MATRIX ANALYSIS – PSI 5794

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4. LINEAR TRANSFORMATIONS

A lin transf is a linear function / mapping of a vec argument yielding another vector argument as a result. It is a function from one vec space to another vec space so that the ~~of~~ original vec space ops are preserved, i.e., vector addition and scalar multiplication.

4.1. Definitions and Examples

Def: Let $(V, \mathbb{F})$ and $(W, \mathbb{F})$ be vector spaces. then a LIN TRANSF $\mathcal{L}$ is a function $\mathcal{L} : V \to W$ that

$$\mathcal{L}(\alpha v_1 + \beta v_2) = \alpha \mathcal{L} v_1 + \beta \mathcal{L} v_2 , \quad \forall \alpha, \beta \in \mathbb{F}, \ v_1, v_2 \in V$$

V is the domain of $\mathcal{L}$ (departure space)
W is the co-domain of $\mathcal{L}$ (arrival space)

$$(V, \mathbb{F}) \xrightarrow{\mathcal{L}} (W, \mathbb{F})$$

A generic LIN TRANSF is also known as a homomorphism

same form
i.e. preserves form
(vec + and scalar ·)

Def: A linear operator is a L.T. $\mathcal{L}: V \to V$, also known as endomorphism. E.g., change of basis for vectors

inside itself

Def: A Lin Trans is injective (or 1-1) if

a) $\mathcal{L} v_1 = \mathcal{L} v_2 \Rightarrow v_1 = v_2$

or

b) $v_1 \neq v_2 \Rightarrow \mathcal{L} v_1 \neq \mathcal{L} v_2$

unique pairs, no ambiguity going back (think in terms of inverse)

(monomorphism)

Def: A L.T. is surjective (or onto) if

$\forall\, w_1 \in W \ \exists\, v_1 \in V \mid \mathcal{L} v_1 = w_1$.

Nothing left on W, it is completely covered

It is also known as epimorphism

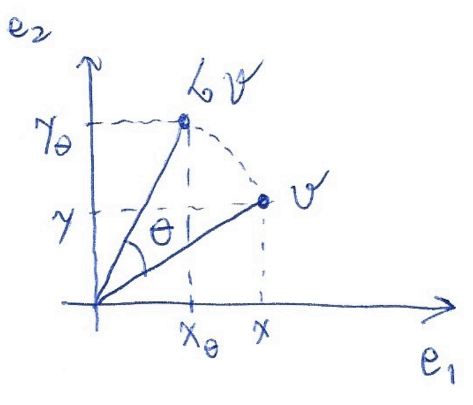
upon/over/on (sobre)

Def: A L.T. is bijective if it is 1-1 and onto, that is, it is an invertible transf., also known as isomorphism

equal

Example 1: Consider a L.T. $\mathcal{L}$ that rotates a given vector of an angle θ about the origin.

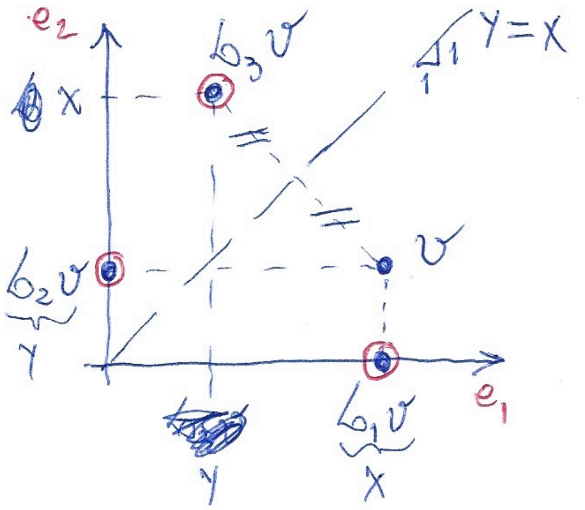
$v = \begin{bmatrix} x \\ y \end{bmatrix}$, $V = W = \mathbb{R}^2$, $\mathcal{L} v = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$



<u>Example 2</u>: Let $V = W = \mathbb{R}^2$, with $v = \begin{bmatrix} x \\ y \end{bmatrix} \in V$.

Define the LT's L_1, L_2, L_3 as follows

$L_1 v = \begin{bmatrix} x \\ 0 \end{bmatrix}$ proj onto x-axis (e_1) $\quad L_2 v = \begin{bmatrix} 0 \\ y \end{bmatrix}$ proj onto y-axis (e_2) $\quad L_3 v = \begin{bmatrix} y \\ x \end{bmatrix}$ reflection about $y = x$ axis



Note that

1) $L_2 L_1 v = L_2\left(\begin{bmatrix} x \\ 0 \end{bmatrix}\right) = 0$; $\therefore L_2 L_1 = \underline{0}$, but $L_1 \neq 0, L_2 \neq 0$

$\Rightarrow$ product (composition) of nonzero LT can be the zero LT

2) $L_3 L_2 v = L_3 \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}$. However,

$L_2 L_3 v = L_2 \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix}$. thus $L_2 L_3 \neq L_3 L_2$,

that is, composition/multiplication of LT's is <u>not</u> commutative

3) $L_1 L_1 v = L_1 \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} = L_1 v$, that is $L_1^2 = L_1$,

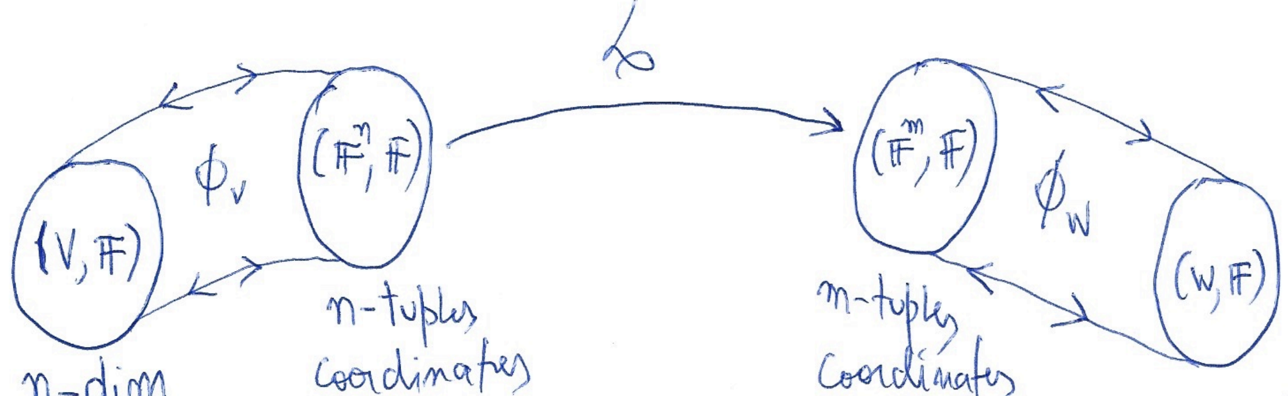
L_1 is an idempotent transformation

4.2. MATRIX REPRESENTATION of A L.T.

the matrix representation of a L.T. $\mathcal{L}$ is useful because we can make calculations and operate on vec spaces using MATRIX ALGEBRA. this is done in terms of the vectors coordinates (a col-vec) and $\mathcal{L}$ coordinates (a matrix) described in properly ordered basis for the departure space V (B_V) and the arrival space W (B_W).

Isomorphism between V and $\mathbb{F}^n$ (W and $\mathbb{F}^m$)

An n-dim vec space V is essentially the same as the vec space of n-tuples from $\mathbb{F}^n$. A bijective mapping (isomorphism) ϕ_V can always be found so that $V \overset{\phi_V}{\Longleftrightarrow} \mathbb{F}^n$ and $W \overset{\phi_W}{\Longleftrightarrow} \mathbb{F}^m$.



n-dim

B_V = list of abstract n vectors

$B_V = \{v_1, v_2, \dots, v_n\}$

n-tuples coordinates in $B_V \equiv$ Matrix $n \times n$

m-tuples Coordinates in B_W = $m \times m$ matrix

m-dim

B_W = list of m abstract vecs

$B_W = \{w_1, w_2, \dots, w_m\}$

BACK

Thus, without loss of generality we may think or work with either $(V, \mathbb{F})$ or $(\mathbb{F}^n, \mathbb{F})$, and with $(W, \mathbb{F})$ or $(\mathbb{F}^m, \mathbb{F})$, as convenient. For that, we introduce the concept of vector coordinates over an ordered basis.

Vector Coordinates and Basis Expansion

If $B_V = \{v_1, v_2, \dots, v_n\}$ is an ordered basis for V, then any vector $v \in V$ has a unique expansion over B_V

any v: Lin comb of the basis vecs

$$v = \sum_{j=1}^{n} \alpha_j v_j = \alpha_1 v_1 + \dots + \alpha_k v_k + \dots + \alpha_n v_n$$

Series expansion or decomposition of v

Component of v w.r.t. B_V

Coordinate of v over B_V

Basis vector

Any basis can be chosen, but once it is fixed, (they are not unique) the representation of a vec v over B_V in terms of its coordinates $\{\alpha_j\}$ is unique, and we denote

$$[v]_{B_V} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \triangleq \alpha$$ (a col vec), the ~~ex~~ vector coordin of v over B_V.

Proof of uniqueness: $v = \sum_{j=1}^{n} \alpha_j v_j$. Now assume there is another description of v over B_V, i.e., $v = \sum_{j=1}^{n} \gamma_j v_j$. Subtracting the two equations

$$v - v = \sum_{j=1}^{n} (\alpha_j - \gamma_j) v_j = 0 \text{, or}$$

$$(\alpha_1 - \gamma_1) v_1 + (\alpha_2 - \gamma_2) v_2 + \dots + (\alpha_n - \gamma_n) v_n = 0 \quad (\mathrm{I})$$

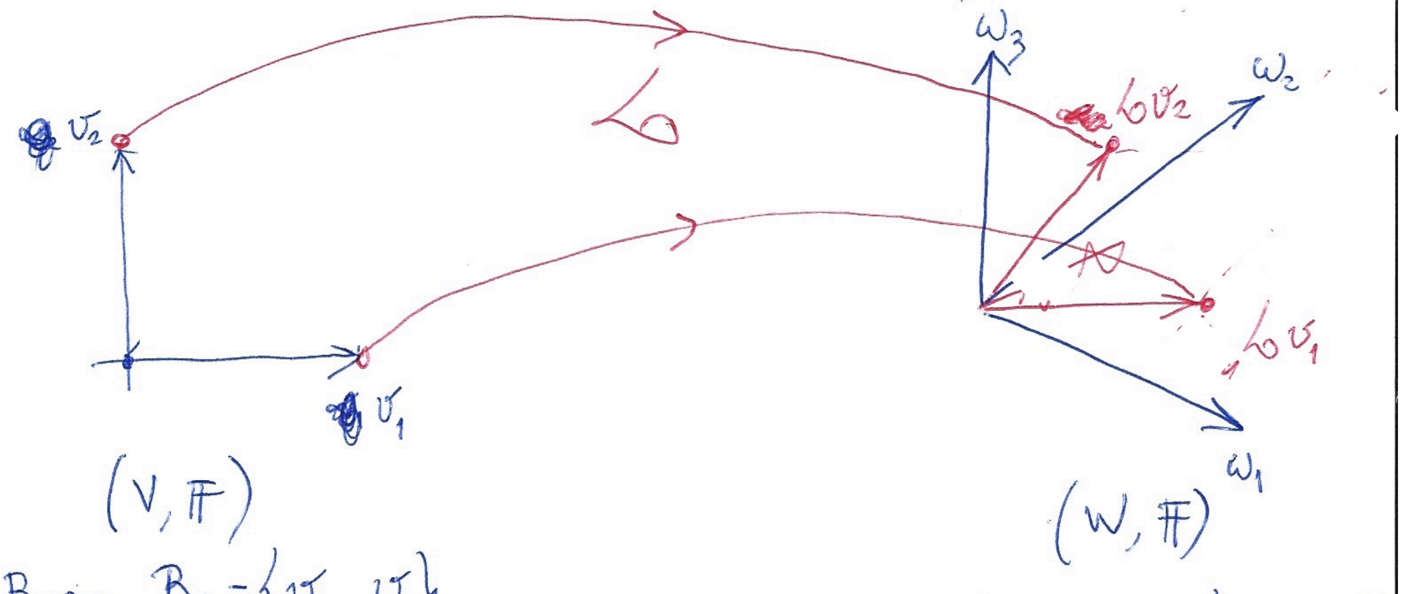
Since the set $\{v_j\}$ is LI (it's a basis), then the unique solution for (I) is the trivial solution, i.e., $\alpha_j - \gamma_j = 0 \;\forall j$, or $\boxed{\alpha_j = \gamma_j}$

The $A = \text{mat } L$ representation

If a basis $B_V = \{v_1, v_2, \dots, v_n\}$ for V and a basis $B_W = \{w_1, w_2, \dots, w_m\}$ for W are chosen, then the matrix representation A for L in terms of B_V and B_W is obtained as:

1) Apply L (known) on <u>each</u> basis vector $v_j \in B_V$, generating n mapped $m \times 1$ vectors $L v_j \in W$. Note that $L v_j \neq w_j$ (in general)

2) Describe each n mapped vectors $L v_j \in W$ in terms of the "local" basis, i.e., B_W



Basis $B_V = \{v_1, v_2\}$

2 n-dim vecs in V

Basis $B_W = \{w_1, w_2, w_3\}$

2 m-dim vecs in W

$$L v_1 = a_{11} w_1 + a_{21} w_2 + a_{31} w_3$$

$$L v_2 = a_{12} w_1 + a_{22} w_2 + a_{32} w_3$$

Each mapped basis vec $\mathcal{L} v_j$ will have a description in basis B_W, that is, will have specific coordinates in W:

$$\mathcal{L} v_1 = a_{11} w_1 + a_{21} w_2 + \cdots + a_{i1} w_i + \cdots + a_{m1} w_m$$

$$\vdots$$

$$\mathcal{L} v_j = a_{1j} w_1 + a_{2j} w_2 + \cdots + a_{ij} w_i + \cdots + a_{mj} w_m$$

Known vec

Coordinates of $\mathcal{L} v_j$ in B_W

$$\vdots$$

$$\mathcal{L} v_n = a_{1n} w_1 + a_{2n} w_2 + \cdots + a_{in} w_i + \cdots + a_{mn} w_m$$

then $[\mathcal{L} v_j]_{B_W} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{bmatrix} = a_j$ (a col vec)

Invoking the iso morphism

$\phi_W : B_W \rightarrow B_W$

set of m abstract vecs → $m \times m$ matrix of the coordinates of the m basis vecs

same for $\phi_V : B_V \rightarrow B_V$

$$\underset{\text{vec}}{\mathcal{L} v_j} = \underset{\text{matrix}}{B_W}\, \underset{\text{vec}}{a_j}$$

as $j = 1, 2, \ldots, n$

$$[\mathcal{L} v_1 \;\; \mathcal{L} v_2 \; \ldots \; \mathcal{L} v_n] = [B_W a_1 \;\; B_W a_2 \; \ldots \; B_W a_n]$$

$$\underbrace{\mathcal{L} B_V}_{\substack{\text{Known matrix} \\ \text{with } n \text{ mapped} \\ \text{vecs in } W \\ \triangleq Y}} = \underbrace{B_W}_{\text{known}} \underbrace{A}_{\text{UNknown}}$$

$$\boxed{Y = B_W A}$$

$$\boxed{A = B_W^{-1} Y \equiv \text{mat}\, \mathcal{L}}$$

or $A = [\mathcal{L}]_{B_V B_W}$

Note

$$\boxed{\mathcal{L} B_V \neq B_W}$$

Example 1: $\mathcal{L}: V \to V = \mathbb{R}^3$. $\mathcal{L}$ projects vectors on xy-plane. If $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathcal{L}v = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$.

Find $A = [\mathcal{L}]_{BB}$ with $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

We have that $Y = B_W A$, with $Y = \mathcal{L}B$ and $B_W = B$.

$$\mathcal{L}B = \mathcal{L}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} = Y$$

then

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} A$$

or

$$A = \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}_{B^{-1}} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}}_{[\mathcal{L}]_{BB}} \quad //$$

FROM LT to MATRIX ALGEBRA

Once we know $A = \text{mat}\,\mathcal{L}$ for given basis B_V and B_W, we may implement $w = \mathcal{L}v$ in terms of their coordinates and matrix algebra.

Let $v = \sum_{j=1}^{n} \alpha_j v_j$ and $w = \mathcal{L}v = \sum_{i=1}^{m} \beta_i w_i$ or

$[v]_{B_V} = \alpha$ and $[w]_{B_W} = \beta$.

Now, $\mathcal{L}v = \mathcal{L}\sum_{j=1}^{n} \alpha_j v_j = \sum_{j=1}^{n} \alpha_j \mathcal{L}v_j$. (I)

but $\mathcal{L}v_j = \sum_{i=1}^{m} a_{ij} w_i$, which plugged in (I)

$$w = \sum_{j=1}^{n} \alpha_j \left(\sum_{i=1}^{m} a_{ij} w_i \right) = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} \alpha_j w_i .$$

$$w = \sum_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{n} a_{ij} \alpha_j \right)}_{} w_i = \sum_{i=1}^{m} \underbrace{(\beta_i)}_{} w_i \quad , \text{or}$$

$\beta_i = \sum_{j=1}^{m} a_{ij} \alpha_j$. As $i = 1, m$, we have

$$\boxed{\beta_{m\times 1} = A_{m\times n}\, \alpha_{m\times 1}} \iff \boxed{[\mathcal{L}v]_{\boxed{B_W}} = [\mathcal{L}]_{B_V \boxed{B_W}} [v]_{B_V}}$$

that is, we may implement $\mathcal{L}$ over any given vec v by Matrix-vec multiplication directly provided that Bases are Compatible.

If they are not, bring $\mathcal{L}v$, v, $\mathcal{L}$ to Compatible Bases first.

4.3. CHANGE OF COORDINATES

Assume vector v is described in a basis $B=\{v_1, v_2, \dots, v_n\}$, i.e., $v=\sum_{i=1}^{n}\alpha_i v_i \iff [v]_B=\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{bmatrix}$

We would like to find coordinates for v in terms of / over another basis $B'=\{z_1, z_2, \dots, z_n\}$, that is, $v=\sum_{j=1}^{n}\beta_j z_j$. First we represent the vecs of the new basis in terms of the old one:

$$z_j=\sum_{i=1}^{n}\underbrace{p_{ij}}_{\text{coordinates of } z_j \text{ over } B} v_i, \text{ but } v=\sum_j \underbrace{\beta_j}_{\text{coordinates of } v \text{ over } B'} z_j. \text{ then}$$

$$v=\sum_j \beta_j\left(\sum_{i=1}^{n} p_{ij}v_i\right)=\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}\beta_j v_i. \text{ But}$$

Originally $v=\sum_{i=1}^{n}\alpha_i v_i$, then, by unicity of coordinates in the same basis for the same vector, $\alpha_i=\sum_{j=1}^{n}p_{ij}\beta_j$.

As $i=1,n$, we have (OLD = P · New)

$$\left.\begin{aligned}\alpha_1&=p_{11}\beta_1+p_{12}\beta_2+\dots+p_{1n}\beta_n\\ &\ \vdots\\ \alpha_n&=p_{n1}\beta_1+p_{n2}\beta_2+\dots+p_{nn}\beta_n\end{aligned}\right\}\quad \alpha=P\beta$$

or

$$\boxed{[v]_B=P_{B'B}[v]_{B'}}$$

For the new basis: $[v]_{B'}=P^{-1}[v]_B$

Where is P? $z_j=\sum_{i=1}^{n}p_{ij}v_i=p_{1j}v_1+p_{2j}v_2+\dots+p_{nj}v_n$

$$z_j=[v_1\ v_2\ \cdots\ v_n]\begin{bmatrix}p_{1j}\\ p_{2j}\\ \vdots\\ p_{nj}\end{bmatrix} \text{ or } z_j=Bp_j. \text{ As } j=1,n$$

$$\boxed{\underbrace{B'}_{\text{matrix of basis vecs coordinates (New)}}=\underbrace{B}_{\text{OLD}}P} \text{ or } \boxed{P=B^{-1}B'}$$

4.4. SIMILARITY TRANSFORMATIONS & MATRICES

We have a L.T. from V to V, ~~then~~ $\mathcal{L}: V \to V$, that is a LIN OPERATOR / ENDOMORPHISM. Let us describe $\mathcal{L}$ using the same basis $B = \{v_1, v_2, \dots, v_n\}$ for both domain and codomain, that is $A = [\mathcal{L}]_{BB}$ or simply $A = [\mathcal{L}]_B$: express the basis transformed vectors ~~the~~ $\mathcal{L} v_j$ in terms of the same basis B

$$\mathcal{L} v_j = \sum_{i=1}^{n} a_{ij} v_i \quad \overset{\text{as } j=\overline{1,n}}{\Longleftrightarrow} \quad \underbrace{\mathcal{L} B}_{\text{known matrix } (\neq B)} = B A$$

So $\boxed{A = B^{-1}(\underbrace{\mathcal{L} B}_{\text{matrix}})} \Rightarrow A = [\mathcal{L}]_{BB}$.

Now let's find the coordinates of the transformed vector $\mathcal{L} v$ over B in terms of the original vector coordinates $[v]_B = \alpha$, $\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$.

We have that $v = \sum_{j=1}^{n} \alpha_j v_j$, which upon $\mathcal{L}$ becomes $\mathcal{L} v = \sum_j \alpha_j \mathcal{L} v_j$. Note that $\mathcal{L} v$ also have a description in B: $\mathcal{L} v = \sum_{i=1}^{n} \beta_i v_i$.

We know from above that $\mathcal{L} v_j = \sum_{i=1}^{n} a_{ij} v_i$. Then

$$\mathcal{L} v = \sum_{j=1}^{n} \alpha_j (\mathcal{L} v_j) = \sum_j \alpha_j \sum_i a_{ij} v_i = \sum_i \Big(\sum_j a_{ij} \alpha_j\Big) v_i$$

but $\mathcal{L} v = \sum_i \beta_i v_i$, thus $\beta_i = \sum_j a_{ij} \alpha_j$. As $i, j = \overline{1,n}$, $\beta = A\alpha$, or $[\mathcal{L} v]_B = [\mathcal{L}]_{BB} [v]_B$ or even

$$\boxed{[\mathcal{L} v]_B = A\, [v]_B}$$

Now assume we change basis: $B \to B'$, for $B' = \{z_1, z_2, \cdots, z_n\}$. We already know that $[v]_B = P[v]_{B'}$, for some matrix P.

If we change basis both on the domain and on the codomain:

$$[bv]_B = A\,[v]_B$$

$$P[bv]_{B'} = A P [v]_{B'}, \quad \text{or}$$

$$[bv]_{B'} = P^{-1} A P [v]_{B'}$$

$$[bv]_{B'} = C\,[v]_{B'}, \quad \text{with } C \triangleq P^{-1}AP.$$

<u>Matrices A and C</u> are said to be <u>similar</u>, in the sense they represent the same LIN OPERATOR under different basis (for both domain and codomain). We can also write

$$[bv]_{B'} = \underbrace{P^{-1}[b]_B P}_{[b]_{B'}} [v]_{B'}$$

$$\therefore \boxed{[b]_{B'} = P^{-1}[b]_B P}$$

4.5. EQUIVALENT TRANSFORMATIONS & MATRICES

We now are in the general scenario again, i.e.,

$\ell : V \to W$ $\qquad B_V = \{v_1, v_2, \dots, v_n\}$

$\quad B_V \quad B_W$ $\qquad B_W = \{w_1, w_2, \dots, w_m\}$

Let us get some shortcuts:

$\ell B_V = B_W A$, and $[\ell v]_{B_W} = [\ell]_{B_V B_W} [v]_{B_V}$;

this implies $\underbrace{[\ell v]_{B_W}}_{\beta_{m\times 1}} = \underbrace{A}_{m\times n} \underbrace{[v]_{B_V}}_{\alpha_{n\times 1}} \Leftrightarrow \beta = A\alpha$

If we change bases in both domain and codomain, $B_V \to B_V'$ and $B_W \to B_W'$, via bases changing matrices Q and P, respectively, $[\ell v]_{B_W} = P[\ell v]_{B_W'}$, $[v]_{B_V} = Q[v]_{B_V'}$. Going back to $[\ell v]_{B_W} = A[v]_{B_V}$, upon changing bases,

$$P[\ell v]_{B_W'} = AQ[v]_{B_V'} \Leftrightarrow [\ell v]_{B_W'} = P^{-1}AQ[v]_{B_V'}$$

or $[\ell v]_{B_W'} = C[v]_{B_V'}$, with $C \triangleq P^{-1}AQ$.

Matrices C and A are said to be equivalent, in the sense they represent the same LIN TRANSF under proper bases changes in both domain and codomain:

$$\boxed{[\ell]_{B_V' B_W'} = P^{-1}[\ell]_{B_V B_W} Q}$$

What is essential in terms of LT for Matrix Analysis?

1) $\forall$ LT $\mathcal{L}: \underset{B_V}{V} \to \underset{B_W}{W}$ $\exists$ $A = \text{mat}[\mathcal{L}]_{B_V B_W}$ or $[\mathcal{L}]_{B_V B_W} = A$

2) Same as vectors that we can work ~~just~~ only using their coordinates for a given basis

$$[v_1]_{B_V} + [v_2]_{B_V} = [v_1 + v_2]_{B_V}$$

$$[\alpha v_1]_{B_V} = \alpha [v_1]_{B_V}$$

3) We can transform any vector using its coordinates and the coordinates of LT, provided that they are expressed in compatible bases

$$v = \mathcal{L} u \qquad \mathcal{L}: \underset{B_V}{V} \to \underset{B_W}{W}$$

$$[v]_{B_W} = [\mathcal{L}]_{B_V B_W} [u]_{B_V}$$

$$y = Ax$$

, i.e., employ matrix algebra to perform the transforms.

<u>CAUTION</u>: $[u]_{B_V} = x$, $[v]_{B_W}$, $[\mathcal{L}]_{B_Z B_W}$

$y \neq Ax$ because $[v]_{B_V} = [\mathcal{L}]_{B_Z B_W} [u]_{B_V}$

$$[v]_{B_W} = [\mathcal{L}]_{B_Z B_W} [u]_{B_V}$$

incompatible

$\therefore$ We cannot proceed via algebraic computations of the coordinates. First, bring all the vecs / $\mathcal{L}$ to compatible bases: $[\mathcal{L}]_{B_Z B_W} \to [\mathcal{L}]_{B_V B_W}$

or $[v]_{B_V} \to [v]_{B_Z}$

1) know that behind any matrix there is an underlying LT ~~given two~~ and vice-versa

$$\mathcal{L}: V \longrightarrow W \quad \Longleftrightarrow \quad A = [\mathcal{L}]_{B_V B_W}$$

$$B_V \qquad B_W$$

$$v = \{u_1, \dots, u_N\} \in V \quad \Longleftrightarrow \quad X = [v]_{B_V}, \; Y = [w]_{B_W}$$

$$w = \{w_1, \dots, w_M\} \in W \quad \Longleftrightarrow$$

$$w = \mathcal{L} v \qquad\qquad Y = AX$$

Meyer ch4: 4.7.8, 4.7.13; HW LEC#05

Also: Consider the change of basis procedure, with original basis $B_V = \{v_1, \ldots, v_N\}$ and new basis $B_Z = \{z_1, \ldots, z_N\}$. Depending on how we ~~ex~~press, OLD = f(NEW), or NEW = g(old), we get different expressions for the change of coordinates equation, in terms of the matrix that performs the change. The same applies for our choice of indexes in the summations, i.e., $v = \sum_i \alpha_i v_i$ & $z = \sum_j \beta_j z_j$; or $v = \sum_j \alpha_j v_j$ and $z = \sum_i \beta_i z_i$. Find all the four different possible equations for the change of coordinates matrices, relating all the matrices. Recall that we adopt column vectors to collect the coordinates of a vector over any given basis.