3.3 Exemplos

13/4/23

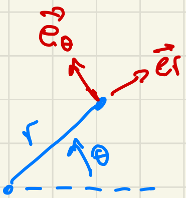
i) Partícula livre: $T = \frac{1}{2} m \vec{v}^{\,2} = \frac{1}{2} m \left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 \right)$

with $q_1 = x \quad q_2 = y \quad q_3 = z$

$L = T \Rightarrow$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] - \frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(m \dot{q}_k \right) \Rightarrow m \ddot{q}_k = 0 \qquad \underline{\text{OK}} \checkmark$$

ii) Partícula num plano c/ potencial $U(r)$ em coordenadas polares:



$$\vec{v} = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta$$

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) \quad ; \quad L = T - U$$

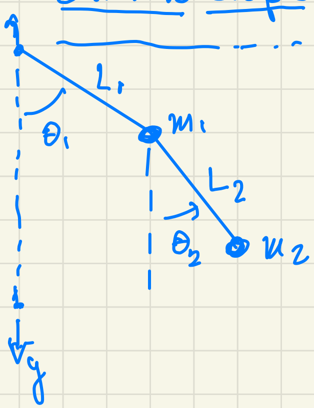
$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{r}}\right] - \frac{\partial L}{\partial r} = 0 = \frac{d}{dt}\left(m\dot{r} \right) = m r \dot{\theta}^2 - \frac{\partial U}{\partial r} \Rightarrow m\left(\ddot{r} - r\dot{\theta}^2 \right) = -\frac{\partial U}{\partial r}$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}\left[- m r^2 \dot{\theta} \right] = 0 \Rightarrow m\left(r^2 \ddot{\theta} + 2\dot{r}\dot{\theta} \right) = 0$$

$\underline{\text{OK}} \checkmark$

iii) Pêndulo duplo:

$$U = -m_1 g L_1 \cos\theta_1 - m_2 g \left(L_1 \cos\theta_1 + L_2 \cos\theta_2 \right)$$



Para o ponto 1: $T_1 = \frac{m_1}{2} L_1^2 \dot{\theta}_1^2$

Para o ponto 2: $T_2 = \frac{m}{2}\left(\dot{x}_2^2 + \dot{y}_2^2 \right)$

$$x_2 = L_1 \sin\theta_1 + L_2 \sin\theta_2$$

$$y_2 = L_1 \cos\theta_1 + L_2 \cos\theta_2$$

$$\dot{x}_2 = L_1 \cos\theta_1 \, \dot{\theta}_1 + L_2 \cos\theta_2 \, \dot{\theta}_2$$

$$\dot{y}_2 = -L_1 \sin\theta_1 \, \dot{\theta}_1 - L_2 \sin\theta_2 \, \dot{\theta}_2$$

$$T_2 = \frac{m_2}{2}\left[L_1^2 \dot{\theta}_1^2 + L_2^2 \dot{\theta}_2^2 + 2 L_1 L_2 \underbrace{\left(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2\right)}_{\cos(\theta_1-\theta_2)} \dot{\theta}_1 \dot{\theta}_2 \right]$$

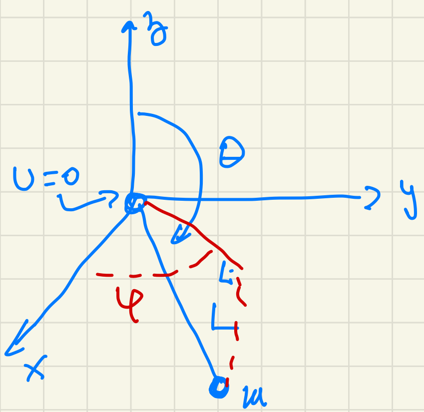
$$= \frac{m_2}{2}\left[L_1^2 \dot{\theta}_1^2 + L_2^2 \dot{\theta}_2^2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1-\theta_2) \right]$$

$$\Rightarrow L = \frac{m_1+m_2}{2} L_1^2 \dot{\theta}_1^2 + \frac{m_2}{2} L_2^2 \dot{\theta}_2^2 + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1-\theta_2)$$

$$+ (m_1+m_2) L_1 g \cos\theta_1 + m_2 L_2 g \cos\theta_2 \qquad \text{¡Ufa!}$$

<u>Exercício:</u> Obtenha as equações de movimento do sistema!

IV) <u>Pêndulo esférico</u>



Posição da massa m determinada por θ e φ do coordenadas esféricas: $q_1 = \theta$ $q_2 = \varphi$

$$\vec{v} = L\dot{\theta}\,\vec{e}_\theta + L\sin\theta\,\dot{\varphi}\,\vec{e}_\varphi$$

$$L = T - U$$

$$L = \frac{m}{2}\left(L^2 \dot{\theta}^2 + L^2 \sin^2\theta \, \dot{\varphi}^2 \right) - mgL\cos\theta$$

✓

Usemos Euler-Lagrange

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] - \frac{\partial L}{\partial q_k} = 0$$

$$k=1 \Rightarrow mL^2\ddot{\theta} - mL^2\sin\theta\cos\theta\,\dot{\varphi}^2 - mgL\,\mathrm{sen}\,\theta = 0$$

$$k=2 \qquad \frac{d}{dt}\left[mL^2\sin^2\theta\,\dot{\varphi}\right] = 0$$

Logo temos uma quantidade conservada

$$J = mL^2\sin^2\theta\,\dot{\varphi}$$

¿Como você interpreta?

Com isso

$$mL^2\ddot{\theta} = \frac{1}{mL^2}\frac{\cos\theta}{\sin^3\theta}J^2 + mgL\,\mathrm{sen}\,\theta$$

$$mL^2\ddot{\theta} = -\frac{\partial V_{ef}}{\partial\theta} \qquad \text{c/} \quad V_{ef} = \frac{1}{2mL^2}\frac{J^2}{\sin^2\theta} + mgL\cos\theta$$

Note que a energia é conservada (multiplicando por $\dot{\theta}$)

$$E = \frac{mL^2}{2}\dot{\theta}^2 + V_{ef}(\theta)$$

Discussão: Mínimo do potencial:

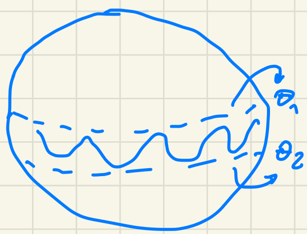
$$\frac{\partial V_{ef}}{\partial\theta} = 0 \Rightarrow -\frac{1}{mL^2}J^2\frac{\cos\theta_0}{\sin^3\theta_0} - mgL\sin\theta_0 = 0 \Rightarrow \frac{J^2}{m^2gL^3}\cos\theta_0 = -\sin^4\theta_0$$

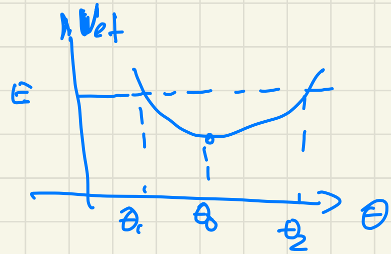
$\Rightarrow \theta_0 > \pi/2$ para conseguir a igualdade. Note J crescendo leva a $\theta_0 \to \pi/2$.

Para $\theta = \theta_0$ o movimento é circular. Note que 'sabemos' obter o movimento dados E e J:

$$dt = \sqrt{\frac{mL^2}{2}} \frac{d\theta}{\sqrt{E - V_{ef}(\theta)}} \quad \text{e} \quad d\varphi = \frac{J}{mL^2 \sin\theta(t)} dt.$$

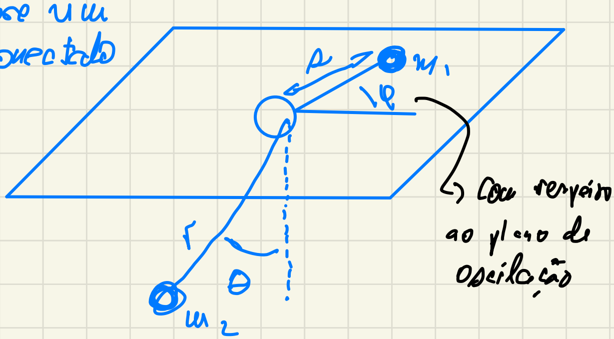
O movimento tem a forma





V) Massa conectada a um pêndulo

Corpo de massa m_1 está sobre um plano sem atrito. Ele está conectado a pêndulo plano como mostra a figura. O fio é ideal.



Vínculos:

1) m_1 no plano horizontal
2) m_2 oscila num plano
3) $r + s = \ell \Rightarrow \dot{r} = -\dot{s}$

Existem apenas 3 graus de liberdade: r, θ, φ.

$$T = \frac{1}{2} m_1 (\dot{s}^2 + s^2\dot{\varphi}^2) + \frac{m_2}{2}(\dot{r}^2 + r^2\dot{\theta}^2) \quad \text{usando o vínculo 3}$$

$$T = \frac{m_1}{2}\left(\dot{r}^2 + (l-r)^2\dot{\varphi}^2\right) + \frac{m_2}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Colocando o zero do potencial gravitacional no plano da mesa

$$U = 0 - m_2 g r\cos\theta$$

Logo: $L = \frac{m_1}{2}\left(\dot{r}^2 + (l-r)^2\dot{\varphi}^2\right) + \frac{m_2}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + m_2 g r\cos\theta$

Eq's de Euler Lagrange:

$k = r$: $$m_1(r-l)\dot{\varphi}^2 + m_2 r\dot{\theta}^2 + m_2 g r\cos\theta =$$

$$= \frac{d}{dt}\left(m_1\dot{r} + m_2\dot{r}\right) \Rightarrow (m_1+m_2)\ddot{r} = m_1(r-l)\dot{\varphi}^2 + m_2 r\dot{\theta}^2 + m_2 g r\cos\theta \quad (1)$$

$k = \varphi$: $$0 = \frac{d}{dt}\left(m_1(l-r)^2\dot{\varphi}\right) \quad (2)$$

$k = \theta$ $$-m_2 g r\sin\theta = \frac{d}{dt}\left(m_2 r^2\dot{\theta}\right)$$

$$\Rightarrow m_2\left(r^2\ddot{\theta} + 2\dot{r}\dot{\theta}r\right) = -m_2 g r\sin\theta \quad (3)$$

A eq. (2) é a conservação do momento angular perpendicular ao plano.