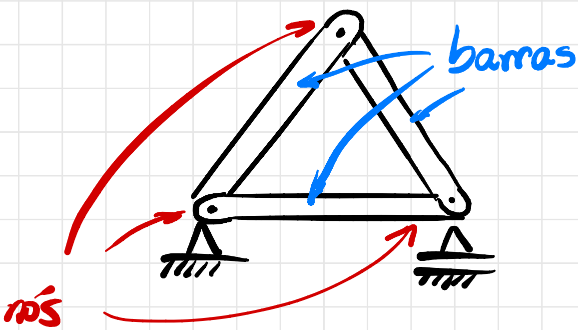
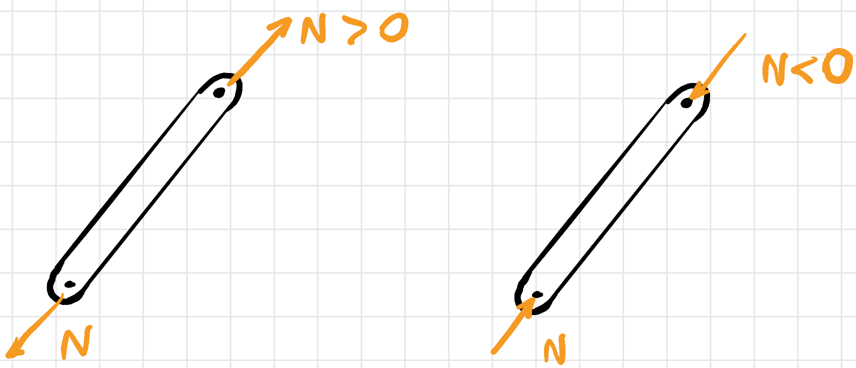
# Treliças

Treliças são caracterizadas por serem conexões articuladas, com carregamentos exclusivamente nos nós. Nós são os pontos de conexão entre as barras.

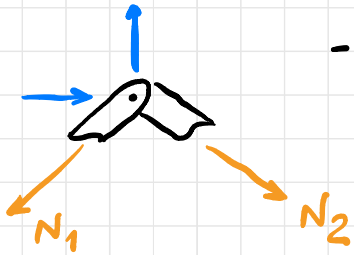



## Convenção de sinais




tração          compressão

# Treliças Planas




- 2 equações de equilíbrio por nó

Logo, para n nós, tem-se 2n equações de equilíbrio

Considerando que existem b barras e v reações vinculares, uma treliça é dita isostática se:

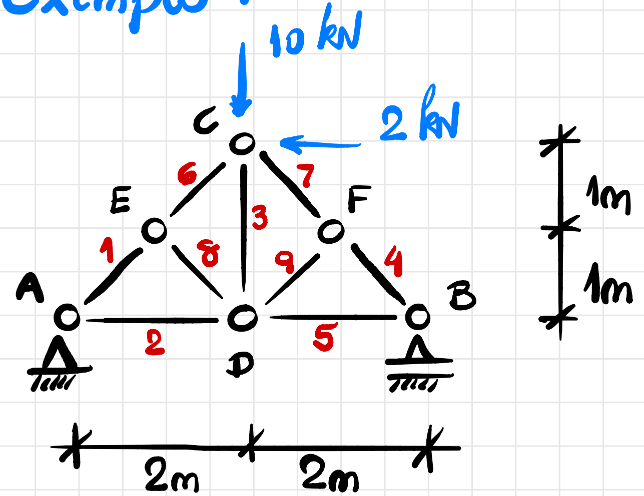
$$\boxed{2n = v + b}$$

Para obter-se as normais nas barras, tem-se dois métodos:

- método dos nós
- método do corte (Corte de Ritter)

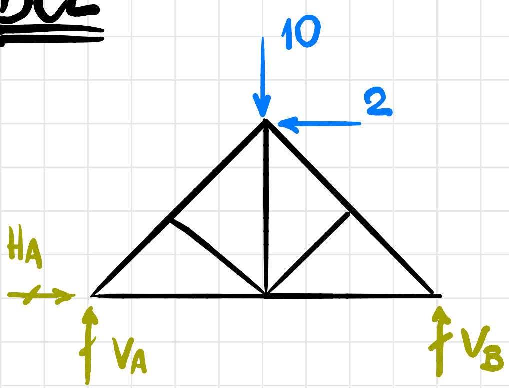
O próximo exemplo mostra o uso do método dos nós.

## Exemplo 1:




| barra | normal [kN] |
|---|---|
| 1 | $-6\sqrt{2}$ |
| 2 | 4 |
| 3 | 0 |
| 4 | $-4\sqrt{2}$ |
| 5 | 4 |
| 6 | $-6\sqrt{2}$ |
| 7 | $-4\sqrt{2}$ |
| 8 | 0 |
| 9 | 0 |

DCL



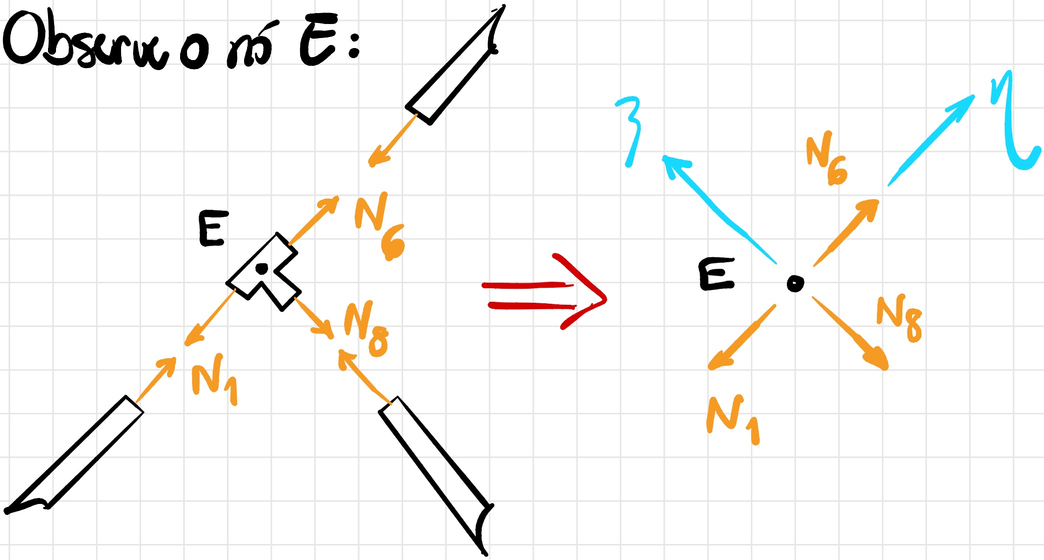

$$\sum F_H = 0: \; H_A - 2 = 0 \Rightarrow \boxed{H_A = 2\,kN}$$

$$\sum F_V = 0: \; V_A + V_B - 10 = 0 \Rightarrow V_A + V_B = 10$$

$$(+)\; \sum M_A = 0: \; -10.2 + 2.2 + V_B.4 = 0 \Rightarrow 4V_B = 16$$

$$\boxed{V_B = 4\,kN} \Rightarrow \boxed{V_A = 6\,kN}$$

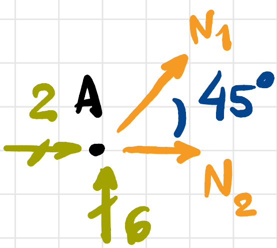
Observo o nó E:




$$\sum F_\eta = 0: \ -N_1 + N_6 = 0 \Rightarrow \underline{N_1 = N_6}$$

$$\sum F_\xi = 0: \ -N_8 = 0 \Rightarrow \underline{N_8 = 0}$$

Nó A:



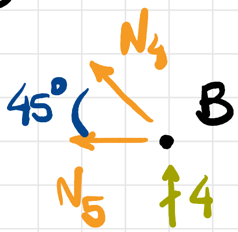

$$\sum F_x = 0: \ 2 + N_1 \cos 45^\circ + N_2 = 0$$

$$\sum F_y = 0: \ N_1 \operatorname{sen} 45^\circ + 6 = 0$$

$$\therefore \ \underline{N_1 = -6\sqrt{2}\ \text{kN}}$$

$$N_2 = -2 - N_1 \cdot 1/\sqrt{2} \Rightarrow \underline{N_2 = 4\ \text{kN}}$$

## Nó B:

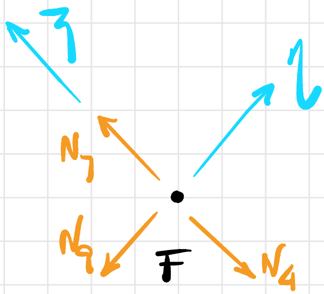



$$\Sigma F_x = 0: \ -N_5 - N_4 \cos 45^\circ = 0$$

$$\Sigma F_y = 0: \ 4 + N_4 \operatorname{sen} 45^\circ = 0$$

$$\underline{N_4 = -4\sqrt{2}\ \text{kN}} \Rightarrow \underline{N_5 = 4\ \text{kN}}$$

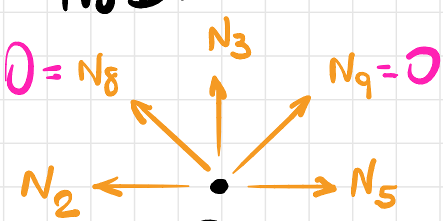
## Nó F:




$$\Sigma F_l = 0: \quad \underline{N_9 = 0}$$

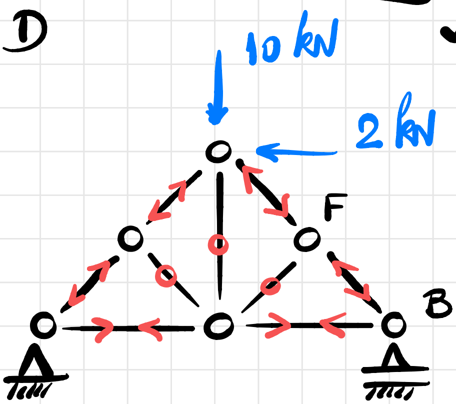
$$\Sigma F_3 = 0: \quad \underline{N_7 = N_4}$$

## Nó D:



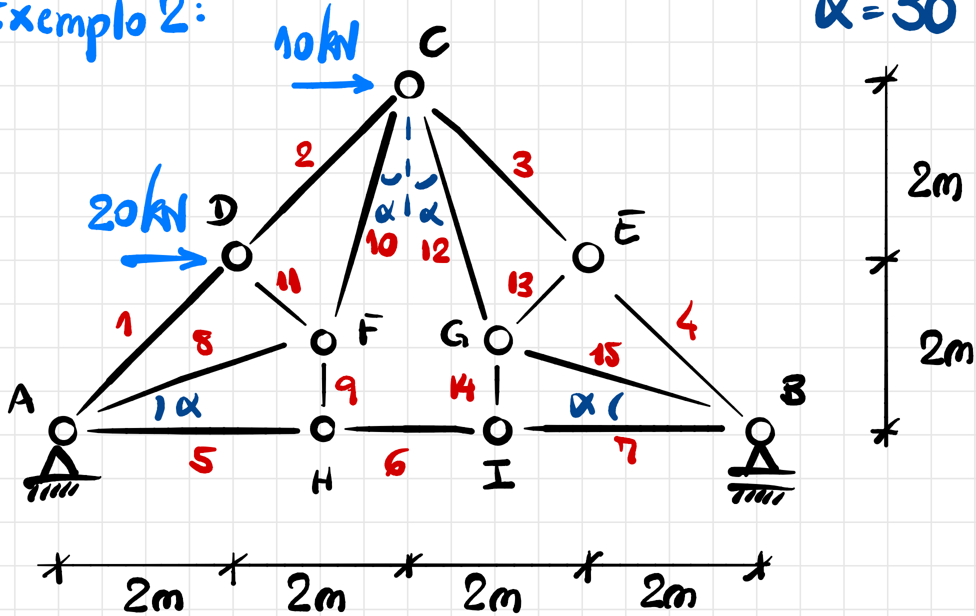

$$\Sigma F_x = 0: \quad \underline{N_2 = N_5}$$

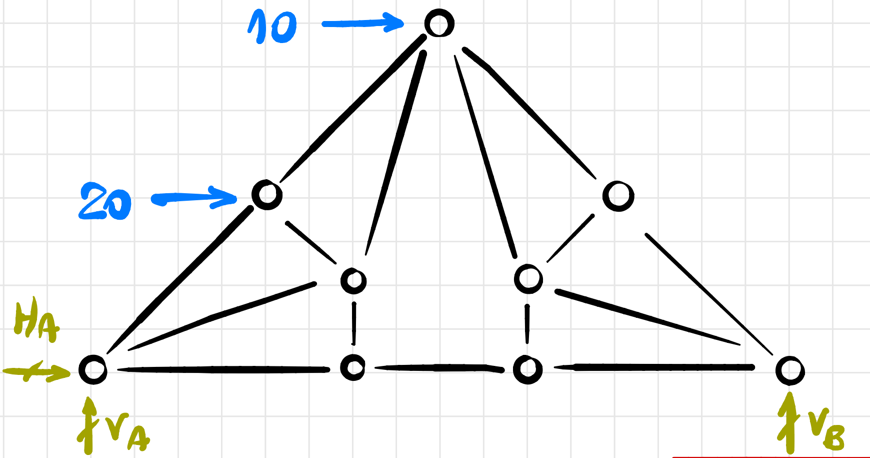
$$\Sigma F_y = 0: \quad \underline{N_3 = 0}$$



## Exemplo 2:

$\alpha = 30^\circ$




$$\sum F_H = 0: \ H_A + 20 + 10 = 0 \Rightarrow \boxed{H_A = -30\,kN}$$

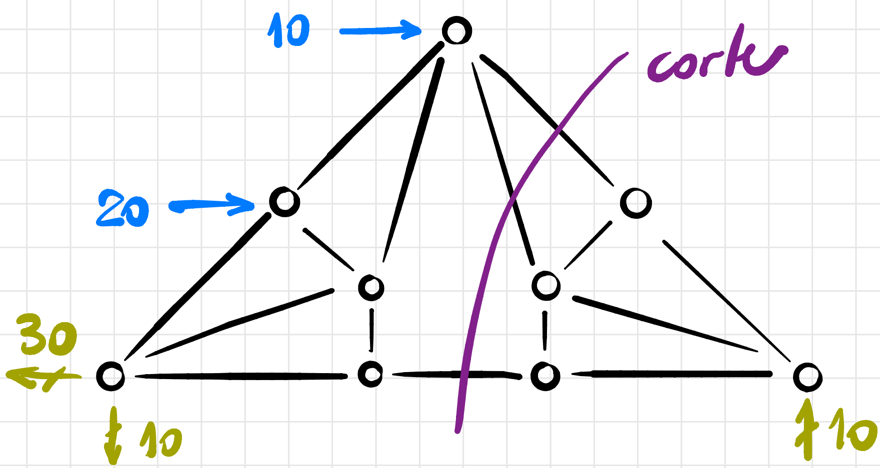
$$\sum F_V = 0: \ V_A + V_B = 0$$

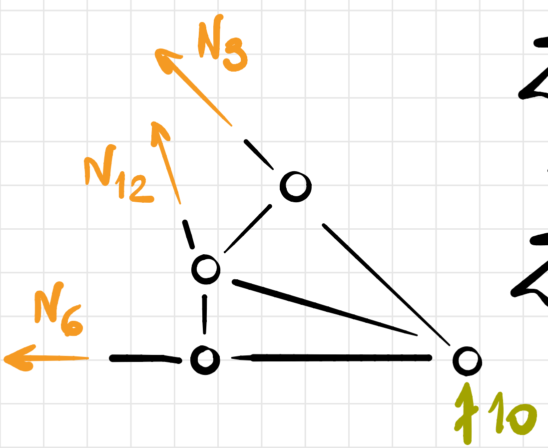
$$(\circlearrowleft +)\ \sum M_A = 0: \ -20.2 - 10.4 + V_B.8 = 0 \Rightarrow \boxed{V_B = 10\,kN}$$

$$V_A = -V_B \Rightarrow \boxed{V_A = -10\,kN}$$

Como fazer? Todos nós possuem pelo menos 3 barras ligadas a ele. ⟹ Corte de Ritter

O corte de Ritter consiste em cortar a estrutura e equilibrar. Para a estrutura do exemplo:




$\sum F_H = 0: -N_6 - N_{12}\,\text{sen}\,\alpha - N_3\,\text{sen}\,45^\circ = 0$

$\sum F_V = 0: 10 + N_{12}\cos\alpha + N_3\cos 45^\circ = 0$

$\curvearrowleft \sum M_C = 0: 10.4 - N_6.4 = 0 \Rightarrow \boxed{N_6 = 10\ kN}$

$$\begin{cases} N_{12} \cdot \dfrac{1}{2} + N_3 \cdot \dfrac{\sqrt{2}}{2} = -10 \\ N_{12} \cdot \dfrac{\sqrt{3}}{2} + N_3 \cdot \dfrac{\sqrt{2}}{2} = -10 \end{cases} \Rightarrow \boxed{N_{12} = 0} \quad \boxed{N_3 = -10\sqrt{2}\ kN}$$

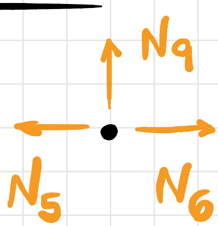
Agora é possível obter as outras forças normais (seja fazendo novos cortes ou usando o método dos nós).

## Diretrizes para usar o método do corte:

- Fazer o corte passar por 3 barras (ou mais barras desde que 3 sejam desconhecidas). Passar por mais leva à sistemas que dependem de uma ou mais variáveis.
- As barras não podem passar pelo mesmo ponto (ou seja, a equação do momento não trivializa).

Com isso, pode-se continuar a solução do exemplo:

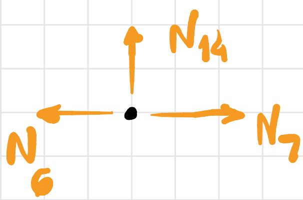
**Nó H:**




$$\sum F_H = 0: \quad \boxed{N_5 = N_6 = 10\,kN}$$

$$\sum F_V = 0: \quad \boxed{N_9 = 0}$$

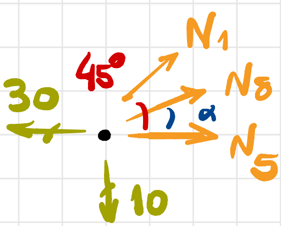
**Nó I:**




$$\sum F_H = 0: \quad \boxed{N_7 = N_6 = 10\,kN}$$

$$\sum F_V = 0: \quad \boxed{N_{14} = 0}$$

**Nó A:**




$$\sum F_H = 0: \ -30 + N_1 . \cos 45° + N_8 \cos 30° + N_5 = 0$$

$$\sum F_V = 0: \ N_1 \operatorname{sen} 45° + N_8 \operatorname{sen} 30° - 10 = 0$$

$$\begin{cases} \frac{1}{\sqrt{2}} N_1 + \frac{\sqrt{3}}{2} N_8 = 20 \\ \frac{1}{\sqrt{2}} N_1 + \frac{1}{2} N_8 = 10 \end{cases}$$

$$\frac{(\sqrt{3}-1)}{2} N_8 = 10 \Rightarrow \boxed{N_8 = \frac{20}{\sqrt{3}-1} = 10(\sqrt{3}+1)\ kN}$$

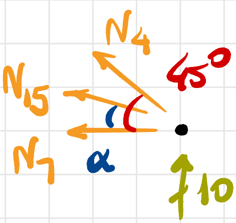
$(27{,}32\ kN)$

$$\frac{1}{\sqrt{2}} N_1 = 20 - \frac{\sqrt{3}}{2} \cdot 10(\sqrt{3}+1) = 20 - 5(3+\sqrt{3})$$

$$\frac{1}{\sqrt{2}} N_1 = 5(1-\sqrt{3}) \Rightarrow \boxed{N_1 = -5\sqrt{2}(\sqrt{3}-1)\ kN}$$

$-5{,}18\ kN$

## Nó B



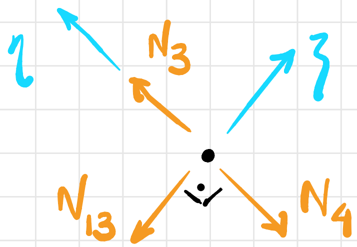

$$\sum F_H = 0 : -N_7 - N_{15}\cos 30^\circ - N_4 \cos 45^\circ = 0$$

$$\sum F_V = 0 : 10 + N_{15}\,\text{sen}\,30^\circ + N_4\,\text{sen}\,45^\circ = 0$$

$$\left\{ \begin{array}{l} \frac{\sqrt{3}}{2} N_{15} + \frac{1}{\sqrt{2}} N_4 = -10 \\ \frac{1}{2} N_{15} + \frac{1}{\sqrt{2}} N_4 = -10 \end{array} \right\} \Rightarrow \boxed{N_{15} = 0}$$

$$\boxed{N_4 = -10\sqrt{2}\ kN}$$

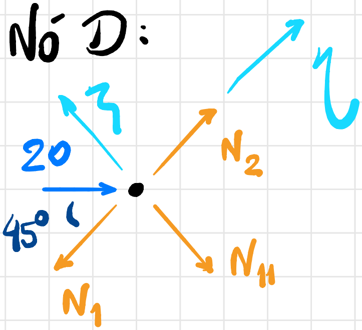
## Nó E




$\sum F_\eta = 0$: $\boxed{N_3 = N_4 = -10\sqrt{2}\ kN}$

$\sum F_\xi = 0$: $\boxed{N_{13} = 0}$

## Nó D:




$\sum F_\eta = 0$: $-N_1 + 20\cos 45° + N_2 = 0$

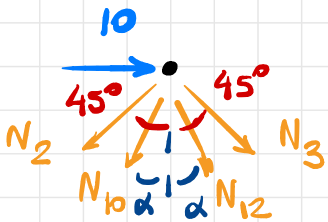
$\sum F_\xi = 0$: $-20\,\text{sen}\,45° - N_{11} = 0$

$\boxed{N_{11} = -10\sqrt{2}\ kN}$

$N_2 = N_1 - 10\sqrt{2} = -5\sqrt{2}(\sqrt{3}-1) - 10\sqrt{2}$

$\therefore \boxed{N_2 = -5\sqrt{2}(\sqrt{3}+1)\ kN}$ $(-19{,}32\ kN)$

## Nó C




$$\sum F_H = 0: \ 10 - N_2 \operatorname{sen} 45^\circ - N_{10} \operatorname{sen} 30^\circ + N_{12} \operatorname{sen} 30^\circ$$

$$+ N_3 \operatorname{sen} 45^\circ = 0$$

$$\sum F_V = 0: -N_2 \cos 45^\circ - N_{10} \cos 30^\circ - N_{12} \cos 30^\circ - N_3 \cos 45^\circ = 0$$

$$\begin{cases} \frac{1}{2} N_{10} - \frac{1}{2} N_{12} = \cancel{10} - \frac{1}{\sqrt{2}}\left[-5\sqrt{2}(\sqrt{3}+1)\right] \cancel{+ \frac{1}{\sqrt{2}}(-10\sqrt{2})} \\ \frac{\sqrt{3}}{2} N_{10} + \frac{\sqrt{3}}{2} N_{12} = -\frac{1}{\sqrt{2}}\left[-5\sqrt{2}(\sqrt{3}+1)\right] - \frac{1}{\sqrt{2}}(-10\sqrt{2}) \end{cases}$$

$$\begin{cases} \frac{1}{2}(N_{10} - N_{12}) = 5\sqrt{3} + 5 \\ \frac{\sqrt{3}}{2}(N_{10} + N_{12}) = 15 + 5\sqrt{3} \end{cases} \Rightarrow \begin{cases} N_{10} - N_{12} = 10 + 10\sqrt{3} \\ N_{10} + N_{12} = 10 + 10\sqrt{3} \end{cases}$$

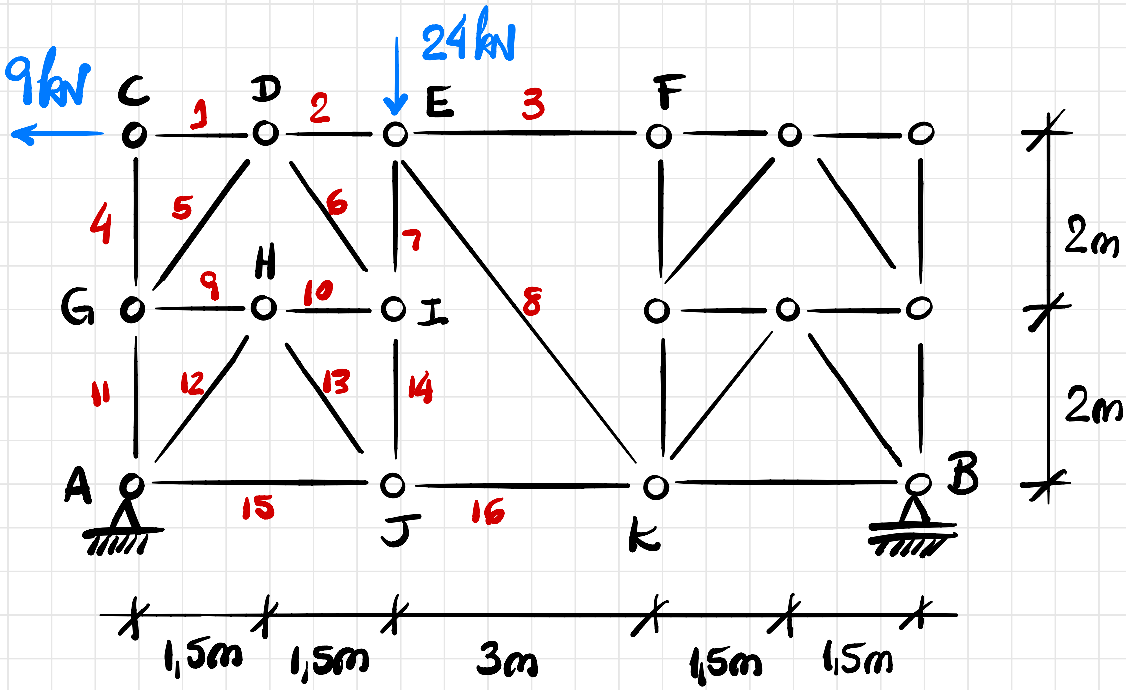
$$\therefore \boxed{N_{10} = 10(\sqrt{3}+1) \text{ kN}} \Rightarrow \boxed{N_{12} = 0}$$

(27,32 kN)

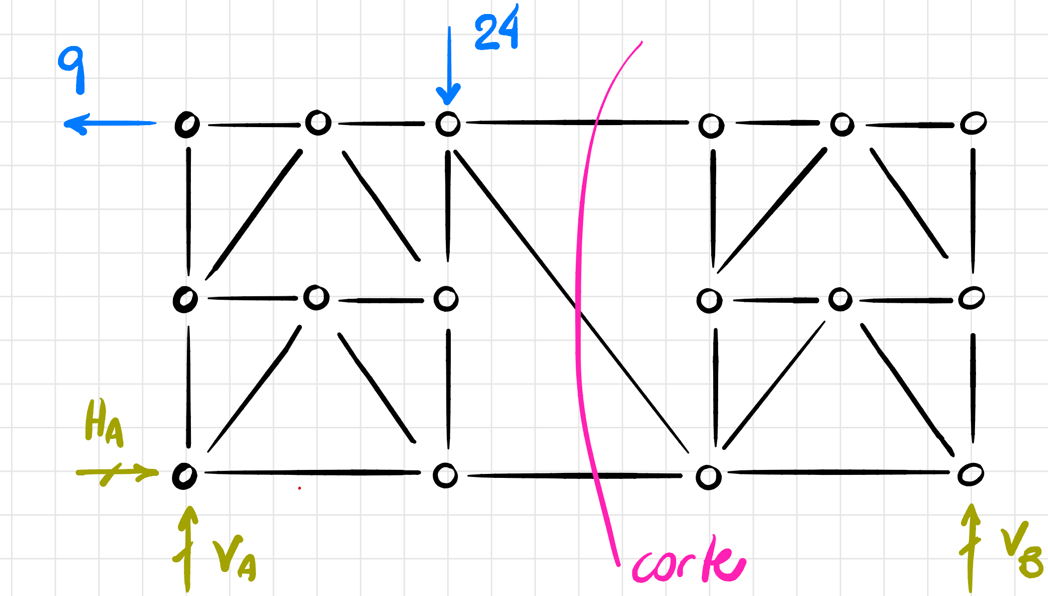
Resumindo :

| Barra | Normal [kN] |
| --- | --- |
| 1 | $-5,18$ |
| 2 | $-19,32$ |
| 3 | $-10\sqrt{2}$ |
| 4 | $-10\sqrt{2}$ |
| 5 | 10 |
| 6 | 10 |
| 7 | 10 |
| 8 | 27,32 |
| 9 | 0 |
| 10 | 27,32 |
| 11 | $-10\sqrt{2}$ |
| 12 | 0 |
| 13 | 0 |
| 14 | 0 |
| 15 | 0 |

Exemplo: Obter as forças normais nas barras 2,3,7 e 8:




| barra | normal [kN] |
|---|---|
| 2 | -6 |
| 3 | -3 |
| 7 | -20 |
| 8 | -5 |



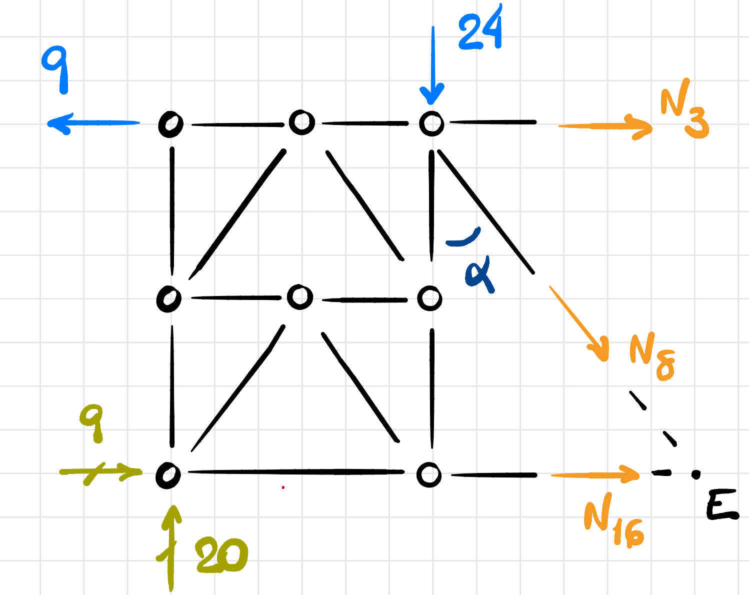

$$\sum F_H = 0: \quad \boxed{H_A = 9\,kN}$$

$$\sum F_V = 0: \quad V_A + V_B = 24$$

$$\circlearrowleft\!+ \sum M_A = 0: \quad 9.4 - 24.3 + V_B.9 = 0$$

$$\therefore \boxed{V_B = 4\,kN} \Rightarrow \boxed{V_A = 20\,kN}$$

Fazendo o corte:




$\operatorname{sen}\alpha = 3/5$

$\cos\alpha = 4/5$

$$\sum F_H = 0:\ 9 - 9 + N_{16} + N_8 \operatorname{sen}\alpha + N_3 = 0$$

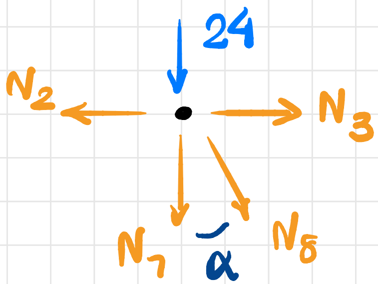
$$\sum F_V = 0:\ 20 - 24 - N_8 \cos\alpha = 0 \quad \text{I}$$

$$(\circlearrowleft +)\ \sum M_E = 0:\ -20 \cdot 6 + 9 \cdot 4 + 24 \cdot 3 - N_3 \cdot 4 = 0 \quad \text{II}$$

De I: $\frac{4}{5} N_8 = -4 \Rightarrow \boxed{N_8 = -5\ \text{kN}}$

De II: $4N_3 = -120 + 36 + 72 \Rightarrow \boxed{N_3 = -3\ \text{kN}}$

Fazendo o equilíbrio no nó E:




$$\sum F_H = 0: \; -N_2 + N_3 + N_8 \operatorname{sen}\alpha = 0$$

$$N_2 = N_3 + N_8 \operatorname{sen}\alpha = -3 - 5\left(3/5\right)$$

$$\boxed{N_2 = -6\,kN}$$

$$\sum F_V = 0: \; -24 - N_7 - N_8 \cos\alpha = 0$$

$$N_7 = -24 - N_8 \cos\alpha = -24 - (-5)\left(4/5\right)$$

$$\therefore \; \boxed{N_7 = -20\,kN}$$