# Fourier Transform: part 1

## SCC0251/5830 – Image Processing

Prof. Moacir Ponti

Instituto de Ciências Matemáticas e de Computação – USP

2021/1

# Agenda

1. Fundamental concepts
   - Representation of functions using points and coefficients
   - Fourier Series and the complex exponential
2. Fourier Transform
   - Motivation, algorithm, examples

# Introduction

Mathematical transformations are used to obtain information not available (or not visible) directly in the original data.

Can be seen as a map between different domains. Although the values in different domains are different, they represent the same data.

# Introduction: same information, different value

$\Leftrightarrow$ (-22.00257, -47.89855) $\Leftrightarrow$

# Introduction: same information, different value

Av.Trabalhador
Saocarlense, 400

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# Introduction: same information, different value

Av.Trabalhador
Saocarlense, 400

⇔ (-22.00257, -47.89855) ⇔

USP São Carlos main entrance

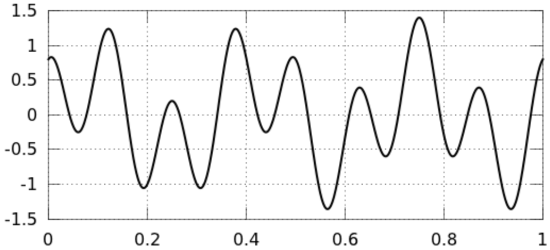
# Introduction

Mathematically a signal/image can be seen as a **function**

There are important (and often non-obvious) information about the function that is not trivial to grasp in their original domains.

# Introduction

- A $1-d$ signal is often represented in the **time domain** in its original form
  - plots are often in terms of time-amplitude



# Introduction

- An image ($2 - d$ signal) is represented in the **space domain**
  - display is in terms of space-amplitude (or space-intensity)

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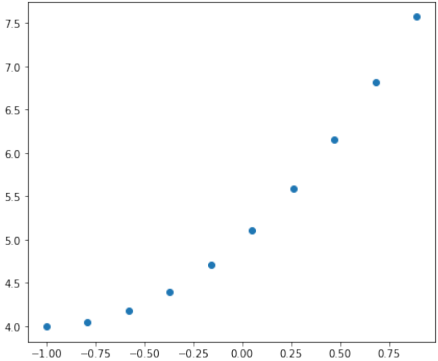
# Representations of a function

Given $n = 10$ unique points:

```
 x       f(x)
-1.0 ,  4.0
-0.79 ,  4.04
-0.58 ,  4.18
-0.37 ,  4.4
-0.16 ,  4.71
0.05 ,  5.1
0.26 ,  5.59
0.47 ,  6.16
0.68 ,  6.82
0.89 ,  7.57
```




Can I represent it using a different set of values?

# Representations of a function

Let us define that

- it is a polynomial of degree 2

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Let us define that

- it is a polynomial of degree 2
- 3 values represent this function
- (since a polynomial of degree $n-1$ has $n$ coefficients!)
- But how to obtain/compute such representation?

# Representations of a function

Build and solve a linear system with the following matrices:

$$A = \begin{bmatrix} x_1^N & x_1^{N-1} & \dots & 1 \\ x_2^N & x_2^{N-1} & \dots & 1 \\ \dots & & & \\ x_n^N & x_n^{N-1} & \dots & 1 \end{bmatrix} \quad Y = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \dots \\ f(x_n) \end{bmatrix}$$

With the coefficients given by:

$$C = (A^T A)^{-1}(A^T Y)$$

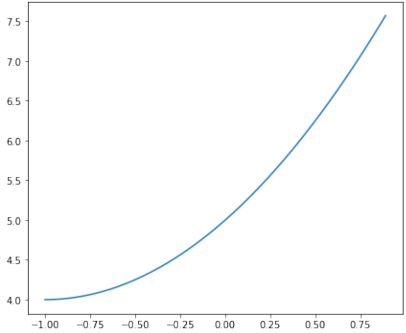
# Representations of a function

In our example

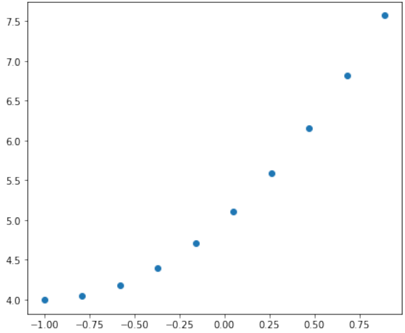
$$A = \begin{bmatrix} 1.0 & -1.0 & 1.0 \\ 0.62 & -0.79 & 1.0 \\ 0.34 & -0.58 & 1.0 \\ 0.14 & -0.37 & 1.0 \\ 0.03 & -0.16 & 1.0 \\ 0.0 & 0.05 & 1.0 \\ 0.07 & 0.26 & 1.0 \\ 0.22 & 0.47 & 1.0 \\ 0.46 & 0.68 & 1.0 \\ 0.79 & 0.89 & 1.0 \end{bmatrix} \qquad Y = \begin{bmatrix} 4.0 \\ 4.04 \\ 4.18 \\ 4.4 \\ 4.71 \\ 5.1 \\ 5.59 \\ 6.16 \\ 6.82 \\ 7.57 \end{bmatrix}$$

$$C = \begin{bmatrix} 1.0 \\ 2.0 \\ 5.0 \end{bmatrix}$$

# Representations of a function




$f(x) = 5 + 2x + x^2$




$f(x) = 4.0, 4.04, 4.18, 4.4, 4.71$
$5.1, 5.59, 6.16, 6.82, 7.57$

# Representations

Representation using coefficients

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}$$

Representation using points

$$f(x) = f(x_1), f(x_2), \cdots f(x_n)$$

the Fourier Transform will take as

**input** points or sampled intervals of a function (i.e. in the way they are acquired),

**output** the coefficients that define the function (its fundamental components).

# Synthesis and Analysis

Two aspects of Fourier Transform:

- **Analysis**: divide the signal (or function) by defining it via simpler parts.
- **Synthesis**: reconstruct the signal (or function) from its parts.

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- **Analysis**: divide the signal (or function) by defining it via simpler parts.
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*Both can be achieved via **linear** operations,*
*i.e. series and integrals.*

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# Fourier Series

- Jean-Baptiste Fourier, 1822, studying heat transfer, claimed that a function of a single variable could be expanded in terms of a series of sinusoids of multiples of the variable.
- After Lagrange and Dirichlet studies using this expansion, it was refered to as **Fourier Series**.

# Fourier Series and Periodicity

*Fourier Series are associated to the mathematical analysis of periodic patterns.*

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## Periodicity

- **Time**: harmonic movement (e.g. of a string)
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  - e.g. heat distribution in a circular object: the temperature repeat itself in cycles.
  - that is why Fourier Analysis is often associated with symmetry.

# Fourier Series and Periodicity

Mathematical descriptors of periodicity

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**Ideas**

- **Time**: frequency — number of pattern repetitions along the time (e.g. 1 second)
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Mathematical descriptors of periodicity

**Ideas**

- **Time**: frequency — number of pattern repetitions along the time (e.g. 1 second)
- **Space**: period (wavelength) — size of the repeating pattern

- In some cases time and space are involved at the same time — e.g. wave movement
  - If we fix the position (in space), we can measure frequency (distribution of the pattern in time)
  - By fixing an instant (in time) we can measure the size (distribution of the pattern in space).

# Fourier Series and Periodicity

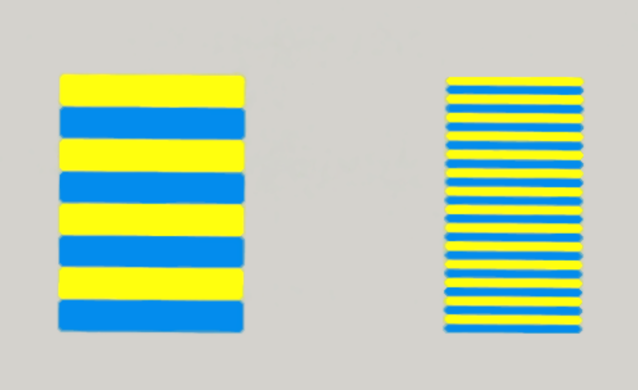
Relationship space (wavelength) and time (frequency)

- $v$ is the velocity (rate) of the wave and $F$ its frequency then:
- $\lambda = v \cdot \frac{1}{F}$, considering one complete wave in $\frac{1}{F}$
- or $F \cdot \lambda = v$

There is a reciprocal relationship between wavelength and frequency

# Wavelength vs Frequency

Let a sequence yellow-blue define the wavelength, then:



# Fourier Series

- There are mathematical functions for which

$$f(t+T) = f(t) \tag{1}$$
$$f(t+nT) = f(t), n = 0, \pm 1, \pm 2, \cdots \tag{2}$$
$$\tag{3}$$

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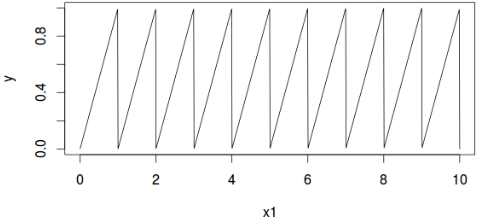
- Why not others: square wave, other periodic signals?



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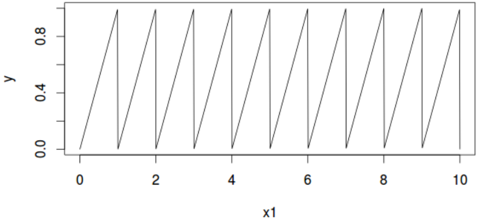

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  - **circle**: $\cos t$ is coordinate $x$ and sin is coordinate $y$ of a unitary circle.

$$\cos(t + 2\pi n) = \cos(t) \tag{4}$$

$$\sin(t + 2\pi n) = \sin(t) \tag{5}$$

# Fourier Series and Periodicity

Can we write an arbitrary function in terms of sinusoids?

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Can we write an arbitrary function in terms of sinusoids?

Must this function we want to write be periodic?

# Fourier Series and Periodicity

Important remarks:

- Functions with the **simplest periodic behaviour**: sines and cosines;

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# Fourier Series

Given a periodic function $f(t)$ of a continuous variable $t$ with period $T$:

$$f(t) = \sum_{n=0}^{\infty} a_n \cos 2\pi nt + \sum_{n=1}^{\infty} b_n \sin 2\pi nt = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t} \tag{6}$$

# [Complex Numbers and Euler's formula]

- A complex number $C$ is defined by

$$c = R + jI, \tag{7}$$

  $R$ and $I$ are real numbers and $j$ is the imaginary $j = \sqrt{-1}$
- Geometric interpretation: a complex Cartesian plane with real axis $R$, and imaginary axis $I$.

# [Complex Numbers and Euler's formula]

- In polar coordinates, we have:
-

$$c = |c|(\cos\omega + j\sin\omega) = a\cos(\omega) + jb\sin(\omega), \quad (8)$$

$|c|$ is the vector size extending from the origin of the complex plane to the point $(R, I)$; and $\omega$ is the angle between the vector and the real axis.

# [Complex Numbers and Euler's formula]

- Euler's formula relates the complex sum of sine and cosine using a complex exponential:

$$e^{j\omega} = \cos\omega + j\sin\omega, \tag{9}$$

we can substitute so that

$$X = |c|(\cos\omega + j\sin\omega), \tag{10}$$

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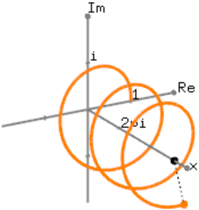
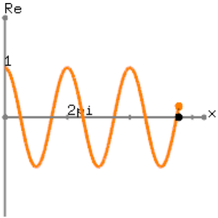
- Example: $x = 1 + j2$
  - in polar coordinates: $\sqrt{5}e^{j\omega}$, with $\omega = 64,4$

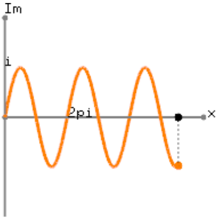
# [Complex Exponential]




$e^{jx}$




$\cos x$




$\sin x$

Thanks to Jim Clay

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# Fourier Transform – interpretation

- A signal can be represented by the independent sum of each number in each point in time: $f(t) = f(t_1) + f(t_2), \ldots$
- instead of summing points, we are going to sum functions cosine and sine with different coefficients.

# Fourier Transform

The Fourier series allows writing a function by a discrete sum of complex exponentials with different frequencies.

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Fourier Transform is the evaluation, for each frequency $\omega$, of its coefficient $c_\omega$

$$F(\omega) = \sum_{t=-\infty}^{\infty} f(t)e^{-j\omega t}$$

# Fourier Transform

- the **functions** cover **all the input axis**:

$$c_\omega e^{j\omega t} = a_\omega \cos(\omega t) + j b_\omega \sin(\omega t)$$

When summing all possible sinusoids with different frequencies, we have a series of values:

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# Fourier Transform

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When summing all possible sinusoids with different frequencies, we have a series of values:

  - $\omega_1$ and its coefficients $a_1, b_1$
  - $\omega_2$ and its coefficients $a_2, b_2$
  - $\omega_3$ and its coefficients $a_3, b_3$
  - ...

# Fourier Transform

- Fourier Transform takes *(a given signal)* from **time/space** domain to the **frequency** domain (per seconds / per measure).
  - signals (time): $f(t)$ to $F(\omega)$
  - images (space): $f(x, y)$ to $F(u, v)$

# Fourier Transform

When plotting the function in the Fourier domain, we use, for each frequency a complex exponential with:

# Fourier Transform

When plotting the function in the Fourier domain, we use, for each frequency a complex exponential with:

- the relative amplitude of the cosine (real part) and of the sine (imaginary part) as a function of $\omega$,
- the representation of the signal in the **frequency domain**:
  - $a_n(\omega) = Re(F(\omega))$
  - $b_n(\omega) = Im(F(\omega))$

# Discrete Fourier Transform

$$F(\omega) = \sum_{t=0}^{N-1} f(t) e^{-j\omega t}$$

- evaluating $F(\omega)$ for different frequencies, we obtain the **amplitudes of** cosines (real part) and sines (imaginary part) so that we can reconstruct $f(t)$ if needed.

# Motivation

- The universe has a lot of periodic phenomena
- Humans often observe time and space phenomena

*the energy propagation of the electromagnetic spectrum is described in waves, including the light that generate images.*

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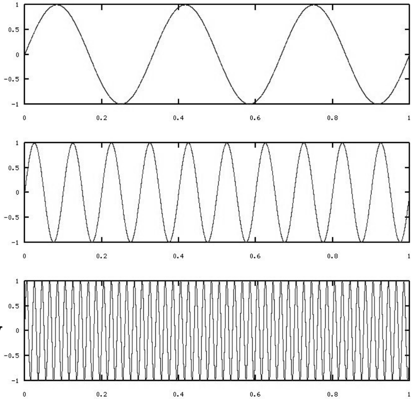
*the energy propagation of the electromagnetic spectrum is described in waves, including the light that generate images.*

- Also, differential equations are key to many applications in science and engineering. Taking signals to the frequency domain makes it easier to solve many problems.

# Information in Frequency

- There is relevant (and often non-obvious) information about the signal in its frequency content.
- It indicates how the amplitude of the signal changes along time or space
  - e.g. is it dominated by abrupt or smooth changes?




```
N = 500  # sample points
Fs = 1.0/1000.0  # frequency of sampling
x = np.linspace(0.0, N*Fs, N) # sampling v

# signal with frequency Fr
Fr = 10
y = np.sin(Fr*2*np.pi*x)
```

# Fourier Transform: translated

$$F(\omega) = \sum_{t=0}^{n-1} f(t) e^{-j\omega t} dt$$

```
for i = 0 to n − 1 do
    multiply: f(t) × e^{−jω_i t},


end for
```

# Fourier Transform: translated

$$F(\omega) = \sum_{t=0}^{n-1} f(t) e^{-j\omega t} dt$$

**for** $i = 0$ to $n - 1$ **do**

    multiply: $f(t) \times e^{-j\omega_i t}$, or $f(t) \times [\cos(\omega_i t) + j \sin(\omega_i t)]$.

**end for**

# Fourier Transform: translated

$$F(\omega) = \sum_{t=0}^{n-1} f(t) e^{-j\omega t} dt = \sum_{t=0}^{n-1} f(t) \cos(\omega t) dt + \sum_{t=0}^{n-1} f(t) j \sin(\omega t) dt$$

1: **for** $i = 0$ to $n - 1$ **do**
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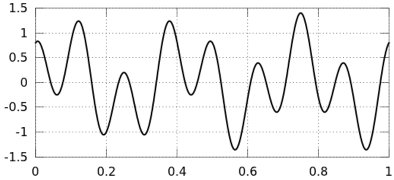
3: sum (integrate) for all $t$ getting coefficients $a$ (real) / $b$ (imag)
4: $F(\omega_i) = a_{\omega_i} + jb_{\omega_i}$
5: **end for**

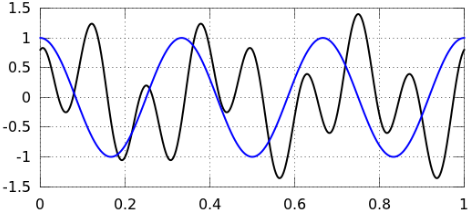
# Frequency analysis

Signal obtained by summing a sine with amplitude 0.6 and frequency 3Hz and a cosine with amplitude 0.8 frequency 8Hz:

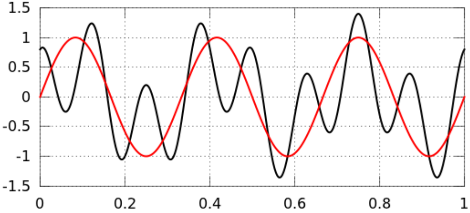
```
f = 0.6*sin((2*pi) * 3 * t) + 0.8*cos((2*pi) * 8 * t)
```



# How the sum behaves in each frequency



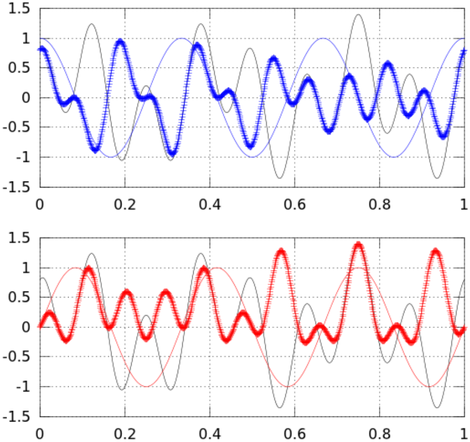

function overlayed with the real part (cosine) in frequency 3Hz




function overlayed with the imaginary part (sine) in frequency 3Hz

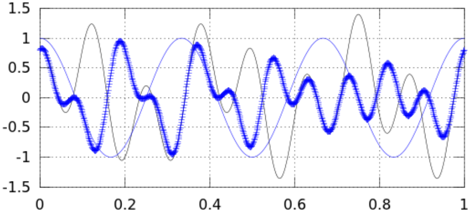
# How the sum behaves in each frequency

Function of the product between the input function and the cosine and sine terms:



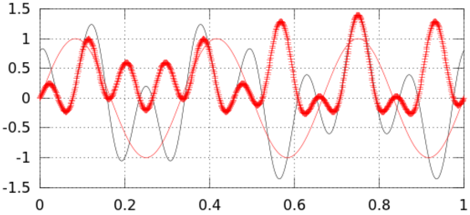
# How the sum behaves in each frequency

After multiply using 3Hz cosine the sum is near zero, since this component **is not part** of the signal (see positive and negatives cancel each other)

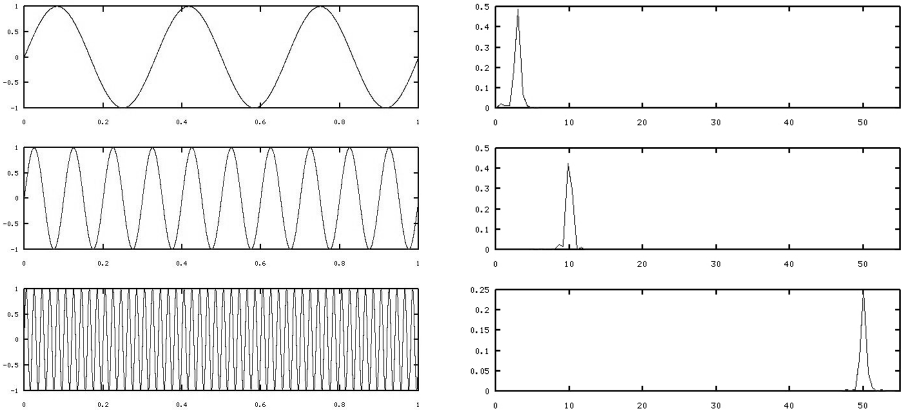


# How the sum behaves in each frequency

On the other hand, for a 3Hz sine, most values are positive because this wave **is part** of the signal.



# Frequency analysis

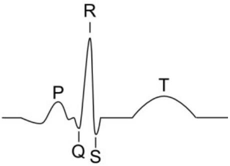



```
yf = scipy.fftpack.fft(y)  # computes Fourier Transform
N2 = N//2                  # gets half of the samples

# reescaled/normalised spectrum
yw = ( 2.0/N*np.abs(yf(:N2)) )
```

# Applications

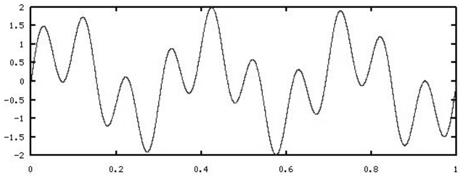
- ECG (electrocardiogram diagnosis)

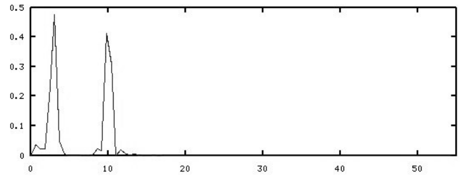




Thanks to Murray Bourne http://www.intmath.com/blog/mathematics/math-of-ecgs-fourier-series-4281

# Frequency analysis in stationary signals

- Fourier analysis suits better stationary signals, e.g. with frequencies 3 and 10 at any point



# Frequency analysis in non-stationary signals

- Signals in which a part ($\sim 75\%$) has frequency 5 Hz and the remaining has frequency 13 Hz, makes it hard to analyse.
- Frequency analysis allow us to see what are the frequencies present in the signal, but **not in which position they occur**.