CHAPTER EIGHTEEN

Optics of Anisotropic Media

18

Optics of Anisotropic Media

18.1 Introduction

In this chapter we discuss wave propagation in anisotropic media. We shall see that in such media the electric vector of a propagating wave is not in general parallel to its polarization direction – defined by the direction of its electric displacement vector. Further, for propagation of plane waves in a particular direction through an anisotropic medium two distinct possible polarization directions exist, and waves having these polarization directions propagate with different velocities. We shall discuss an ellipsoidal surface called the indicatrix and show how with its aid the allowed polarization directions and their corresponding refractive indices can be determined for wave propagation in a given direction. Other three-dimensional surfaces related to the indicatrix and their use in describing different optical properties of anisotropic media are also discussed. We shall concentrate our attention primarily on uniaxial crystals, which have optical properties that can be referred to an indicatrix with two equal axes, and will discuss how such crystals can be used to control the polarization characteristics of light.

Important anisotropic optical media are generally crystalline and their optical properties are closely related to various symmetry properties possessed by crystals. To assist the reader who is not familiar with basic ideas of crystal symmetry, Appendix 5 summarizes a number of aspects of this subject that should be helpful in reading this chapter and a number of those succeeding it.

18.2 The Dielectric Tensor

An isotropic media the propagation characteristics of electromagnetic waves are independent of their propagation direction. This generally implies that there is no direction within such a medium which is any different from any other. Clearly then, we can class gases and liquids, but not liquid crystals, as isotropic media, provided there are no externally applied fields present. Such a field would, of course, imply the existence of a unique direction in the medium - that of the field. As an example of a situation where an isotropic medium becomes anisotropic in an external field, we mention the case of a gas in a magnetic field, where the gas changes the polarization characteristics of a wave which propagates in the field direction. This phenomenon is called the Faraday effect. In most circumstances, cubic crystals of the highest symmetry, crystal classes m3m and 432, also behave as isotropic media.

In an isotropic medium the electric displacement vector **D** and its associated electric field **E** are parallel, in other words we write

$$\mathbf{D} = \epsilon_r \epsilon_o \mathbf{E} \tag{18.1}$$

where ϵ_r is the scalar dielectric constant, which in the general case is a function of frequency. This is equivalent to saying that the polarization induced by the field and the field itself are parallel

$$\mathbf{P} = \epsilon_o \chi \mathbf{E} \tag{18.2}$$

where χ is the scalar susceptibility. We will restrict ourselves in what follows to materials that are neither absorbing nor amplifying, so that both ϵ_r and χ are real.

In an anisotropic medium **D** and **E** are no longer necessarily parallel and we write

$$\mathbf{D} = \overline{\overline{\epsilon}}_r \epsilon_o \mathbf{E} \tag{18.3}$$

where $\overline{\overline{\epsilon}}_r$ is the dielectric tensor, which in matrix form referred to three arbitrary orthogonal axes is

$$\overline{\overline{\epsilon}}_r \equiv \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix} \tag{18.4}$$

So for example:

$$D_x = \epsilon_o(\epsilon_{xx}E_x + \epsilon_{xy}E_y + \epsilon_{xz}E_z)$$
$$D_y = \epsilon_o(\epsilon_{yx}E_x + \epsilon_{yy}E_y + \epsilon_{yz}E_z)$$
$$D_z = \epsilon_o(\epsilon_{zx}E_x + \epsilon_{zy}E_y + \epsilon_{zz}E_z)$$

By making the appropriate choice of axes the dielectric tensor can be diagonalized. With this choice of axes, called the principal axes of the material, Eq. (18.5) in matrix form becomes

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \epsilon_o \begin{pmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{18.6}$$

where ϵ_x, ϵ_y and ϵ_z are called the principal dielectric constants.

Alternatively, we can describe the anisotropic character of the medium with the aid of the susceptibility tensor $\overline{\overline{\chi}}$

$$\mathbf{P} = \epsilon_o \, \overline{\overline{\chi}} \, \mathbf{E} \tag{18.7}$$

where $\overline{\overline{\chi}}$ has the matrix form when referred to three arbitrary orthogonal axes

$$\overline{\overline{\chi}} \equiv \begin{pmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{yx} & \chi_{yy} & \chi_{yz} \\ \chi_{zx} & \chi_{zy} & \chi_{zz} \end{pmatrix} \tag{18.8}$$

and in the principal coordinate system

$$\overline{\overline{\chi}} \equiv \begin{pmatrix} \chi_x & 0 & 0 \\ 0 & \chi_y & 0 \\ 0 & 0 & \chi_z \end{pmatrix} \tag{18.9}$$

Since $\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P}$, it is clear that in a principal coordinate system

$$\epsilon_x = 1 + \chi_x, \text{ etc.} \tag{18.10}$$

We can understand why crystals with less than cubic symmetry have susceptibilities which depend on the direction of the applied field by considering the physical relationship between polarization and applied field. When a field $\mathbf{E}$ is applied to a crystal, it displaces both electrons and nuclei from their equilibrium positions in the lattice and induces a net dipole movement per unit volume (polarization), which we can write in the form:

$$\mathbf{P} = \sum_j N_j e_j \Delta\mathbf{r}_j \tag{18.11}$$

where N_j is the density of species j with charge e_j in the crystal and $\Delta\mathbf{r}_j$ is the displacement of this charged species from its equilibrium position. We should note that when the applied field is at optical frequencies, only electrons make any significant contribution to this polarization, particularly the most loosely bound outer valence electrons of the ions within the lattice. The ions themselves are too heavy to follow the rapidly oscillating applied field. If the applied electric field has components E_x, E_y and E_z, then in equilibrium we can write

$$E_x = -k_{jx}(\mathbf{\Delta r}_j)_x \tag{18.12}$$

and two other similar equations, where k_{jx} is a restoring force constant appropriate to the x component of the displacement of the charge j from its equilibrium position; thus

$$\mathbf{\Delta r}_j = (\frac{E_x}{k_{jx}}\hat{\imath} + \frac{E_y}{k_{jy}}\hat{\jmath} + \frac{E_z}{k_j z}\hat{\mathbf{k}}) \tag{18.13}$$

which is a vector parallel to $\mathbf{E}$ only if all three force constants are equal. The equality of these force constants for displacement along three orthogonal axes would imply an arrangement with cubic symmetry of the ions in the lattice about the charged particle being considered.

18.3 Stored Electromagnetic Energy in Anisotropic Media

If we wish the stored energy density in an electromagnetic field to be the same in an anisotropic medium as it is in an isotropic one then we require

$$U = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}) \tag{18.14}$$

which gives

$$U = \frac{1}{2}(\overline{\overline{\epsilon}}_r\ \epsilon_0\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H}) \tag{18.15}$$

Now, in any medium the net power flow into unit volume is *

$$\begin{aligned} \frac{\partial U}{\partial t} &= -\nabla\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\frac{\partial\mathbf{B}}{\partial t} \\ &= \epsilon_o\mathbf{E}\cdot\ \overline{\overline{\epsilon}}_r\ \frac{\partial\mathbf{E}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} \end{aligned} \tag{18.16}$$

where we have assumed that the medium is non-conductive so that $\mathbf{j} = 0$. Eq. (18.16) gives the rate of change of stored energy within unit volume, which must also be given by the time derivative of Eq. (18.15)

$$\frac{\partial U}{\partial t} = \frac{1}{2}(\epsilon_o\ \overline{\overline{\epsilon}}_r\ \frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E}) + \frac{\partial}{\partial t}(\mathbf{B}\cdot\mathbf{H})) \tag{18.17}$$

Most optical crystals are not, or are only very slightly, magnetic so we can assume that $\mathbf{B} = \mu_o\mathbf{H}$ since $\mu_r \simeq 1$. In this case

$$\frac{1}{2}\frac{\partial}{\partial t}\mathbf{B}\cdot\mathbf{H} = \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} \tag{18.18}$$

so comparing the first term on the R.H.S. of Eq. (18.16) with the first two terms on the R.H.S. of (18.17)

$$\epsilon_o\mathbf{E}\cdot\ \overline{\overline{\epsilon}}_r\ \frac{\partial\mathbf{E}}{\partial t} = \frac{1}{2}(\epsilon_o\ \overline{\overline{\epsilon}}_r\ \mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \epsilon_o\ \overline{\overline{\epsilon}}_r\ \frac{\partial\mathbf{E}}{\partial t}\cdot\mathbf{E}) \tag{18.19}$$

* See Appendix xxx.

which implies that

$$\overline{\overline{\epsilon}}_r\ \mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} = \overline{\overline{\epsilon}}_r\ \frac{\partial\mathbf{E}}{\partial t}\cdot\mathbf{E}. \tag{18.20}$$

Written out in full Eq. (18.20) is

$$\begin{aligned}
&\epsilon_{xx}E_x\dot{E}_x + \epsilon_{xy}E_y\dot{E}_x + \epsilon_{xz}E_z\dot{E}_x + \epsilon_{yx}E_x\dot{E}_y + \epsilon_{yy}E_y\dot{E}_y \\
&\quad +\epsilon_{yz}E_z\dot{E}_y + \epsilon_{zx}E_x\dot{E}_z + \epsilon_{zy}E_y\dot{E}_z + \epsilon_{zz}E_z\dot{E}_z \\
&\quad = \epsilon_{xx}\dot{E}_xE_x + E_{xy}\dot{E}_yE_x + \epsilon_{xz}\dot{E}_zE_x + \epsilon_{yx}\dot{E}_xE_y \\
&+\epsilon_{yy}\dot{E}_yE_y + \epsilon_{yz}\dot{E}_zE_y + \epsilon_{zx}\dot{E}_xE_z + \epsilon_{zy}\dot{E}_yE_z + \epsilon_{zz}\dot{E}_zE_z
\end{aligned} \tag{18.20a}$$

Clearly $\epsilon_{xy} = \epsilon_{yx}; \epsilon_{xz} = \epsilon_{zx}$; etc. so the dielectric tensor only has 6 independent terms. By working in the principal coordinate system, all the off-diagonal terms of the dielectric tensor become zero, which greatly simplifies consideration of the wave propagation characteristis of anisotropic crystals. In this case we can write

$$D_x = \epsilon_o\epsilon_x E_x;\quad D_y = \epsilon_o\epsilon_y E_y;\quad D_z = \epsilon_o\epsilon_z E_z \tag{18.21}$$

where we are writing $\epsilon_x = \epsilon_{xx}$, etc., for simplicity and the electrical energy density in the crystal becomes

$$U_E = \frac{1}{2}\mathbf{E}\cdot\mathbf{D} = \frac{1}{2\epsilon_0}\left(\frac{D_x^2}{\epsilon_x} + \frac{D_y^2}{\epsilon_y} + \frac{D_z^2}{\epsilon_z}\right) \tag{18.22}$$

which shows that the electric displacement vectors from a given point that correspond to a constant stored electrical energy describe an ellipsoid.

18.4 Propagation of Monochromatic Plane Waves in Anisotropic Media

Let us assume that a monochromatic plane wave of the form

$$\mathbf{D} = \mathbf{D_o}\exp[i(\omega t - \mathbf{k}\cdot\mathbf{r})] = D_o e^{i\phi}$$

can propagate through an anisotropic medium. The direction of **D** specifies the direction of polarization of this wave. The wave vector **k** of this plane wave is normal to the wavefront (the plane where the phase of the wave is everywhere equal) and has magnitude $|\,\mathbf{k}\,| = \omega/c; c$ is the phase velocity of the wave, which is related to the velocity of light *in vacuo*, c_o, by the refractive index n experienced by the wave according to $c = c_o/n$. The phase velocity of the wave in a particular wave vector direction in the crystal can also be written as $c = \omega/|\mathbf{k}|$.

We assume that Maxwell's equations still hold, so that in the absence of currents or free charges

$$\text{div } \mathbf{D} = 0 \tag{18.23}$$

$$\text{div } \mathbf{B} = 0 \tag{18.24}$$

$$\text{curl } \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t} \tag{18.25}$$

$$\text{curl } \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \tag{18.26}$$

We stress that because **D** and **E** are now related by a tensor operation, div **E** is no longer zero.

Taking the curl of both sides of Eq. (18.25)

$$\text{curl curl}\mathbf{E} = -\frac{\partial}{\partial t}(\text{ curl } \mu_o \mathbf{H}) = -\mu_o \frac{\partial}{\partial t}(\frac{\partial \mathbf{D}}{\partial t}) = -\mu_o \frac{\partial^2}{\partial t^2}(\epsilon_o \stackrel{=}{\epsilon}_r \mathbf{E}) \tag{18.27}$$

and using the vector identity curl curl $\mathbf{E} =$ grad (div $\mathbf{E}$) $- \nabla^2 \mathbf{E}$ gives

$$\nabla^2 \mathbf{E} - \text{ grad } (\text{div}\mathbf{E}) = \epsilon_o \mu_o \stackrel{=}{\epsilon}_r \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{18.28}$$

In the principal coordinate system the x component of Eq. (18.28) is, for example

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} - \frac{\partial}{\partial x}(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}) = \epsilon_o \mu_o \epsilon_x \frac{\partial^2 E_x}{\partial t^2} \tag{18.29}$$

For a plane wave of the form above, again in the principal coordinate system,

$$\epsilon_0 \epsilon_x E_x = D_x = (\mathbf{D}_o)_x \exp[i(wt - (k_x x + k_y y + k_z z))] \tag{18.30}$$

where k_x, k_y and k_z are the three orthogonal components of the wave vector and ϵ_x depends on the angular frequency ω; with similar equations for E_y and E_z. Substituting for $E_x = D_x/\epsilon_o \epsilon_x$ in Eq. (18.29) gives

$$\frac{D_x}{\epsilon_x}(k_x^2 + k_y^2 + k_z^2) - k_x(\frac{k_x D_x}{\epsilon_x} + \frac{k_y D_y}{\epsilon_y} + \frac{k_z D_z}{\epsilon_z}) = \epsilon_o \mu_o \omega^2 D_x \tag{18.31}$$

and two other similar equations. Recognizing that

$$\epsilon_o \mu_o = \frac{1}{c_o^2} \tag{18.32}$$

and

$$(k_x^2 + k_y^2 + k_z^2) = \mid \mathbf{k} \mid^2, \tag{18.33}$$

and writing

$$\frac{\epsilon_o c_o^2}{\mid \mathbf{k} \mid^2}(k_x E_x + k_y E_y + k_z E_z) = P^2 \tag{18.34}$$

Eq. (18.31) becomes

$$D_x = \frac{-k_x P^2}{c^2 - c_x^2} \tag{18.35}$$

and similarly

$$D_y = \frac{-k_y P^2}{c^2 - c_y^2} \tag{18.36}$$

$$D_z = \frac{-k_z P^2}{c^2 - c_z^2} \tag{18.37}$$

where $c_x = c_o/\epsilon_x^{1/2}$; $c_y = c_o/\epsilon_y^{1/2}$ and $c_z = c_o/\epsilon_z^{1/2}$ are called the principal phase velocities of the crystal. Now, since div $\mathbf{D} = -i(k_x D_x + k_y D_y + k_z D_z) = 0$, we have

$$\frac{k_x^2}{c^2 - c_x^2} + \frac{k_y^2}{c^2 - c_y^2} + \frac{k_z^2}{c^2 - c_z^2} = 0. \tag{18.38}$$

Multiplying both sides of Eq. (18.38) by c^2 and rearranging gives

$$\frac{k_x^2 n_x^2}{n^2 - n_x^2} + \frac{k_y^2 n_y^2}{n^2 - n_y^2} + \frac{k_z^2 n_z^2}{n^2 - n_z^2} = 0, \tag{18.39}$$

where we have put $n_x = \epsilon_x^{1/2}$, $n_y = \epsilon_y^{1/2}$, and $n_z = \epsilon_z^{1/2}$; n_x, n_y and n_z are called the principal refractive indices of the crystal. Eq. (18.38) and Eq. (18.39), which is called Fresnel's equation, are quadratic in c^2 and n^2 respectively. Thus in general there are two possible solutions c_1, c_2 and n_1, n_2, respectively, for the phase velocity and refractive index of a monochromatic wave propagating through an anisotropic medium with wave vector **k**. However, when **k** lies in certain specific directions both roots of Eqs. (18.38) and (18.39) become equal. These special directions within the crystal are called optic axes. The fact that, for example, $(n_1^2)^{1/2}$ has two roots $+n_1$ and $-n_1$ merely corresponds to each solution of Eq. (18.39) allowing a wave to propagate in a given direction either with wave vector **k** or $-\mathbf{k}$.

18.5 The Two Possible Directions of D for a Given Wave Vector are Orthogonal

We can show that the two solutions of Eq. (18.38) and (18.39) correspond to two different possible linear polarizations of a wave propagating with wave vector **k** and that these two solutions have mutually othogonal polarization. Let us first illustrate this by considering some special cases.

If we send a monochromatic wave into an anisotropic crystal travelling in the z direction, but linearly polarized in the x direction, then from Maxwell's Eqs. (18.25) and (18.26)

$$\frac{\partial E_x}{\partial z} = -\mu_r \mu_o \frac{\partial H_y}{\partial t} \tag{18.40}$$

$$\frac{\partial H_y}{\partial z} = -\epsilon_x \epsilon_o \frac{\partial E_x}{\partial t} \tag{18.41}$$

where we have taken the axes x, y, and z to correspond to the principal coordinate system of the crystal. Taking the z derivative of Eq. (18.40) and substituting for $\partial H_y/\partial z$ from (18.41) gives

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_r \mu_o \epsilon_x \epsilon_o \frac{\partial^2 E_x}{\partial t^2} \tag{18.42}$$

which is a one-dimensional wave equation with a solution of the form

$$E_x = (\mathbf{E_o})_x e^{i(\omega t - k_x z)} \tag{18.43}$$

where $k_x = \omega(\mu_r \mu_o \epsilon_x \epsilon_o)^{1/2}$

Thus the wave propagates along the z axis with a phase velocity $c_1 = c_o/n_x$, where $n_x^2 = \epsilon_x$.

If we repeat this derivation, but for a wave propagating in the z direction and polarized in the y direction we find that the wave now propagates with a phase velocity $c_z = c_o/n_y$ where $n_y^2 = \epsilon_y$. This illustrates, that, at least in this special case, a wave propagating in the z direction of the principal coordinate system has two possible orthogonal allowed linear polarizations, E_x and E_y, that travel with respective phase velocities $c_1 = c_o/n_x$ and $c_2 = c_o/n_y$. It is straightforward to show that for propagation in the x direction there are two possible orthogonal linear polarizations E_y and E_z with corresponding phase velocities c_o/n_y and c_o/n_z with similar behavior for a wave propagating in the y direction. Clearly, in these special cases where a wave propagates along one principal axis, and is polarized along a second, both **D** and **E** are parallel and the two possible orientations of **D** for a given wave vector are orthogonal. However, for propagation in an arbitrary direction **D** and **E** are no longer parallel. To find the angular relationship between the two directions of **D** which are allowed for a particular wave vector we use Eqs. (18.35), (18.36) and (18.37). If we designate quantities which refer to the two solutions by the subscripts 1 and 2, then

$$\begin{aligned} D_{j_1} &= \frac{-k_j P_1^2}{c_1^2 - c_j^2} \\ D_{j_2} &= \frac{-k_j P_2^2}{c_2^2 - c_j^2} \end{aligned} \qquad j = x, y, z \tag{18.44}$$

and

$$\mathbf{D}_1 \cdot \mathbf{D}_2 = \sum_{j=x,y,z} \frac{k_j^2 P_1^2 P_2^2}{(c_1^2 - c_j^2)(c_2^2 - c_j^2)}$$
$$= \frac{P_1^2 P_2^2}{(c_2^2 - c_1^2)} \sum_{j=x,y,z} \frac{k_j^2}{(c_1^2 - c_j^2)} - \frac{k_j^2}{(c_2^2 - c_j^2)} \tag{18.45}$$

which is zero by virtue of Eq. (18.38). Thus the two possible directions of **D** for a given wave vector are orthogonal.

18.6 Angular Relationships between D, E, H, k and the Poynting vector S

The electric displacement vector **D** and wave vector **k** are, by definition and by virtue of Eq. (18.23), mutually perpendicular. Further, since the values of **D**, **E** and **H** are constant over the phase front of a plane wave, the phase factor ϕ must be the same for all these three vectors. We can show that this is so provided certain angular relationships exist between them. For **D**, **E** and **H** with a phase dependence of the form $\exp i(\omega t - \mathbf{k} \cdot \mathbf{r})$ we can replace the operation $\partial/\partial t$ by multiplication by $i\omega$ and the operation $\partial/\partial x$ by multiplication by $-ik_x$. So for example

$$\frac{\partial \mathbf{D}}{\partial t} = i\omega \mathbf{D} \tag{18.46}$$

$$\text{curl } \mathbf{E} = i(\mathbf{E} \times \mathbf{k}), \tag{18.47}$$

which can be verified by writing out both sides in cartesian coordinates, and

$$\text{div } \mathbf{E} = -i(\mathbf{E} \cdot \mathbf{k}) \tag{18.48}$$

with similar relations for curl **H**, etc.

Thus from Eqs. (18.47) and (18.25)

$$\mathbf{E} \times \mathbf{k} = \omega \mu \mu_0 \mathbf{H}, \tag{18.49}$$

and similarly

$$\mathbf{H} \times \mathbf{k} = \omega \mathbf{D}. \tag{18.50}$$

Thus, **H** is normal to both **D**, **E** and **k**, and the latter three are coplanar. **D** and **E** make an angle α with one another where

$$\alpha = \arccos\left(\frac{\mathbf{E} \cdot \mathbf{D}}{|\mathbf{E}||\mathbf{D}|}\right) = \arccos\left(\frac{\epsilon_o(\overline{\overline{\epsilon}}_r\ \mathbf{E}) \cdot \mathbf{E}}{|\mathbf{E}||\mathbf{D}|}\right) \tag{18.51}$$

These angular relationships are illustrated in Fig. (18.1). The direction of the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is, by definition, perpendicular to

Fig. 18.1.

both **E** and **H**. **S** defines the direction of energy flow within the medium and we identify it as the direction of the *ray* of familiar geometric optics. Whereas in isotropic media the ray is always parallel to the wave vector, and is therefore perpendicular to the wavefront, this is no longer so in anisotropic media, except for propagation along one of the principal axes.

To summarize, we have shown that transverse electromagnetic plane waves can propagate through anisotropic media, but for propagation in a general direction two distinct allowed linear polarizations specified by the direction of **D** can exist for the wave. We have shown that these two allowed polarizations are orthogonal and that the wave propagates with a phase velocity (the velocity of the surface of constant phase - the wavefront) which depends on which of these two polarizations it has. Clearly, a wave of arbitrary polarization which enters such an anisotropic medium will not in general correspond to one of the allowed polarizations, and will therefore be resolved into two linearly polarized components polarized along the allowed directions. Each component propagates with a different phase velocity. All that remains for us to do to fully characterize these allowed polarization directions, is to specify their orientation with respect to the principal axes of the medium. This is done with the aid of a geometric figure called the index ellipsoid or indicatrix.

18.7 The Indicatrix

The indicatrix, wave-normal, or index ellipsoid is an ellipsoid with the equation

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1, \tag{18.52}$$

which allows us to determine the refractive index for monochromatic plane waves as a function of their direction of polarization. It is, apart from a scale factor, equivalent to the surface mapped out by the **D** vectors corresponding to a constant energy density at a given frequency. This ellipsoid can be visualized as oriented inside a crystal consistent with the symmetry axes of the crystal. For example, in any crystal with perpendicular symmetry axes, such as those belonging to the cubic, tetragonal, hexagonal, trigonal or orthorhombic crystal systems, the axes of the ellipsoid, which are the principal axes of the crystal, are parallel to the three axes of symmetry of the crystal. For the orientation of the indicatrix to be consistent with the symmetry of the crystal, planes of mirror symmetry within the crystal must coincide with planes of symmetry of the indicatrix: namely the xy, yz and zx planes. In the monoclinic system, crystal symmetry is referred to three axes, two of which, the a and c axes of crystallographic terminology, intersect at acute and obtuse angles and a third, the b axis, is perpendicular to the a and c axes. In such crystals, one of the axes of the indicatrix must coincide with the b axis, but the other two axes have any orientation, although this is fixed for a given crystal and wavelength. In triclinic crystals whose symmetry is referred to three unequal length, non-orthogonal axes, the indicatrix can take any orientation, although this is fixed for a given crystal and wavelength.

We use the geometrical properties of the indicatrix to determine the refractive indices and polarizations of the two monochromatic plane waves which can propagate through the crystal with a given wave vector. This is illustrated in Fig. (18.2), which shows the direction of the wavevector **k** for a monochromatic wave propagating through a crystal, drawn relative to the orientation of the indicatrix in the crystal. The plane surface that is orthogonal to the wavevector, and that passes through the center of the indicatrix, intersects this ellipsoid in an ellipse called the intersection ellipse. The semi-axes of this ellipse define the directions of the two allowed **D** polarizations which can propagate through the crystal with the given wave vector **k**, and the lengths of these semi-axes give the refractive indices experienced by these two polarizations. That this interpretation is consistent follows for a number of reasons. For a wave of given intensity propagating through the crystal, the stored energy density must be independent of polarization direction otherwise we could either absorb or extract energy from the wave merely by rotating the crystal. This cannot be so as we have already specified that the crystal is transparent. Consequently, for a given wave vector the many possible **D**

Fig. 18.2.

polarizations trace out an ellipse, equivalent to the intersection ellipse. Further, it can be shown that the lengths of the semiaxes of the intersection ellipse corresponding to a given wavevector $\mathbf{k} = k_x\hat{\imath} + k_y\hat{\jmath} + k_z\hat{\mathbf{k}}$ are the two roots n_1 and n_2 of Fresnel's Eq. (18.39). Thus, only the two **D** vectors parallel to these semiaxes simultaneously satisfy the condition of both being on the ellipse and having the appropriate refractive indices to satisfy Fresnel's equations.

In the general case there are two **k** vector directions through the center of the indicatrix for which the intersection ellipse is a circle. This is a fundamental geometric property of ellipsoids. These two directions are called the principal optic axes and are fixed for a given crystal and frequency of light. Waves can propagate along these optic axes with any arbitrary polarization, as in these directions the refractive index is not a function of polarization.

In cubic crystals the indicatrix is a sphere called the isotropic indicatrix, there are no specific optic axes as the indicatrix is anaxial and the propagation of waves is independent of both the directions of **k** and **D**.

In crystals belonging to the tetragonal, hexagonal and trigonal crystal systems the crystal symmetry requires that $n_x = n_y$ and the indicatrix reduces to an ellipsoid of revolution. In this case there is only one optic axis, oriented along the axis of highest symmetry of the crystal, the z axis (or c axis). These crystals classes, listed in Table (18.1), are said to be uniaxial: a discussion of their properties is considerably simpler than for crystals belonging to the less symmetric orthohombic, monoclinic and triclinic crystal systems, which are *biaxial.*

18.8 Uniaxial Crystals

Table 18.1 *Uniaxial Crystal Classes*

Hexagonal	Trigonal	Tetragonal
$\bar{6}$2m	3m	$\bar{4}$2m
6mm	32	4mm
622	3	422
$\bar{6}$		$\bar{4}$
6		4

Fig. 18.3.

The equation of the uniaxial indicatrix is

$$\frac{x^2 + y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1 \tag{18.53}$$

where n_o is the index of refraction experienced by waves polarized perpendicular to the optic axis, called *ordinary* or O - waves; n_e is the index of refraction experienced by waves polarized parallel to the optic axis, called *extraordinary* or E - waves. If $n_e > n_o$ the indicatrix is a prolate ellipsoid of revolution as shown in Fig. (18.3) and such a crystal is said to be *positive uniaxial*.

If $n_e < n_o$ the indicatrix is an oblate ellipsoid of revolution as shown in Fig. (18.3) and the crystal is said to be *negative uniaxial.* Fig. (18.4) shows the orientation of this ellipsoid inside a negative uniaxial crystal of calcite.

Because uniaxial crystals have indicatrices which are circularly symmetric about the z (optic) axis, their optical properties depend only on the polar angle θ that the wave vector $\mathbf{k}$ makes with the optic axis and not on the azimuthal orientation of $\mathbf{k}$ relative to the x and y axes. Thus, we can illustrate all their optical characteristics by considering propaga-

Fig. 18.4.

Fig. 18.5.

tion in any plane containing the optic axis, as shown in Fig. (18.5). For propagation with the wave vector in the direction ON, the two allowed polarizations are as indicated: perpendicular to the optic axis, and in the plane containing ON and the optic axis OZ. OQ is the major semi-axis of the intersection ellipse in this case. The length OQ gives the refractive index for waves polarized parallel to OQ (E - waves). The refractive index of the O - wave is given by the minor semi-axis of the intersection ellipse, which in a uniaxial crystal is independent of direction and is equal to the length OR. In Fig. (18.5) the O - wave with wave vector in the direction ON propagates with phase velocity $c_o/n_o = c_o/1.5$, while the E - wave propagates with phase velocity $c_o/OQ = c_o/1.68$. Thus, in this positive crystal the O - wave travels faster than the E - wave.

In a uniaxial crystal the O - waves have no component of **D** along the z axis, thus from Eq. (18.6)

$$D_x = \epsilon_o \epsilon_x E_x \tag{18.54}$$

$$D_y = \epsilon_o \epsilon_y E_y \tag{18.55}.$$

In uniaxial crystals, since $\epsilon_x = \epsilon_y$, **D** and **E** are parallel for the O - wave and the O - ray is parallel to the O - wave vector. However, the ray and wave vector of the E - wave are not parallel, except for propagation along, or perpendicular to, the optic axis. We can find the direction of the E - ray by a simple geometric construction shown in Fig. (18.5). The wavevector is in the direction ON, and TQ is the major axis of the intersection ellipse. The tangent to the ellipse at the point T is parallel to the ray direction OP. Note that the tangent to the ellipse at point P is orthogonal to the wave vector. The wave index of refraction of the extraordinary wave with wave vector **k** is given by the length OT. The length TM gives the ray index of refraction which is a measure of the velocity with which energy flows along the ray OP.

Light propagating along the ray OP as an E - wave is not polarized perpendicular to the ray direction, however, the **E** vector of this wave is perpendicular to the ray, parallel to the direction MT in Fig. (18.5). Since light propagating along the ray OP with wave vector **k** travels a distance OP in the time the wave front (which is parallel to the direction PN) travels a distance ON, it is clear that the velocity of light along the ray is greater than the wave-normal velocity (the phase velocity). If the wave vector is in the direction OR in Fig. (18.5) then both wave-normal and ray are parallel for the E and O - waves. As indicated by the dots, the O - wave propagates more quickly in this positive crystal.

18.9 Index Surfaces

The refractive index $n_e(\theta)$ of an E-wave propagating at angle θ to the optic axis in a uniaxial crystal can be calculated with reference to Fig. (18.6). For this wave the value of $n_e(\theta)$ is given by the length OX in the figure, the major semi-axis of the intersection ellipse. The caretesian coordinates of the point X relative to the origin O are

$$\begin{aligned} x &= -n_e(\theta)\cos\theta \\ z &= n_e(\theta)\sin\theta \\ y &= 0 \end{aligned} \tag{18.56}$$

This point lies on the indicatrix so

$$\frac{n_e^2(\theta)\cos^2(\theta)}{n_o^2} + \frac{n_e^2(\theta)\sin^2\theta}{n_e^2} = 1 \tag{18.57}$$

Fig. 18.6.

which gives

$$n_e(\theta) = \frac{n_o n_e}{(n_e^2 \cos^2\theta + n_o^2 \sin^2\theta)^{1/2}} \tag{18.58}$$

We can use this relationship to specify a surface called the extraordinary index surface, which shows geometrically the index of refraction of extraordinary waves in a uniaxial crystal as a function of their direction of propagation. The cartesian coordinates on this surface must satisfy

$$n_e^2(\theta) = x^2 + y^2 + z^2 \tag{18.59}$$

and also

$$\sin^2\theta = \frac{x^2 + y^2}{x^2 + y^2 + z^2} \tag{18.60}$$

$$\cos^2\theta = \frac{z^2}{x^2 + y^2 + z^2} \tag{18.61}$$

Substituting from Eqs. (18.59), (18.60) and (18.61) in Eq. (18.58) gives

$$\frac{x^2 + y^2}{n_e^2} + \frac{z^2}{n_o^2} = 1. \tag{18.62}$$

We might have expected this, since for an E-wave propagating perpendicular to the optic axis $z = 0$ and $(x^2 + y^2)^{1/2} = n_e(\theta) = n_e$. For an E-wave propagating along the z axis (which in this case is actually also an O-wave) $x^2 + y^2 = 0$ and $z = n_e(\theta) = n_0$.

The index surface for O-waves is, of course, a sphere since the index of refractive of such waves is independent of their propagation direction. The equation of this surface is

$$x^2 + y^2 + z^2 = n_o^2 \tag{18.63}$$

Sections of the ordinary and extraordinary index surfaces for both positive and negative uniaxial crystals which contain the optic axis are shown in Fig. (18.7). Such sections are called principal sections.

Fig. 18.7.

18.10 Other Surfaces Related to the Uniaxial Indicatrix

There are several other surfaces related to the uniaxial indicatrix whose geometrical properties describe various aspects of the propagation characteristics of uniaxial crystals.

The wave-velocity surface describes the velocity of waves in their direction of propagation, this surface is, like the index surface, a two-shelled surface. The wave-velocity surface for the O-waves is clearly

$$n_o^2(x^2+y^2+z^2)=c_o^2 \tag{18.64}$$

while for the E-waves

$$n_e^2(\theta)(x^2+y^2+z^2)=c_o^2 \tag{18.65}$$

which since $\sin\theta=\frac{x^2+y^2}{x^2+y^2+z^2}$; $\cos^2\theta=\frac{z^2}{x^2+y^2+z^2}$, from Eq. (18.58) gives

$$\frac{x^2+y^2}{n_e^2}+\frac{z^2}{n_o^2}=\frac{(x^2+y^2+z^2)^2}{c_o^2} \tag{18.66}$$

This surface is not an ellipsoid but an ovaloid of revolution.

The ray-velocity surface describes the velocity of rays in their direction of propagation, which for E-rays is, as we have mentioned already, not the same as the direction of their wave vectors except for propagation along a principal axis. Along the x and y principal axes the extraordinary ray and wave velocities are both c_o/n_e, while along the z axis they both have the value c_o/n_o. It is left as an exercise to the reader to show that the ray velocity surface for extraordinary rays is an ellipsoid of revolution with semi-axes c_o/n_e and c_o/n_e satisfying the equation

$$\frac{x^2+y^2}{1/n_e^2}+\frac{z^2}{1/n_o^2}=c_o^2 \tag{18.67}$$

Fig. 18.8.

The ray-velocity surface for O-rays is clearly a sphere with equation

$$x^2 + y^2 + z^2 = \frac{c_o^2}{n_o^2} \tag{18.68}$$

Fig. (18.8) shows principal sections of ray-velocity surfaces in both positive and negative crystals.

18.11 Huygenian Constructions

When a linearly polarized wave of arbitrary polarization direction enters an anisotropic medium, it will be resolved into two components polarized along the two allowed polarization directions determined by the direction of the wave vector relative to the axes of the indicatrix. In a uniaxial crystal these two components propagate as O and E-waves respectively. At the entry surface of the anisotropic medium refraction of these waves, and of their corresponding rays, occurs. For normal incidence at the boundary of the anisotropic medium no refraction of either the O- or E-wave vectors occurs so that within the medium the wave vectors of both O and E waves remain parallel. However, except for propagation along a principal axis, the E-ray deviates from the common direction of the O and E-wave vectors and the O-ray. Thus at the exit surface of the medium, where refraction renders and O and E-rays parallel once again, the E-ray will have been laterally displaced from the O-ray. In simple terms we can say that the E-ray is not refracted at the surface of the medium according to Snell's law. For other than normal incidence the O- and E- wave vectors refract separately according to Snell's law. For the O-wave calculation of the wave vector direction inside the medium is straightforward. For angle of incidence θ_i the angle of refraction obeys

$$\sin\theta_i = n_o \sin\theta_r$$

However, for the E-wave the refraction of the wave vector obeys

$$\sin\theta_i = n_e(\theta)\sin\theta_r$$

where θ is the angle that the refracted wave vector makes with the optic axis. To illustrate these geometric optical properties of anisotropic media it is instructive to use *Huygenian constructions* using the ray-velocity surfaces for the O and E-rays. In Huygenian constructions we treat each point on the entrance boundary of the anisotropic media as a secondary emitter of electromagnetic waves. In a given time the distance travelled by all the O or E-rays leaving this point as a function of direction traces out the appropriate ray velocity surface. The geometric paths of the O and E-rays are perpendicular to the envelope of the ray velocity surfaces which arise from these secondary emitters. To illustrate this, consider first the simple case shown in Fig. (18.9) where light enters a negative uniaxial crystal normally travelling along the optic axis. Secondary emitters at A and B give rise to the ray velocity surfaces shown and contribute to the envelope PP' which in this case is both the wavefront and the ray front (the surface normal to the ray direction). Fig. (18.10) illustrates the situation that results when light is incident normally on a positive uniaxial crystal travelling in a plane perpendicular to the optic axis. In this case, the ray-velocity surfaces which arise from secondary emitters at A and B define two envelopes, the surfaces OO' and EE' which are respectively the ordinary wave and ray-front and the extraordinary wave and ray-front. The O-ray travels faster than the E-ray as indicated by the dots and arrows. In both Figs. (18.9) and (18.10) the wave-vector and ray, which are perpendicular to their respective fronts, remain parallel. When the incident radiation does not travel along a principal axis in the crystal the O and E rays are no longer coincident, as shown in Fig. (18.11), which shows light travelling at an angle to the optic axis striking a planar uniaxial crystal slab cut normal to the optic axis. In this case the utility of the ray-velocity surface in determining the ray directions within the crystal is clearly apparent. The incident wavefronts are parallel to PP'; by the time the wavefront reaches point A the secondary emitter at B has given rise to the O and E ray-velocity surfaces shown. The O and E ray-fronts are the tangents from A to the respective ray-velocity surfaces. The ray directions are perpendicular to these ray fronts. Once again, in this positive crystal the O-ray travels more quickly than the E-ray. On leaving the crystal the O

Fig. 18.9.

Fig. 18.10.

and E-rays become parallel to each other once more but the E-ray has been laterally displaced. This phenomenon is called *double refraction.* If the input to the crystal were a narrow laser beam, linearly polarized at some angle between the two allowed polarization directions within the crystal, it would be resolved into an O and an E-ray and emerge from the crystal as two separate orthogonally polarized beams.

Further examples of double refraction are shown in Figs. (18.12) and (18.13) with the paths of the O and E-rays determined by Huygenian constructions. Fig. (18.13) shows that, even in normal incidence, if the input wave vector is not along a principal axis the E-ray is displaced laterally from the O-ray. The refraction of the O-ray in every case follows the familiar refraction laws of geometric optics, with the angles of incidence θ_i and refraction θ_r at the crystal boundary obeying Snell's law:

Fig. 18.11.

Fig. 18.12.

Fig. 18.13.

$$n_o = \frac{\sin\theta_i}{\sin\theta_r} \tag{18.69}$$

The path of the E-ray is controlled by the anisotropic character of the

Fig. 18.14.

medium and must be determined from a non-spherical ray-velocity surface.

The separation of the O and E-rays within an anisotropic medium is the basic phenomenon operative in the various polarizing devices used for controlling the direction of polarization of light beams, particularly laser beams. The construction, and mode of operation of some common types of laser polarizer are illustrated in Fig. (18.14).

18.12 Retardation

Unless a light beam is travelling in the direction of a principal axis, and is polarized parallel to a principal axis, when it is incident normally on a planar uniaxial crystal slab, it will be resolved within this material into O and E waves. If the direction of the wave vector is at angle θ to the optic axis then the wave velocities of these waves will be c_o/n_o and $c_o/n_e(\theta)$, respectively. In a positive crystal the O-wave will travel faster than the E-wave and in a negative crystal vice-versa.

The wave vector of the ordinary wave is

$$| \mathbf{k}_o |= k_o = \frac{\omega n_o}{c_o} = \frac{2\pi n_o}{\lambda_o} \tag{18.70}$$

and of the extraordinary wave

$$| \mathbf{k}_e |= k_e = \frac{\omega n_e(\theta)}{c_o} = \frac{2\pi n_e(\theta)}{\lambda_o} \tag{18.71}$$

where ω and λ_o are the angular frequency and wavelength *in vacuo*, respectively, of the incident wave.

On passing through a crystal of thickness L the phase changes for the

O and E-waves, respectively, are

$$\phi_o = k_o L = \frac{2\pi n_o L}{\lambda_o} \tag{18.72}$$

$$\phi_e = k_e L = \frac{2\pi n_e(\theta) L}{\lambda_o} \tag{18.73}$$

The phase difference (*retardation*) introduced by the crystal is

$$\Delta\phi = \phi_e - \phi_o = \frac{2\pi L}{\lambda_o}(n_e(\theta) - n_o) \tag{18.74}$$

For an incident wave of the form

$$D = A\cos(\omega t - kr) \tag{18.75}$$

linearly polarized at an angle β to the ordinary polarization direction, The ordinary and extraordinary waves will be

$$D_o = A\cos\beta\cos(\omega t - k_o r) \tag{18.76}$$

$$D_e = A\sin\beta\cos(\omega t - k_e r) \tag{18.77}$$

and at the exit face of the crystal

$$D_o = A\cos\beta\cos(\omega t - \phi_o) \tag{18.78}$$

$$D_e = A\sin\beta\cos(\omega t - \phi_e) \tag{18.79}$$

If the input to the crystal is a narrow beam of light, these two orthogonally polarized output beams will in general be displaced laterally from one another and will not recombine to form a single output beam. However, if the input to the crystal is a plane wave, or a narrow beam travelling perpendicular to the optic axis, these two electric vectors recombine to form a resultant single displacement vector with magnitude

$$D_{\text{out}} = (D_o^2 + D_e^2)^{1/2} \tag{18.80}$$

which makes an angle α with the ordinary polarization direction, where

$$\tan\alpha = \frac{D_e}{D_o} = \tan\beta\frac{\cos(\omega t - \phi_e)}{\cos(\omega t - \phi_o)} \tag{18.81}$$

as illustrated in Fig. (18.15).

In the simplest case where $\phi_e - \phi_o = 2n\pi$, with n being any positive or negative integer or zero, $\tan\alpha = \tan\beta$ and the output wave is linearly polarized in the same direction as the input.

If $\phi_e - \phi_o = (2n+1)\pi$, then $\tan\alpha = -\tan\beta$; $\alpha = -\beta$ and the output wave is linearly polarized but rotated by 2β from its original polarization direction. This rotation occurs through a rotation by angle β towards the O-polarization direction followed by a further rotation through angle

Fig. 18.15.

β *. Since a retardation $\phi = (2n+1)\pi$ is equivalent to a path difference of $(2n+1)\lambda/2$ a crystal which rotates the plane of linear polarization by 2β is called a ($(2n+1)$th order) half-wave (retardation) plate. It is most usual to cut such a crystal so its faces are parallel to the optic axis and to polarize the input at 45° to the optic axis, in which case the output is linearly polarized and rotated 90° from the input. When the input is polarized in this manner and $\phi = \phi_e - \phi_o = (2n+1)\pi/2$ then $\tan\alpha = \pm\tan(\omega t - \phi_o)$.

For n even $\tan\alpha = \omega t - \phi_o$.

For n odd $\tan\alpha = -(\omega t - \phi_o)$.

In both these cases the resultant displacement vector has magnitude, from Eqs. (18.78), (18.79) and (18.80), with $\beta = 45°$

$$\begin{aligned} D_{\text{out}} &= \frac{A}{\sqrt{2}}[\cos^2(\omega t + \phi_o) + \cos^2(\omega t - \phi_o - \phi)] \\ &= \frac{A}{\sqrt{2}} \end{aligned} \tag{18.82}$$

so the output displacement vector has a constant magnitude but rotates about the direction of propagation with constant angular velocity ω. This is circularly polarized light. If the electric vector rotates in a clockwise direction when viewed in the direction of propagation, as when n is odd above, the light is said to be *left-hand circularly polarized*, as illustrated in Fig. (18.16). When the electric vector rotates counterclockwise the light is *right-hand circularly polarized.* A crystal which introduces a retardation ϕ of $(2n+1)\pi/2$ is called a *quarter-wave plate* (strictly a $(2n+1)$th order quarter-wave plate).

* For this reason a waveplate alone cannot be used to make an optical isolator as can a Faraday rotator.

Fig. 18.16.

In the general case when ϕ is not an integral number of half-wavelengths, or if the input to a quarter-wave plate is not polarized at 45° to the optic axis, the resultant output will be elliptically polarized, as illustrated also in Fig. (18.16). In this case the displacement vector, and the electric vector, trace out an ellipse as they rotate in time as the wave propagates. To prove this we rewrite Eqs (18.78) and (18.79) as

$$D_o = p = a\cos z \tag{18.83}$$

$$D_e = q = b\cos(z-\phi) = b\cos z\phi + b\sin z\sin\phi \tag{18.84}$$

where $a = A\cos\beta; b = A\sin\beta$ and $z = (\omega t - \phi_o)$.

From Eq. (18.84)

$$\frac{q^2}{b^2} - \frac{2q}{b}\cos z\cos\phi + \cos^2 z\cos^2\phi = \sin^2 z\sin^2\phi \tag{18.85}$$

and substituting from (18.83)

$$\frac{q^2}{b^2} - \frac{2pq}{ab}\cos\phi + \frac{p^2}{a^2}\cos^2\phi = (1-\frac{p^2}{a^2})\sin^2\phi \tag{18.86}$$

giving finally

$$\frac{p^2}{a^2} + \frac{q^2}{b^2} - \frac{2pq}{ab}\cos\phi - \sin^2\phi = 0. \tag{18.87}$$

This is the equation of an ellipse, one axis of which makes an angle ψ with the O-polarization direction, where

$$\tan 2\psi = \frac{2ab\cos\phi}{a^2-b^2} \tag{18.88}$$

This ellipse has its axes coincident with the O and E vibration directions if $\psi = (2n+1)\pi/2$. Thus, a quarter wave plate produces elliptically polarized light if the incident light is not polarized exactly half-way between the O and E allowed polarization directions.

A first-order quarter - wave plate is very thin, as can be seen from

Eq. (18.74), its thickness L is

$$L = \frac{\lambda_o}{4(n_e - n_o)} \tag{18.89}$$

For calcite (Iceland spar), a mineral form of calcium carbonate that is commonly used to make polarizing optics $n_o = 1.658, \;\; n_e = 1.486$. so for a wavelength of 500 nm the thickness of a first-order quarter-wave plate is only $726.7nm \sim 0.0007$ mm. It is not practical to cut a crystalline slab so thin as this, except for a birefringent material such as mica which cleaves readily into very thin slices. $(2n+1)$th-order quarter-wave plates can be of practical thickness for large values of n but suffer severely from the effects of termperature: the plate only has to expand or contract very slightly and it will cease to be a quarter-wave plate at the wavelength for which it was designed. To overcome this drawback, temperature compensated plates can be made. These consist of one $(2n+1)$ th-order quarter-wave plate stacked on top of a plate which produces a retardation of $n\pi$ but whose optic axis is perpendicular to the optic axis (E-wave polarization direction) of the quarter-wave plate. The total retardation of the combination is

$$\phi = \phi_1 - \phi_2 = (2n+1)\pi/2 - n\pi = \pi/2 \tag{18.95}$$

so it is equivalent to a first-order quarter-wave plate but is not sensitive to temperature changes.

18.13 Biaxial Crystals

The optical properties of biaxial crystals can be related to an ellipsoid with three unequal axes, the biaxial indicatrix, whose equation is

$$\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1 \tag{18.96}$$

where n_x, n_y and n_z are the three principal refractive indices of the material. It is the normal convention to label the axes so that $n_x < n_y < n_z$. When this is done the two optic axes, those directions through the crystal along which the direction of propagation of waves is independent of their polarization direction, lie in the xz plane. If the two optic axes are closer to the z axis then the x axis the crystal is said to be *positive*, otherwise it is *negative*. Some of the important features of positive and negative biaxial indicatrices are illustrated in Fig. (18.17). The acute angle between the two optic axes is labelled $2V$ - the *optic angle*. Whichever axis bisects this acute angle is called the *acute bisectrix*, this

Fig. 18.17.

is the z axis in positive crystals, the x axis in negative crystals. The other axis in each case is called the *obtuse bisectrix.*

The angle between one of the optic axes and the z axis is V_z and is given by the expression

$$V_z = \tan^{-1}\left(\frac{\frac{1}{n_x^2} - \frac{1}{n_y^2}}{\frac{1}{n_y^2} - \frac{1}{n_z^2}}\right)^{1/2} \tag{18.97}$$

which is an angle $< 45°$ in a positive crystal.

Because of the lack of rotational symmetry of the biaxial indicatrix, the optical properties of biaxial crystals are more complicated than for uniaxial ones. For wave propagation along the three principal axes wave-normals and rays of both of the two allowed polarization directions are parallel. For propagation in a symmetry plane of the biaxial indicatrix, as shown in Fig. (18.18), the two allowed polarization directions are perpendicular to, and in, the plane of symmetry. The component polarized perpendicular to the symmetry plane has its polarization direction parallel to a principal axis and therefore has its **E** direction parallel to **D**. Thus, for this polarization the wave-normal direction ON and the ray OS_1 are parallel. Both the wave and ray refractive indices for this polarization have the value n_y. The ray direction for the wave polarized in the symmetry plane is the direction OS, which is found by a construction similar to the one described for finding the direction of the E - ray in a uniaxial crystal. The wave refractive index for this polarization is given by the length OT and its ray refractive index by the length OM; OM is less than OT as the ray travels further than the wave in the same period of time. There are two directions for which $OM = n_y$ in Fig. (18.18). These are ray directions for which the ray velocity is the same for both polarization directions; of course the wave vectors of these two waves are

Fig. 18.18.

Fig. 18.19.

not co-linear with their rays. These directions of equal ray velocity are called the secondary optic axes and are generally within two degrees of the primary optic axes, which are directions of equal wave-velocity. For propagation in an arbitrary direction not lying in a symmetry plane, both allowed polarization directions have rays which are not parallel to their wave normals. This is illustrated in Fig. (18.19) which shows the two ray directions OS_1, OS_2 which correspond to the two allowed polarization directions OP_1, OP_2 of a wave propagating in the general direction ON.

The surfaces which are related to the indicatrix of a biaxial crystal are more complicated than for uniaxial crystals, although for propagation in symmetry planes of the indicatrix they are somewhat like, but not quite the same, as sections of uniaxial surfaces. Fig. (18.20) illustrates the ray-velocity surface of a biaxial crystal. This is a two-shelled surface, one shell for each polarization direction corresponding to a given ray

Fig. 18.20.

Fig. 18.21.

direction. The four dimples where one shell cuts through the other (two are visible in Fig. (18.20)) are the directions of the secondary optic axes.

18.14 Intensity Transmission Through Polarizer/Waveplate/Polarizer Combinations

A wave is passed through a linear polarizer (P) whose preferred direction is at angle β to the O-direction of a succeeding waveplate W, as shown in Fig. (18.21), and is then transmitted through a second linear polarizer (A) whose preferred axis makes an angle ψ with the O-direction of the waveplate. The second polarizer, generally called the analyzer, will not in general permit all the radiation emerging from the waveplate to pass. This PWA combination serves as an adjustable attenuator of a light beam.

From Eqs. (18.78) and (18.79) it is easy to see that the **D** vector transmitted through the analyzer has magnitude

$$D = A\cos\beta\cos(\omega t - \phi_0)\cos\psi + A\sin\beta\cos(\omega t - \phi_e)\sin\psi \tag{18.98}$$

which can be rearranged to give

$$D = D_0\cos(\omega t - \phi_e + \chi) \tag{18.99}$$

where

$$D_0 = A[(\cos\beta\cos\psi\cos\Delta\phi + \sin\beta\sin\psi)^2 + (\cos\beta\cos\psi\sin\Delta\phi)^2]^{1/2} \tag{18.100}$$

and

$$\tan\chi = \frac{\cos\beta\cos\psi\sin\Delta\phi}{\cos\beta\cos\psi\cos\Delta\phi + \sin\beta\sin\psi} \tag{18.101}$$

$\Delta\phi = \phi_e - \phi_0$ is the retardation produced by the wave plate.

Table 2. *Need Table Caption*

P/A	W	Transmittance
$\parallel$	0	1
$\parallel$	$\lambda/4$	1/2
$\parallel$	$\lambda/2$	0
$\perp$	0	0
$\perp$	$\lambda/4$	1/2
$\perp$	$\lambda/2$	1

18.15 Examples

To illustrate the value of Eq. (18.100) some specific examples are in order:

(1) The input and output polarizers are parallel: $\psi = \beta$. In this case

$$D_0 = A[1 - \tfrac{1}{2}\sin^2 2\beta(1 - \cos\Delta\phi)]^{1/2}$$

If $\Delta\phi = 0$ (or if $\Delta\phi$ is any multiple of 2π) then $D_0 = A$: all the light is transmitted as we would have expected.

If $\beta = 45°$ then

$$D_0 = A[1 - \tfrac{1}{2}(1 - \cos\Delta\phi)]^{1/2}$$

If $\Delta\phi = \pi$ (or any odd multiple of π) then $D_0 = 0$. If $\Delta\phi = \pi/2$ (or any odd multiple of $\pi/2$) then $D_0 = A/\sqrt{2}$. Since the transmitted intensity is proportional to D_0^2 half of the incident light is transmitted.

(2) The input and output polarizers are crossed: $\psi = \beta \pm \pi/2$. In this case

$$D_0 = \frac{1}{\sqrt{2}}\sin 2\beta(1 - \cos\Delta\phi)^{1/2}$$

If $\beta = 45°$ then

$$D_0 = \frac{1}{\sqrt{2}}(1 - \cos\Delta\phi)^{1/2}$$

If $\Delta\phi = 0$ then $D_0 = 0$.

If $\Delta\phi = (2n + 1)\pi$, the waveplate is a half wave plate, and $D_0 = A$. All the incident light is transmitted.

If $\Delta\phi = (2\pi + 1)\pi/2$, the waveplate is a quarter wave plane and $D_0 = \frac{A}{\sqrt{2}}$. Half the incident light is transmitted.

These findings are summarized in Table (18.2).

18.16 The Jones Calculus

Over fifty years ago R. Clark Jones described a very useful technique for describing the change in polarization state of light wave as it passed through an optical system containing various interfaces and polarizing elements.[18.] The *Jones Calculus* treats the optical system as a linear system describable by an appropriate Jones matrix that transforms vectors describing the polarization state of the wave. In this sense the Jones Calculus is analogous to paraxial ray analysis.

18.16.1 The Jones Vector

The electric field of a light wave linearly polarized along the x axis can be written as

$$\mathbf{E}_x = A_x \sin(\omega t + \phi_x)\hat{\simeq}, \tag{18.102}$$

which is complex exponential notation is

$$\mathbf{E}_x = A_x e^{i\phi_x} e^{i\omega t}. \tag{18.103}$$

On the other hand a light wave linearly polarized along the y axis can be written as

$$\mathbf{E}_y = A_y e^{i\phi_y} e^{i\omega t} \tag{18.104}$$

The superposition of the electric fields $\mathbf{E}_x$ and $\mathbf{E}_y$ leads in general to elliptically polarized light with electric field

$$\mathbf{E} = \left(A_x e^{i\phi_x} + A_y e^{i\phi_y}\right) e^{i\omega t} \tag{18.105}$$

The complex amplitudes of the x and y components of this wave form the 2 elements of the Jones vector $\mathbf{J}$, where

$$\mathbf{J} = \begin{pmatrix} A_x e^{i\phi_x} \\ A_y e^{i\phi_y} \end{pmatrix} \tag{18.106}$$

Clearly, light linearly polarized along the x axis has

$$\mathbf{J} = \begin{pmatrix} A_x e^{i\phi_x} \\ 0 \end{pmatrix} \tag{18.107}$$

with a similar result for linearly polarized along y.

Light linearly polarized at angle β to the axis has

$$\mathbf{J}_\beta = \begin{pmatrix} E_0 \cos\beta \\ E_0 \sin\beta \end{pmatrix} \tag{18.108}$$

Right handd cciircularly polarized light has

$$\mathbf{J}_{\mathrm{rcp}} = \begin{pmatrix} E_0 e^{i\phi_x} \\ E_0 e^{i(\phi_x + \pi/2)} \end{pmatrix} \tag{18.109}$$

and left hand circularly polarized light has

$$\mathbf{J}_{\text{rcp}} = \begin{pmatrix} E_0 e^{i\phi_x} \\ E_0 e^{i(\phi_x - \pi/2)} \end{pmatrix} \tag{18.110}$$

In an isotopic medium the intensity of the wave is

$$I\alpha \mathbf{E}^* \cdot \mathbf{E}, \tag{18.111}$$

which in terms of the Jones vector gives

$$I\alpha \mathbf{J}^* \cdot \mathbf{J} = A_x^2 + A_y^2 \tag{18.112}$$

It is generally simple and convenient to use the Jones vector in its normalized form, in which

$$\mathbf{J}^* \cdot \mathbf{J} = 1, \tag{18.113}$$

which would correspond in a practical sense to assigning an intensity of 1W/m^2 to the wave. In this case the three vectors above in Eqs. (18.108), (18.109) and (18.110) would become

$$\mathbf{J}_p = \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} \tag{18.114}$$

$$\mathbf{J}_{\text{rcp}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\pi/2} \end{pmatrix} \text{ or } \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\pi/4} \\ e^{i\pi/4} \end{pmatrix} \tag{18.115}$$

$$\mathbf{J}_{\ell\text{cp}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{-i\pi/2} \end{pmatrix} \text{ or } \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi/4} \\ e^{i\pi/4} \end{pmatrix} \tag{18.116}$$

The second description for circularly polarized light is an alternative symmetrical way of writing the column vector, since only the phase difference between the x and y component issignificant.

In this notation general elliptically polarized beam has a Jone vector that can be written as

$$\mathbf{J} = \begin{pmatrix} \cos\beta e^{-i\Delta/2} \\ \sin\beta e^{i\Delta/2} \end{pmatrix}. \tag{18.117}$$

18.16.2 The Jones Matrix

In a linear system description the output Jones vector of a light wave after it has interacted with an optical system has components that are linearly related to its input components.

If

$$\mathbf{J}_{\text{in}} = E_x \hat{\simeq} + Ey\hat{\mathbf{j}} \tag{18.118}$$

and

$$\mathbf{J}_{\text{out}} = E'_x \hat{\mathbf{i}} + Ey\hat{\mathbf{j}} \tag{18.119}$$

where phase factors are included in the complex amplitudes, we can now write

$$E'_x = m_{11}E_x + m_{12}E_y$$
$$E'_y = m_{21}E_x + m_{22}E_y \tag{18.120}$$

or in matrix form

$$\begin{pmatrix} E'_x \\ E'_y \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \tag{18.121}$$

Eq. (18.121) introduce the Jones matrix **M** withe elements m_{ij}, it can be rewritten as

$$\mathbf{J}_{\text{out}} = \mathbf{M}\mathbf{J}_{\text{in}} \tag{18.122}$$

The determination of the Jones matrix for common optical elements is straight forward. This can be demonstrated with a few examples.

(a) Isotropic element: since $\mathbf{J}_{\text{out}} = \mathbf{J}_{\text{in}}$ clearly

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{18.123}$$

(b) Linear polarizer oriented along with x axis

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{18.1124}$$

(c) Linear polarizer oriented at angle θ to the x axis

$$\mathbf{M} = \begin{pmatrix} co^2\theta \sin\theta\cos\theta \\ \sin\theta\cos\theta \sin^2\theta \end{pmatrix} \tag{18.125}$$

(d) A waveplante that produces a retardation $\phi_x - \phi_y = \Gamma$. This case is worthy of a closer consideration. If the waveplate has its principal axis parallel to the x y axis then the refractive induces seemly the x and y components of the wave are n_x, n_y respectively. In this case

$$E'_x = E_x e^{ik_0\ell n_x}$$
$$E'_y = E_y e^{-ik_0\ell n_y}, \tag{18.126}$$

where $k_0 = 2\pi/\lambda_0$ and ℓ is the thickness of the retarder. Since $\Gamma = k_0\ell(n_y - n_x)$, Eq. (18.) can be written as

$$E'_x = E_x e^{-ik_0\ell n_y} e^{i\Gamma}$$
$$E^1_y = E_y e^{-ik_0\ell n_y} \tag{18.127}$$

or in symmetrical form

$$E'_x = E_x e^{ie\phi i\Gamma/2}$$
$$E'_y = E_y e^{-i\phi} e^{-i\Gamma/2} \tag{18.128}$$

where $\phi = k_0\ell(n_x t n_y)/2$. Therefore, the Jones matrix of the waveplate is

$$M = e^{i\phi}\begin{pmatrix} e^{i\Gamma/2} & 0 \\ 0 & e^{-i\Gamma/2}\end{pmatrix} \tag{18.129}$$

The phase factor $e^{-i\phi}$ in Eq. (18.129) can be omitted for most practical purposes.

If a $\lambda/4$ plate has

$$\mathbf{M} = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4}\end{pmatrix} \tag{18.130}$$

A linearly polarized wave with

$$\mathbf{J} = \begin{pmatrix} 1/\sqrt{2} \\ 1\sqrt{2}\end{pmatrix} \tag{18.131}$$

is linearly polarized at 45° to the fast (and slow) axis of the waveplate. Its electric field is in the $\mathbf{E}_1$ direction in Fig. (18.). The output Jones vector is

$$\begin{aligned}\mathbf{J}_{\text{out}} &= \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2}\end{pmatrix},\end{aligned} \tag{18.132}$$

which has transformed the linearly polarized input light to left hand circular polarization.

If the input linearly polarized light is rotated 90° with respect to the axis of the $\lambda/4$ plate to the $\mathbf{E}_2$ direction in Fig. (18.), then the output Jones vector will be

$$\begin{aligned}\mathbf{J}_{\text{out}} &= \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4}\end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2}\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\pi/4} \\ e^{i3\pi/4}\end{pmatrix} \\ &= \frac{1}{\sqrt{2}}e^{i\pi/2}\begin{pmatrix} e^{-i\pi/4} \\ e^{i\pi/4}\end{pmatrix}.\end{aligned} \tag{18.133}$$

This is right hand circularly polarized light, so a $\lambda/4$ plate will convert linearly polarized light into left or right hand circularly polarized light depending on its orientation.

It is of value to view the action of the $\lambda/4$ plate in a coordinate system that is rotated by 45°. What is now the Jones matrix for the plate for an input linearly polarized wave whose Jones vector is

$$\mathbf{J}_{\text{in}} = \begin{pmatrix} 1 \\ 0\end{pmatrix}$$

Such a wave is polarized in the x' direction in Fig. (18.22). Now the

Fig. 18.22. Coordinate system used in the Jones matrix description of a $\lambda/4$ plate.

transformation between the xy and $x'y'$ coordinate system is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{S} \begin{pmatrix} x \\ y \end{pmatrix} \tag{18.134}$$

and

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{S}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \tag{18.135}$$

In terms of the rotation matrix $\mathbf{S}$, in the $x'y'$ coordinate system

$$\mathbf{J}_{\text{out}} = S\mathbf{J}_{\text{out}} \tag{18.136}$$

in the xy coordinate system

$$\mathbf{J}_{\text{out}} = \mathbf{M}\mathbf{J}_{\text{in}} \tag{18.137}$$

where

$$\mathbf{M} = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \tag{18.138}$$

and

$$\mathbf{J}_{\text{in}} = \mathbf{S}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{S}^{-1}\mathbf{J}_{\text{in}}^{-1} \tag{18.139}$$

Combining Eqs. (18.136), (18.137) and (18.139)

$$\mathbf{J}'_{\text{out}} = \mathbf{S}\mathbf{M}\mathbf{S}^{-1}\mathbf{J}'_{\text{in}} \tag{18.140}$$

so the new Jones matrix in the $x'y'$ coordinate system is

$$\mathbf{M}' = \mathbf{S}\mathbf{M}\mathbf{S}^{-1} \tag{18.141}$$

In this case

$$\begin{aligned} \mathbf{M}' &= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \end{aligned} \tag{18.142}$$

We can generalize from this example: if coordinate system $x'y'$ is obtained by a counterclockwise rotation by angle α from coordinate system xy, then the transformation matrix is

$$\mathbf{S} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{18.143}$$

The transformation of the Jones matrix from xy to $x'y'$ obeys Eq. (18.141).

(e) Faraday Rotator. A Faraday rotator that rotates the plane of linear polarization in a counterclockwise direction by angle θ has a Jones matrix

$$M_{\text{fv}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{18.144}$$

This brief survey provides the essentials that are needed to describe the polarization change that occurs in a multi-element optical system. The one additional matrix method and graphical techniques that can also be used to provide similar information, in particular the Mueller Calculus and the Poincaré sphere. The interest reader is referred to the specialized literature.

18.17 Problem for Chapter 18

(1) Prove that in a uniaxial crystal the maximum angular separation of 0- and ϵ-rays occurs when the wavevector makes an angle θ with the optic axis that satisfies

$$\theta = \text{arc}\ \tan(n_e/n_0)$$

REFERENCES

R. Clark Jones, "A New Calculus for the Treatment of Optical Systems, I, Desscription and Discussion of the Calculus," J. Opt. Soc. Am., 31, 488-493 (1941); see also R. Clark Jones, J. Opt. Soc. Am., 31, 493-499 (1941); 31, 500-503 (1941); 32, 486-493 (1942); 37, 107-110 (1942); 38, 671-685 (1948); 46, 126-131 (1956).

D.S. Kliger, J.W. Lewis, and C.E. Randall, *Polarized Light in Optics and Spectroscopy*, Academic Press, San Diego, 1990.

W.A. Sherchoff, *Polarized Light: Production and Use*, Harvard University Press, Cambridge, MA, 1962.

Further Reading

R.M.A. Azzam and N.M. Bashara, *Ellipsometry and Polarized Light*, North-Holland, Amsterdam, 1977.

E. Wahlstrom, *Optical Crystallography*, 3rd Edition, Wiley, New York, 1960.

A. Yariv and P. Yeh, *Optical Waves in Crystals*, Wiley, New York, 1984.