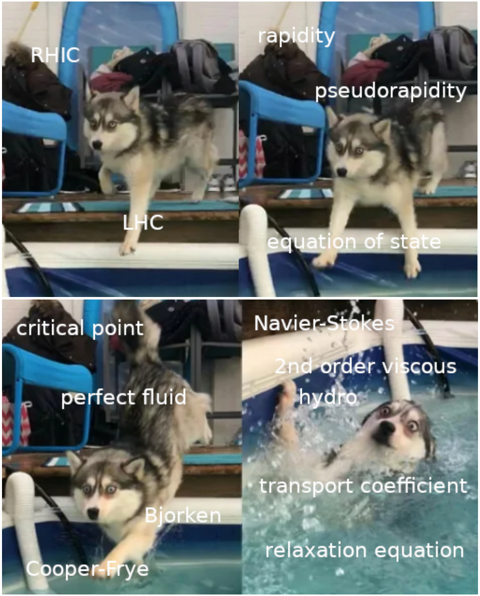
Lecture 8
Hydrodynamics

Part II



Bjorken model: fast derivation

We use: 1) $y = \eta_s \Leftrightarrow v_z = z/t$, 2) no dependence on η_s [maintained by hydro. evolution]

We solve around $z = 0 \Leftrightarrow y = 0$. So $v_z = z/t = 0$, $\partial_z v_z = 1/t$, $\gamma = 1$

$\frac{\partial n u^0}{\partial t} + \frac{\partial n u^z}{\partial z} = 0 = \frac{\partial n}{\partial t} + \frac{\partial n v_z}{\partial z} = \frac{\partial n}{\partial t} + n\frac{\partial v_z}{\partial z} + v_z\frac{\partial n}{\partial_z} = \frac{\partial n}{\partial t} + \frac{n}{t}$

So $n(t) = n(t_0)t/t_0$ which can be re-writen $\boxed{n(\tau) = n(\tau_0)\tau_0/\tau}$.

To get the solution for another z (with $|z| < t$), we do a Lorentz boost, changing y by a constant, this leaves τ constant and the expression of $n(\tau)$ is unchanged.

Similarly $\partial_\mu T^{\mu 0} = \partial_t[(\epsilon + p) - p] + \partial_z[(\epsilon + p)v_z] = \partial_t \epsilon + (\epsilon + p)/t$ which leads to $\boxed{\epsilon(\tau) = \epsilon(\tau_0)(\tau_0/\tau)^{4/3}}$ for $p = \epsilon/3$

Also, since $\partial_\mu T^{\mu\nu} = 0 \Rightarrow \partial_\mu(s u^\mu) = 0$, $\boxed{s(\tau) = s(\tau_0)\tau_0/\tau}$.

Exercise:

Compute $\epsilon(\tau)$ for Bjorken model with MIT bag equation of state

We use $\epsilon = g_{qgp} \frac{\pi^2}{30} T^4 + B$, $p = g_{qgp} \frac{\pi^2}{p0} T^4 - B$

Near $z = 0$, to first order in v_z

$\partial_\mu T^{\mu 0} = \partial_t[(\epsilon + p) - p] + \partial_z[(\epsilon + p)v_z] = \partial_t \epsilon + (\epsilon + p)/t$

We note that: $\epsilon + p = (4/3)(g_{qgp} \frac{\pi^2}{30} T^4) = (4/3)(\epsilon - B)$

So $\partial_t \epsilon + (\epsilon + p)/t = 0 = \partial_t(\epsilon - B) + (\epsilon - B)/t$

$\Rightarrow \epsilon(\tau) = [(\epsilon(\tau_0) - B] \left(\frac{\tau}{\tau_0}\right)^{4/3} + B$

Simple estimates with Bjorken model

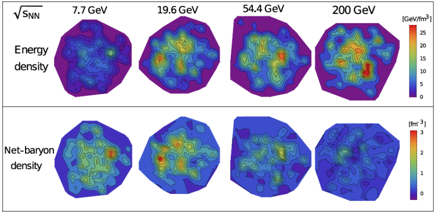
It is possible to get estimates of the initial particle number, entropy density, energy density, etc. Here we show an example of the latter.

In the rest frame ($y = 0$): $dV = S_\perp dz = \tau S_\perp dy$ and for a central collision $S_\perp = \pi R^2$. So $\epsilon(\tau_0) = \epsilon(\tau)(\tau/\tau_0)^{4/3} = (\tau/\tau_0)^{4/3} \frac{dE}{\tau \pi R^2 dy}$

[Note that $(\tau/\tau_0)^{4/3} \neq \tau/\tau_0$ (energy per unit rapidity is not conserved)]

Assume $dE \sim< m_\perp > dN$, $\tau_0 = 1\, fm$, $\tau =$ some fm.
For a central Au+Au collision, at 200 GeV, $dN_\pi/dy \sim 3 \times 300$ (see p. 7 in lecture 4) and $< m_\perp > \sim < p_\perp > \sim 0.5\,\mathrm{GeV}$
So $\epsilon(\tau_0) \sim 3\, GeV/fm^3 >>$ deconfinement value from lattice QCD (p.7 lecture 6).

Modern values for initial energy density:



Results obtained with NeXSPheRIO code by R.N.Hirayama

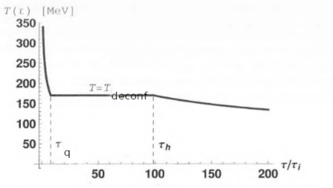
Bjorken model with a first order phase transition

We can learn a lot from $s(\tau) = s(\tau_0)\tau_0/\tau$.

We use the MIT bag toy model to describe the transition.

- For $\tau_0 \leq \tau \leq \tau_q$: fluid in the QGP phase and $s_{qg} = g_{qg}\frac{4}{3}\frac{\pi^2}{30}T^3$
- For $\tau_h \leq \tau$: fluid in the pion phase and $s_\pi = g_\pi \frac{4}{3}\frac{\pi^2}{30}T^3$

$s(\tau)\tau = s(\tau_0)\tau_0 \Rightarrow g_q\tau_q = g_h\tau_h$ so $\tau_h = (g_{qg}/g_\pi)\,\tau_q >> \tau_q$ since g_{qg} =37 ($N_f = 2$) and 47.5 ($N_f = 3$), while g_π =3 [Large volume increase]
Also $s_\pi = s_{qg}(\tau_q/\tau_h) = s_{qg}(g_\pi/g_{qg}) << s_{qg}$ i.e. jump in entropy density.



- For $\tau_q \leq \tau \leq \tau_h$: fluid is mixture of QGP and pions

$s(\tau) = f(\tau)g_\pi\frac{4}{3}\frac{\pi^2}{30}T^3_{dec} + [1 - f(\tau)]g_{qg}\frac{4}{3}\frac{\pi^2}{30}T^3_{dec} = \frac{\tau_0 s(\tau_0)}{\tau}$ where $f(\tau)$ is the volume fraction of fluid in the pion phase.

$\Rightarrow f(\tau) = \frac{1-(\tau_q/\tau)}{1-g_\pi/g_{qg}}$

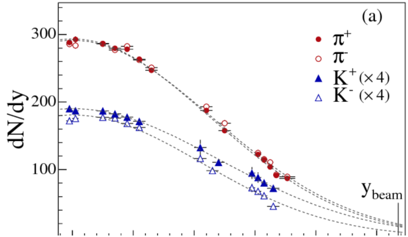
When can we use the Bjorken hypothesis $v_z = z/t$ given the gaussian shape of dN/dy's?

Answer: amazingly, v_z does not deviates that much from this. Let us see this.

Consider longitudinal expansion only (which dominates initially) and a slice of matter at $z = 0$.
Assume that initially the entropy density is gaussian:

$$s(x, y, \eta_s, \tau_0) \propto exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{\eta_s^2}{2\sigma_{\eta s}^2}\right)$$

This is motivated by the gaussian shape of dN/y:



Rapidity distribution for central Au+Au collisions at 200 GeV: BRAHMS Phys. Rev. Lett. 94 (2005) 162301 with gaussian fit and $\sigma_{\eta_s} \sim 2.3\ fm$

To first order in $v_z = z/t$: $\partial_\mu T^{\mu z} = 0 = \frac{\partial}{\partial t}[(\epsilon + p)v_z] + \frac{\partial}{\partial z} p$ so

$\frac{\partial v_z}{\partial t} = -\frac{1}{\epsilon+p}\frac{\partial p}{\partial z} = -\frac{1}{\epsilon+p}\frac{dp}{d\epsilon}\frac{\partial}{\partial z}\epsilon = -\frac{1}{Ts} c_s^2 T \frac{\partial}{\partial z} s = -c_s^2 \frac{\partial \ln s}{\partial z}$

Near $z = 0$, $dt \sim d\tau$, $dz \sim \tau d\eta_s$, $v_z \sim y$ is small, so

$\frac{\partial y}{\partial \tau} = -\frac{c_s^2}{\tau}\frac{\partial \ln s}{\partial \eta_s}$

$\Rightarrow y(\tau) = \left(1 + \frac{c_s^2 \ln(\tau/\tau_0)}{\sigma_{\eta s}^2}\right)\eta_s$.

$\longrightarrow$ y is proportional to η_s.

For $c_s^2 = 1/3$ and $\sigma_{\eta_s} \sim 2.3$: $y(\tau) = [1 + 0.06\ln(\tau/\tau_0)]\eta_s$ i.e. very close to η_s.

So $v_z = \tanh y \sim \tanh \eta_s = z/t$

Challenge

In the case of a 1+1 dimensional expansion along the collision axis, show that for solutions independent of proper time, $y = \eta_s \pm \mathtt{tanh}^{-1} c_s$. Compute v_z. What kind of solution is this?

Homework

Consider a plasma of gluons and two flavors of massless quarks at $T_0 = 300\text{MeV}$ for $\tau_0 = 1\ fm$. In the Bjorken model, compute the proper time when the plasma will reach the deconfinement temperature $T \sim 150\text{MeV}$ and its energy density then.

Other references on this topic

- J.-Y.ollitrault Relativistic hydrodynamics for heavy-ion collisions Eur.J.Phys.29 (2008) 275, arXiv:0708.2433
- R. Vogt, Ultrarelativistic Heavy-ion Collisions, Elsevier, 2007
- W. Florkowski, Phenomenology of Ultra-Relativistic Heavy-Ion Collisions, World Scientific, 2010