Circuitos LC
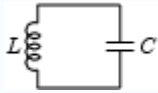
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LC circuits behave as electronic resonators, which are a key component in many applications such as oscillators, filters, tuners and frequency mixers.

An LC circuit consists of an inductor and a capacitor. The electrical current will alternate between them at an angular frequency of

$$\omega = \sqrt{\frac{1}{LC}},$$

where **L** is the inductance in henries, and **C** is the capacitance in farads. The angular frequency has units of radians per second.

An LC circuit is an idealized model since it assumes there is no dissipation of energy due to resistance. For a model incorporating resistance see RLC circuit.

Resonant frequency

The resonant frequency of the LC circuit (in radians per second) is

$$\omega = \sqrt{\frac{1}{LC}}$$

The equivalent frequency in the more familiar unit of hertz is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

Circuit analysis

By Kirchoff's voltage law, we know that the voltage across the capacitor, V_C must equal the voltage across the inductor, V_L:

$$V_C = V_L$$

Likewise, by Kirchoff's current law, the current through the capacitor plus the current through the inductor must equal zero:

$$i_C + i_L = 0$$

From the constitutive relations for the circuit elements, we also know that

$$V_L(t) = L\frac{di_L}{dt}$$

and

$$i_C(t) = C\frac{dV_C}{dt}$$

After rearranging and substituting, we obtain the second order [differential equation](#)

$$\frac{d^2i(t)}{dt^2} + \frac{1}{LC}i(t) = 0$$

We now define the parameter ω as follows:

$$\omega = \sqrt{\frac{1}{LC}}$$

With this definition, we can simplify the differential equation:

$$\frac{d^2i(t)}{dt^2} + \omega^2 i(t) = 0$$

The associated polynomial is $s^2 + \omega^2 = 0$, thus

$$s = +j\omega$$

or

$$s = -j\omega$$

where j is the [imaginary unit](#).
Thus, the complete solution to the differential equation is

$$i(t) = Ae^{+j\omega t} + Be^{-j\omega t}$$

and can be solved for A and B by considering the initial conditions.
Since the exponential is complex, the solution represents a sinusoidal [alternating current](#).
If the initial conditions are such that $A = B$, then we can use [Euler's formula](#) to obtain a real

$$\omega = \sqrt{\frac{1}{LC}}$$

[sinusoid](#) with [amplitude](#) $2A$ and [angular frequency](#) .
Thus, the resulting solution becomes:

$$i(t) = 2A\cos(\omega t)$$

The initial conditions that would satisfy this result are:

$$i(t = 0) = 2A$$

and

$$\frac{di}{dt}(t=0) = 0$$

Impedance of LC circuits

Series LC

First consider the [impedance](#) of the series LC circuit. The total impedance is given by the sum of the inductive and capacitive impedances:

$$Z = Z_L + Z_C$$

By writing the inductive impedance as $Z_L = i\omega L$ and capacitive impedance as $Z_C = \frac{-i}{\omega C}$ and substituting we have:

$$Z = i\omega L + \frac{-i}{\omega C}$$

Writing this expression under a common denominator gives:

$$Z = \frac{(\omega^2 LC - 1)i}{\omega C}$$

Note that the numerator implies if $\omega^2 LC = 1$ the total impedance Z will be zero and otherwise non-zero. Therefore the series connected circuit will act as a [band-pass filter](#) having zero impedance at the resonant frequency of the LC circuit.

Parallel LC

The same analysis may be applied to the parallel LC circuit. The total impedance is then given by:

$$Z = \frac{Z_L Z_C}{Z_L + Z_C}$$

and after substitution of Z_L and Z_C we have:

$$Z = \frac{\frac{L}{C}}{\frac{(\omega^2 LC - 1)i}{\omega C}}$$

which simplifies to:

$$Z = \frac{-L\omega i}{\omega^2 LC - 1}$$

Note that $\lim_{\omega^2 LC \to 1} Z = \infty$ but for all other values of $\omega^2 LC$ the impedance is finite (and therefore less than infinity). Hence the parallel connected circuit will act as band-stop filter having infinite impedance at the resonant frequency of the LC circuit.

Selectivity

LC circuits are often used as filters; the L/C ratio determines their selectivity. For a series resonant circuit, the higher the inductance and the lower the capacity, the narrower the filter bandwidth. For a parallel resonant circuit the opposite applies.

Applications

LC circuits behave as electronic resonators, which are a key component in many applications:

- Oscillators
- Filters
- Tuners
- Mixers
- Foster-Seeley discriminator

    The **Foster-Seeley discriminator** is an FM detector circuit that works on the same principle as most commonly used FM detectors, which is through variations in frequency. It uses a twice-tuned RF transformer to convert frequency changes into amplitude changes.

    This is accomplished by the use of resonance networks on the secondary of the transformer, tuned to the frequency of the carrier. If the input equals the carrier frequency, the inductive and capacitative responses should be equal in amplitude and opposite in phase which produces zero voltage on the output. Depending on whether the incoming FM signal varies below or above the carrier, the resultant inductive and capacitive vectors are summed and the result is a voltage proportional to the original phase shift from the carrier.

    Foster-Seeley discriminators are sensitive to both frequency and amplitude variations, unlike some detectors. Therefore a limiter must be used which is basically a circuit tuned so that at lower amplitudes it acts as a Class-A amplifier and at higher amplitudes acts like a Class-C amplifier which clips off the peaks.

Circuitos RLC

An **RLC circuit** (sometimes known as resonant or tuned circuit) is an electrical circuit consisting of a resistor (R), an inductor (L), and a capacitor (C), connected in series or in parallel. A RLC circuit is called a *second-order* circuit as any voltage or current in the circuit can be described by a second-order differential equation.

Fundamental Parameters

There are two fundamental parameters that describe the behavior of *RLC circuits*: the resonant frequency and the damping factor. In addition, there are several other parameters that can be derived from these first two (see next section).

Resonant frequency

The undamped resonance or natural frequency of an *RLC circuit* (in radians per second) is:

$$\omega_o = \frac{1}{\sqrt{LC}}$$

In the more familiar unit hertz, the natural frequency becomes

$$f_o = \frac{\omega_o}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

Resonance occurs when the complex impedance Z_{LC} of the LC resonator becomes zero:

$$Z_{LC} = Z_L + Z_C = 0$$

Both of these impedances are functions of complex angular frequency s:

$$Z_C = \frac{1}{Cs}$$

$$Z_L = Ls$$

Setting these expressions equal to one another and solving for s, we find:

$$s = \pm j\omega_o = \pm j\frac{1}{\sqrt{LC}}$$

where the resonance frequency ω_o is given in the expression above.

Damping factor

The damping factor of the circuit (in radians per second) is:

$$\zeta = \frac{R}{2L}$$

For applications in oscillator circuits, it is generally desirable to make the damping factor as small as possible, or equivalently, to increase the quality factor (Q) as much as possible. In practice, this requires decreasing the resistance R in the circuit to as small as physically possible. In this case, the *RLC circuit* becomes a good approximation to an ideal LC circuit, which is not realizable in practice. (Even if the resistor is removed from the circuit, there is always a small but non-zero amount of resistance in the wiring and interconnects between the other circuit elements that can never be eliminated entirely).

Alternatively, for applications in bandpass filters, the value of the damping factor is chosen based on the desired bandwidth of the filter. For a wider bandwidth, a larger value of the damping factor is required (and vice versa). In practice, this requires adjusting the relative values of the resistor R and the inductor L in the circuit.

Derived Parameters

The derived parameters include **Bandwidth**, **Q factor**, and **damped resonance frequency**.

Bandwidth

The *RLC circuit* may be used as a [bandpass](#) or [band-stop](#) filter, and the [bandwidth](#) (in radians per second) is

$$\Delta\omega = 2\zeta = \frac{R}{L}$$

Alternatively, the bandwidth in hertz is

$$\Delta f = \frac{\Delta\omega}{2\pi} = \frac{\zeta}{\pi} = \frac{R}{2\pi L}$$

The bandwidth is a measure of the width of the frequency response at the two *half-power* frequencies. As a result, this measure of bandwidth is sometimes called the **full-width at half-power**. Since electrical [power](#) is proportional to the square of the circuit voltage (or current), the frequency response will drop to $\frac{1}{\sqrt{2}}$ at the half-power frequencies.

Quality or Q factor

The Quality of the series tuned circuit, or [Q factor](#), is calculated as the ratio of the resonance frequency ω_o to the bandwidth $\Delta\omega$ (in radians per second):

$$Q_s = \frac{\omega_o}{\Delta\omega} = \frac{\omega_o}{2\zeta} = \frac{L}{R\sqrt{LC}} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Or in hertz:

$$Q_s = \frac{f_o}{\Delta f} = \frac{2\pi f_o L}{R} = \frac{1}{\sqrt{R^2 C/L}} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

For the parallel tuned circuit:

$$Q_p = \frac{1}{Q_s}$$

Q is a dimensionless quantity.

Damped Resonance

The [damped](#) resonance frequency derives from the natural frequency and the damping factor. If the circuit is *underdamped*, meaning $\zeta < \omega_o$ then we can define the damped resonance as

$$\omega_d = \sqrt{\omega_o^2 - \zeta^2}$$

In an oscillator circuit

$$\zeta \ << \ \omega_o.$$

As a result $\omega_d \ = \ \omega_o$ (approx).

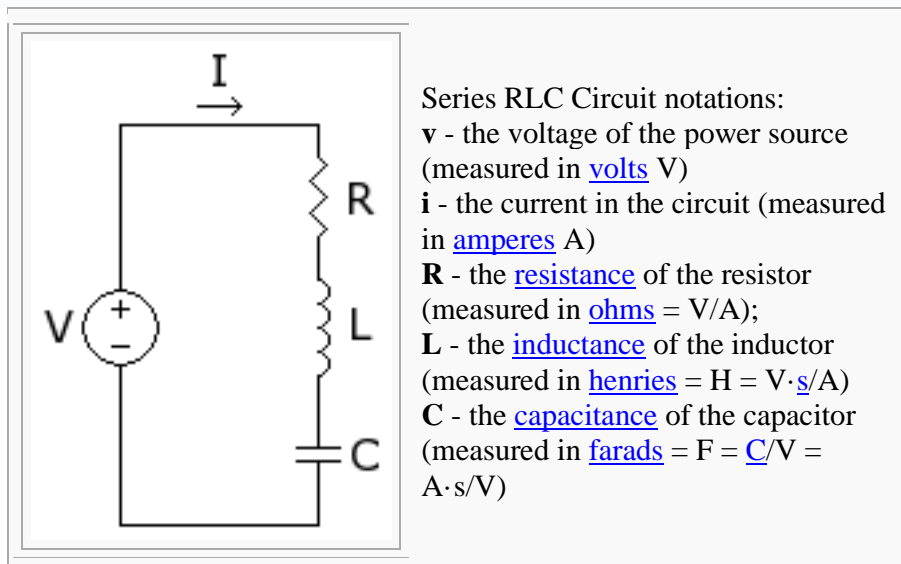See discussion of underdamping, overdamping, and critical damping, below.

Configurations

Every RLC circuit consists of two components: a *power source* and *resonator*. There are two types of power sources – Thévenin and Norton. Likewise, there are two types of resonators – series LC and parallel LC. As a result, there are four configurations of RLC circuits:
- Series LC with Thévenin power source
- Series LC with Norton power source
- Parallel LC with Thévenin power source
- Parallel LC with Norton power source.

Circuit Analysis

Series RLC with Thévenin power source
In this circuit, the three components are all in series with the voltage source.

Series RLC Circuit notations:
v - the voltage of the power source (measured in volts V)
i - the current in the circuit (measured in amperes A)
R - the resistance of the resistor (measured in ohms = V/A);
L - the inductance of the inductor (measured in henries = H = V·s/A)
C - the capacitance of the capacitor (measured in farads = F = C/V = A·s/V)

Given the parameters v, R, L, and C, the solution for the current (I) using Kirchoff's voltage law is:

$$v_R + v_L + v_C = v$$

For a time-changing voltage *v(t)*, this becomes

$$Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau = v(t)$$

Rearranging the equation gives the following second order differential equation:

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i(t) = \frac{1}{L}\frac{dv}{dt}$$

We now define two key parameters:

$$\zeta = \frac{R}{2L}$$

and

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

both of which are measured as radians per second.
Substituting these parameters into the differential equation, we obtain:

$$\frac{d^2i}{dt^2} + 2\zeta\frac{di}{dt} + \omega_0^2 i(t) = \frac{1}{L}\frac{dv}{dt}$$

The Zero Input Response (ZIR) solution

Setting the input (voltage sources) to zero, we have:

$$\frac{d^2i}{dt^2} + 2\zeta\frac{di}{dt} + \omega_o^2 i(t) = 0$$

with the initial conditions for the inductor current, $I_L(0)$, and the capacitor voltage $V_C(0)$. In order to solve the equation properly, the initial conditions needed are $I(0)$ and $I'(0)$.
The first one we already have since the current in the main branch is also the current in the inductor, therefore

$$i(0) = i_L(0)$$

The second one is obtained employing KVL again:

$$v_R(0) + v_L(0) + v_C(0) = 0$$

$$\Rightarrow i(0)R + i'(0)L + v_C(0) = 0$$

$$\Rightarrow i'(0) = \frac{1}{L}\left[-v_C(0) - I(0)R\right]$$

We have now a homogeneous second order differential equation with two initial conditions.
Substituting the two parameters ζ and ω_0, we have

$$i'' + 2\zeta i' + \omega_0^2 i = 0$$

We now convert the form of this equation to its characteristic polynomial
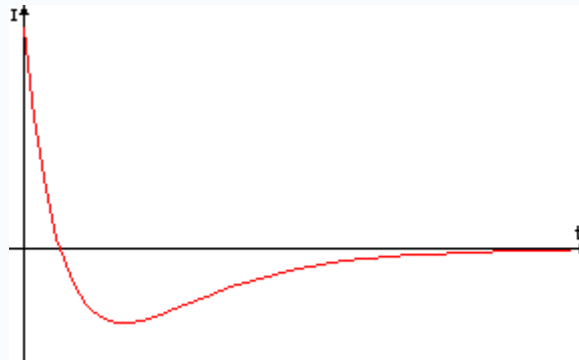
$$\lambda^2 + 2\zeta\lambda + \omega_0^2 = 0$$

Using the quadratic formula, we find the roots as

$$\lambda = -\zeta \pm \sqrt{\zeta^2 - \omega_0^2}$$

Depending on the values of α and ω_0, there are three possible cases:
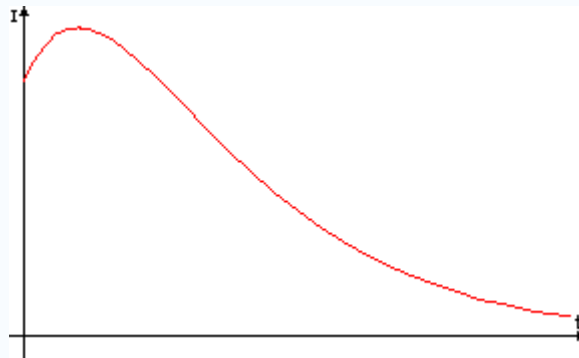
Over-damping

RLC series Over-Damped Response

$$\zeta > \omega_0 \Rightarrow RC > 4\frac{L}{R}$$

In this case, the characteristic polynomial's solutions are both negative real numbers. This is called "over-damping".
Two negative real roots, the solutions are:

$$I(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

Critical damping

RLC series Critically Damped

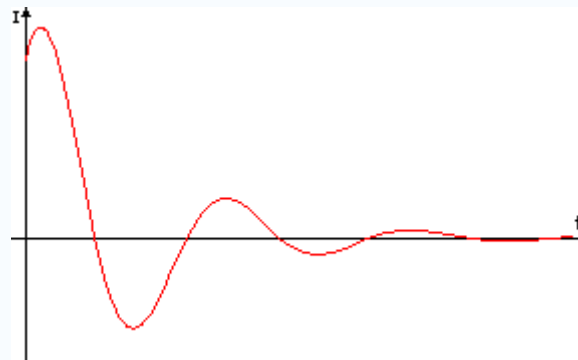$$\zeta = \omega_0 \Rightarrow RC = 4\frac{L}{R}$$

In this case, the characteristic polynomial's solutions are identical negative real numbers. This is called "critical damping".
The two roots are identical ($\lambda_1 = \lambda_2 = \lambda$), the solutions are:

$$I(t) = (A + Bt)e^{\lambda t}$$

for arbitrary constants A and B

Under-damping

RLC series Under-Damped

$$\zeta < \omega_0 \Rightarrow RC < 4\frac{L}{R}$$

In this case, the characteristic polynomial's solutions are complex conjugate and have negative real part. This is called "under-damping" and results in oscillations or *ringing* in the circuit. The solution consists of two conjugate roots

$$\lambda_1 = -\zeta + i\omega_c$$

and

$$\lambda_2 = -\zeta - i\omega_c$$

where

$$\omega_c = \sqrt{\omega_o^2 - \zeta^2}$$

The solutions are:

$$i(t) = Ae^{-\zeta + i\omega_c} + Be^{-\zeta - i\omega_c}$$

for arbitrary constants A and B.

Using Euler's formula, we can simplify the solution as

$$i(t) = e^{-\zeta t}\left[C\sin(\omega_c t) + D\cos(\omega_c t)\right]$$

for arbitrary constants C and D.

These solutions are characterized by *exponentially decaying sinusoidal response*. The time required for the oscillations to "die out" depends on the Quality of the circuit, or Q factor. The higher the Quality, the longer it takes for the oscillations to decay.

The Zero State Response (ZSR) solution

This time we set the initial conditions to zero and use the following equation:

$$\begin{cases} \dfrac{d^2 I}{dt^2} + \dfrac{R}{L}\dfrac{dI}{dt} + \dfrac{1}{LC}I(t) = \dfrac{1}{L}\dfrac{dV}{dt} \\ \\ I(0^-) = I'(0^-) = 0 \end{cases}$$

$$\frac{d^2 i}{dt^2} + 2\zeta\frac{di}{dt} + \omega_o i(t) = \frac{1}{L}\frac{dv}{dt}$$

There are two approaches we can take to finding the ZSR: (1) the Laplace Transform, and (2) the Convolution Integral.

Laplace Transform
We first take the Laplace transform of the second order differential equation:

$$(s^2 + 2\zeta s + \omega_o^2)I(s) = \frac{s}{L}V(s)$$

where *V(s)* is the Laplace Transform of the input signal:

$$V(s) = \mathcal{L}\left\{v(t)\right\}$$

We then solve for the complex admittance *Y(s)* (in siemens):

$$Y(s) = \frac{I(s)}{V(s)} = \frac{s}{L(s^2 + 2\zeta s + \omega_o^2)}$$

We can then use the admittance *Y(s)* and the Laplace transform of the input voltage *V(s)* to find the complex electrical current *I(s)*:

$$I(s) = Y(s) \times V(s)$$

Finally, we can find the electrical current in the time domain by taking the inverse Laplace Transform:

$$i(t) = \mathcal{L}^{-1}\left\{I(s)\right\}$$

Example:
Suppose $v(t) = Au(t)$, where $u(t)$ is the [Heaviside](#) [step function](#).
Then

$$V(s) = \frac{A}{s}$$

$$I(s) = \frac{A}{L(s^2 + 2\zeta s + \omega_o^2)}$$

Convolution Integral

A separate solution for every possible function for V(t) is impossible. However, there is a way to find a formula for I(t) using [convolution](#). In order to do that, we need a solution for a basic input - the [Dirac](#) [delta function](#).
In order to find the solution more easily we will start solving for the [Heaviside step function](#) and then using the fact that our circuit is a [linear system](#), its derivative will be the solution for the delta function.
The equation will be therefore, for t > 0:

$$\begin{cases} \frac{d^2 I_u}{dt^2} + \frac{R}{L}\frac{dI_u}{dt} + \frac{1}{LC}I_u(t) = 0 \\ I(0^+) = 0 \qquad I'(0^+) = \frac{1}{L} \end{cases}$$

Assuming λ_1 and λ_2 are the roots of

$$P(\lambda) = \lambda^2 + 2\zeta\lambda + \omega_o^2$$

then as in the ZIR solution, we have 3 cases here:

Over-damping
Two negative real roots, the solution is:

$$I_u(t) = \frac{1}{L(\lambda_1 - \lambda_2)}\left[e^{\lambda_1 t} - e^{\lambda_2 t}\right]$$

$$\Rightarrow I_\delta(t) = \frac{1}{L(\lambda_1 - \lambda_2)}\left[\lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t}\right]$$

Critical damping
The two roots are identical ($\lambda_1 = \lambda_2 = \lambda$), the solution is:

$$I_u(t) = \frac{1}{L}te^{\lambda t}$$

$$\Rightarrow I_\delta(t) = \frac{1}{L}(\lambda t + 1)e^{\lambda t}$$

Under-damping

Two conjugate roots ($\lambda_1 = \bar{\lambda}_2 = \zeta + i\omega_c$), the solution is:

$$I_u(t) = \frac{1}{\omega_c L} e^{\zeta t} \sin(\omega_c t)$$

$$\Rightarrow I_\delta(t) = \frac{1}{\omega_c L} e^{\zeta t} \left[\zeta \sin(\omega_c t) + \omega_c \cos(\omega_c t) \right]$$

(to be continued...)

Frequency Domain

The series RLC can be analyzed in the frequency domain using complex impedance relations. If the voltage source above produces a complex exponential wave form with amplitude V(s) and angular frequency $s = \sigma + i\omega$, KVL can be applied:

$$V(s) = I(s) \left(R + Ls + \frac{1}{Cs} \right)$$

where I(s) is the complex current through all components. Solving for I:

$$I(s) = \frac{1}{R + Ls + \frac{1}{Cs}} V(s)$$

And rearranging, we have

$$I(s) = \frac{s}{L \left(s^2 + \frac{R}{L} s + \frac{1}{LC} \right)} V(s)$$

Complex Admittance

Next, we solve for the complex admittance Y(s):

$$Y(s) = \frac{I(s)}{V(s)} = \frac{s}{L \left(s^2 + \frac{R}{L} s + \frac{1}{LC} \right)}$$

Finally, we simplify using parameters α and ω_o

$$Y(s) = \frac{I(s)}{V(s)} = \frac{s}{L \left(s^2 + 2\alpha s + \omega_o^2 \right)}$$

Notice that this expression for *Y(s)* is the same as the one we found for the Zero State Response.

Poles and Zeros

The zeros of *Y(s)* are those values of *s* such that *Y(s) = 0*:
$s = 0$ and $s = \infty$

The [poles](#) of *Y(s)* are those values of *s* such that $Y(s) = \infty$:

$$s = -\zeta \pm \sqrt{\zeta^2 - \omega_o^2}$$

Notice that the poles of *Y(s)* are identical to the roots λ_1 and λ_2 of the characteristic polynomial.

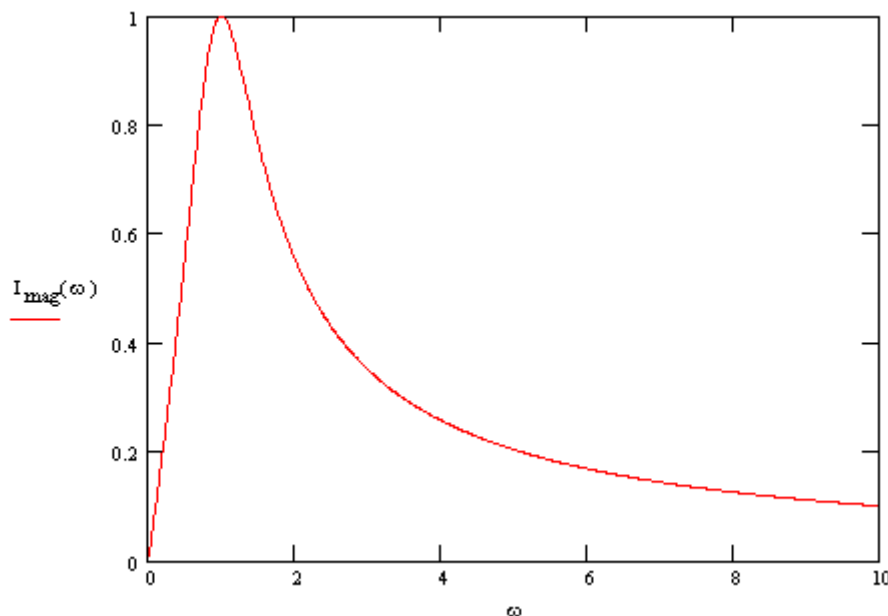### *Sinusoidal Steady State*

If we now let $s = i\omega$....

Taking the magnitude of the above equation:

$$|Y(s = i\omega)| = \frac{1}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

Next, we find the magnitude of current as a function of ω

$$|I(i\omega)| = |Y(i\omega)| \times |V(i\omega)|$$

If we choose values where $R = 1$ ohm, $C = 1$ farad, $L = 1$ henry, and $V = 1.0$ volt, then the graph of magnitude of the current *I* (in amperes) as a function of ω (in radians per second) is:
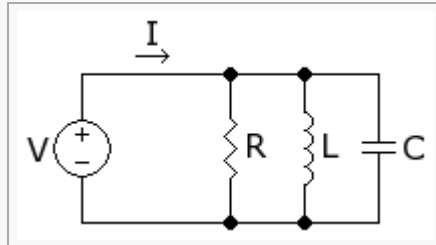

Sinusoidal steady-state analysis

Note that there is a peak at $I_{mag}(\omega) = 1$. This is known as the [resonant frequency](#). Solving for this value, we find:

$$\omega_o = \frac{1}{\sqrt{LC}}$$

Parallel RLC circuit

A much more elegant way of recovering the circuit properties of an RLC circuit is through the use of [nondimensionalization](.).

Parallel RLC Circuit notations:
V - the voltage of the power source (measured in [volts](.) V)
I - the current in the circuit (measured in [amperes](.) A)
R - the [resistance](.) of the resistor (measured in [ohms](.) = V/A);
L - the [inductance](.) of the inductor (measured in [henries](.) = H = V·[s](.)/A)
C - the [capacitance](.) of the capacitor (measured in [farads](.) = F = [C](.)/V = A·s/V)

For a parallel configuration of the same components, where Φ is the magnetic flux in the system

$$C\frac{d^2\Phi}{dt^2} + \frac{1}{R}\frac{d\Phi}{dt} + \frac{1}{L}\Phi = I_0\cos(\omega t) \Rightarrow \frac{d^2\chi}{d\tau^2} + 2\zeta\frac{d\chi}{d\tau} + \chi = \cos(\Omega\tau)$$

with substitutions

$$\Phi = \chi x_c, \ t = \tau t_c, \ x_c = LI_0, \ t_c = \sqrt{LC}, \ 2\zeta = \frac{1}{R}\sqrt{\frac{L}{C}}, \ \Omega = \omega t_c.$$

The first variable corresponds to the maximum magnetic flux stored in the circuit. The second corresponds to the period of resonant oscillations in the circuit.

Similarities and differences between series and parallel circuits

The expressions for the bandwidth in the series and parallel configuration are inverses of each other. This is particularly useful for determining whether a series or parallel configuration is to be used for a particular circuit design. However, in circuit analysis, usually the reciprocal of the latter two variables are used to characterize the system instead. They are known as the [resonant frequency](.) and the [Q factor](.) respectively.

Applications of tuned circuits

There are many applications for tuned circuits especially in radio and communication systems. They can be used to select a certain narrow range of frequencies from the total [spectrum](.) of radio waves.