing into V . The Poynting theorems (9) and (10) are statements of energy conservation. For example, for a plane wave,

$$\mathbf{E} = \hat{x} f(z - vt), \quad \mathbf{H} = \hat{y} \sqrt{\frac{\epsilon}{\mu}} f(z - vt), \quad (11)$$

then

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \hat{z} \sqrt{\frac{\epsilon}{\mu}} f^2(z - vt). \quad (12)$$

Also,

$$U_E + U_H = \frac{1}{2} \epsilon f^2(z - vt) + \frac{1}{2} \epsilon f^2(z - vt) = \epsilon f^2(z - vt), \quad (13)$$

Therefore,

$$\mathbf{S} = \hat{z} \frac{1}{\sqrt{\mu\epsilon}} \epsilon f^2(z - vt) = \hat{z} v (U_E + U_H). \quad (14)$$

Hence, the velocity times the total energy density stored equals the power density flow in a plane wave.