Como obter geodésicas com cálculo variacional?

1. Escrever o elemento de linha, ds^2, em coordenadas adequadas;
2. Integrar ds, obtendo uma integral (funcional);
3. Aplicar Euler-Lagrange ao integrando do funcional;
4. Resolver as equações provenientes de 3.

Exemplo 1. $\mathbb{R}^2$ (plano cartesiano)

1. $ds^2 = dx^2 + dy^2 \Rightarrow ds = \sqrt{dx^2+dy^2}$ → integramos isso

2. $\int ds = \int \sqrt{dx^2+dy^2} = \int \sqrt{dx^2\left(1+\frac{dy^2}{dx^2}\right)} = \int_{x_1}^{x_2} \underbrace{\sqrt{1+(y'(x))^2}}_{\doteq f(y,y',x)}\, dx = S[y]$ → minimizamos isso

3. $\cancelto{0}{\frac{\partial f}{\partial y}} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \Rightarrow \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'}} = C_1$ → agora resolvemos essa equação

4. $\Rightarrow y'^2 = C_1^2(1+y'^2) \Rightarrow y'^2 = \frac{C_1^2}{1-C_1^2} \equiv a^2 \Rightarrow \frac{dy}{dx} = a \Rightarrow \boxed{y(x) = ax+b}$

Exemplo 2 $\mathbb{R}^2$ (polares) → resultado que devemos esperar: uma reparametrização da reta em (r,θ):

1. $ds^2 = dr^2 + r^2 d\theta^2 \Rightarrow ds = \sqrt{dr^2+r^2d\theta^2}$

2. $\int ds = \int_{r_1}^{r_2} \underbrace{\sqrt{1+r^2(\theta'(r))^2}}_{f(\theta,\theta',r)}\, dr$

3. $\cancelto{0}{\frac{\partial f}{\partial \theta}} - \frac{d}{dr}\left(\frac{\partial f}{\partial \theta'}\right) = 0 \Rightarrow \frac{r^2\theta'}{\sqrt{1+r^2\theta'^2}} = C \Rightarrow \theta' = \left[\frac{C^2}{(r^4-r^2C^2)}\right]^{1/2} \Rightarrow$

4. $\Rightarrow \theta = -\sin^{-1}\left(\frac{C}{r}\right) + k \Rightarrow r = \frac{C}{\sin(k-\theta)} = \frac{b}{\sin\theta - a\cos\theta}$ → mesmos "a" e "b" do exemplo anterior

Exemplo 3 S^2 (esfera de raio fixo $= R$)

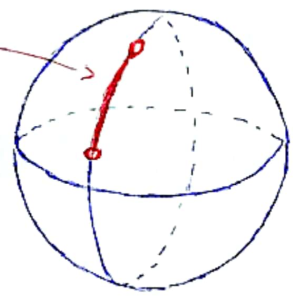
① $ds^2 = \cancelto{0}{dr^2} + R^2(d\theta^2 + \text{sen}^2\theta\, d\phi^2) \Rightarrow ds = R\sqrt{d\theta^2 + \text{sen}^2\theta\, d\phi^2}$

② $\displaystyle\int ds = R\int_{\theta_1}^{\theta_2} d\theta\, \underbrace{\sqrt{1+\text{sen}^2\theta\,(\phi'(\theta))^2}}_{f(\phi,\phi',\theta)} \equiv S[\phi]$

③ $\displaystyle\cancelto{0}{\frac{\partial f}{\partial \phi}} - \frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right) = 0 \Rightarrow \frac{\text{sen}^2\theta\ \phi'(\theta)}{\left\{1+\text{sen}^2\theta[\phi'(\theta)]^2\right\}^{1/2}} = C$

④ $\displaystyle\phi' = \frac{C}{\text{sen}\theta\sqrt{\text{sen}^2\theta - C^2}} \overset{\text{integrando}}{\Rightarrow} \phi = -\cos^{-1}\left(\frac{C\cot\theta}{\sqrt{1-C^2}}\right) + k \Rightarrow$

$\Rightarrow C\cot\theta = \overset{\to \cos k}{A}\cos\phi + \overset{\to\,\text{sen}\,k}{B}\,\text{sen}\phi$

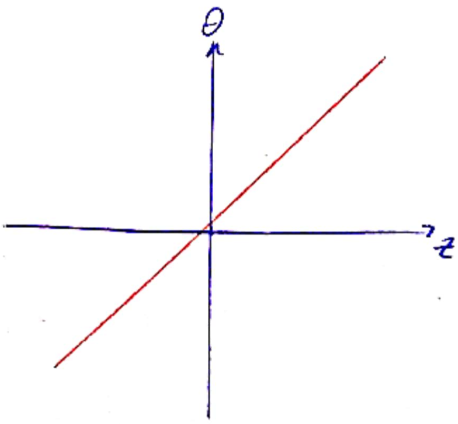


Exemplo 4 cilindro

① $ds^2 = dz^2 + R^2 d\theta^2 \Rightarrow ds = \sqrt{dz^2 + R^2 d\theta^2}$

② $\displaystyle\int ds = S[\theta] = \int_{z_1}^{z_2} dz\, \underbrace{\sqrt{R^2(\theta'(z))^2 + 1}}_{f(\theta,\theta',z)}$

③ $\displaystyle\cancelto{0}{\frac{\partial f}{\partial \theta}} - \frac{d}{dz}\underbrace{\left(\frac{\partial f}{\partial \theta'}\right)}_{=C} = 0 \Rightarrow \theta' = \underbrace{\left[\frac{C^2}{(R^4 - R^2C^2)}\right]}_{\text{constante}}$ ④ $\Rightarrow \theta = az + b$



$\Rightarrow$