27/06/2022 - Gabarito P2 - MAP0151 Monitor

1. Encontrar a parábola que melhor se ajuste aos pontos: $(-1,-2)$, $(0,-1)$, $(1/2, 1/4)$ e $(1,2)$ pelo critério dos mínimos quadrados.

R:

x	-1	0	$1/2$	1
y	-2	-1	$1/4$	2

e equação geral da parábola é: $f(x) = ax^2 + bx + c$

$$\Rightarrow \begin{cases} a(-1)^2 + b(-1) + c = -2 \\ a(0)^2 + b(0) + c = -1 \\ a(1/2)^2 + b(1/2) + c = 1/4 \\ a(1)^2 + b(1) + c = 2 \end{cases} \Rightarrow \begin{cases} 1\cdot a - 1\cdot b + c = -2 \\ 0\cdot a + 0\cdot b + c = -1 \\ (1/4)\cdot a + (1/2)\cdot b + c = 1/4 \\ 1\cdot a + 1\cdot b + c = 2 \end{cases} \Rightarrow$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1/4 & 1/2 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\vec{a}} = \underbrace{\begin{bmatrix} -2 \\ -1 \\ 1/4 \\ 2 \end{bmatrix}}_{y}$$

. Faremos: $A^t \cdot A \cdot \vec{a} = A^t \cdot y$.

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1/4 & 1 \\ -1 & 0 & 1/2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1/4 & 1/2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 33/16 & 1/8 & 9/4 \\ 1/8 & 9/4 & 1/2 \\ 9/4 & 1/2 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1/4 & 1 \\ -1 & 0 & 1/2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -1 \\ 1/4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/16 \\ 33/8 \\ -3/4 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 33/16 & 1/8 & 9/4 \\ 1/8 & 9/4 & 1/2 \\ 9/4 & 1/2 & 4 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1/16 \\ 33/8 \\ -3/4 \end{bmatrix}$$ => por MEG, temos:

$$\left[\begin{array}{ccc:c} 33/16 & 1/8 & 9/4 & 1/16 \\ 1/8 & 9/4 & 1/2 & 33/8 \\ 9/4 & 1/2 & 4 & -3/4 \end{array}\right] \xrightarrow[L_3' = L_3 - \left(\frac{9}{4}\cdot\frac{16}{33}\right)L_1 \;\to 12/11]{L_2' = L_2 - \left(\frac{1}{8}\cdot\frac{16}{33}\right)L_1 \;\to 2/33} \left[\begin{array}{ccc:c} 33/16 & 1/8 & 9/4 & 1/16 \\ 0 & \frac{74}{33} & 4/11 & 136/33 \\ 0 & 4/11 & 17/11 & -9/11 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc:c} 33/16 & 1/8 & 9/4 & 1/16 \\ 0 & \frac{74}{33} & 4/11 & 136/33 \\ 0 & 4/11 & 17/11 & -9/11 \end{array}\right] \xrightarrow{L_3'' = L_3' - \left(\frac{4}{11}\cdot\frac{33}{74}\right)L_2' \;\to 6/37} \left[\begin{array}{ccc:c} 33/16 & 1/8 & 9/4 & 1/16 \\ 0 & \frac{74}{33} & 4/11 & \frac{136}{33} \\ 0 & 0 & 55/37 & -55/37 \end{array}\right]$$

$\Rightarrow c = (-55/37)(37/55) = \underline{-1}$,

$b = (136/33 - (4/11)\cdot(-1))\cdot(33/74) = \underline{2}$,

$a = (1/16 - (9/4)(-1) - (1/8)\cdot(2))\cdot(16/33) = \underline{1}$,

$\therefore f(x) = \underline{x^2 + 2x - 1}$ //

2. Dada uma tabela com três pontos $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$ e uma família de funções $\{\varphi_1(x), \varphi_2(x), \varphi_3(x)\}$:

$$\begin{cases} (\varphi_1(x_1), \varphi_1(x_2), \varphi_1(x_3)) = (1,1,1) \\ (\varphi_2(x_1), \varphi_2(x_2), \varphi_2(x_3)) = (2,1,-3) \\ (\varphi_3(x_1), \varphi_3(x_2), \varphi_3(x_3)) = (1,0,0) \end{cases}$$

Substitua $\varphi_3(x)$ por $\phi_3(x) = a\varphi_1(x) + b\varphi_2(x) + c\varphi_3(x)$ tal que $\{\varphi_1, \varphi_2, \phi_3\}$ seja ortogonal em relação à tabela dada.

R.: $\begin{cases} (\varphi_1(x_1), \varphi_1(x_2), \varphi_1(x_3)) = (1,1,1) \\ (\varphi_2(x_1), \varphi_2(x_2), \varphi_2(x_3)) = (2,1,-3) \\ (\varphi_3(x_1), \varphi_3(x_2), \varphi_3(x_3)) = (1,0,0) \end{cases}$,

x	x_1	x_2	x_3
y	y_1	y_2	y_3

e $\phi_3(x) = a\varphi_1(x) + b\varphi_2(x) + c\varphi_3(x)$. Queremos a, b e c tais que $\{\varphi_1, \varphi_2, \phi_3\}$ seja ortogonal em relação à tabela de (x_i, y_i), $i \in \{1,2,3\}$

$\cdot \langle \varphi_1, \varphi_2 \rangle = 1\cdot 2 + 1\cdot 1 + 1\cdot(-3) = 3-3 = 0$

$\cdot \langle \varphi_1, \phi_3 \rangle = 1\cdot(a\cdot 1 + b\cdot 2 + c\cdot 1) + 1\cdot(a\cdot 1 + b\cdot 1 + c\cdot 0) + 1\cdot(a\cdot 1 + b\cdot(-3) + c\cdot 0) = 3a + 0\cdot b + 1\cdot c$. Queremos $= 0$

$\cdot \langle \varphi_2, \phi_3 \rangle = 2\cdot(a\cdot 1 + b\cdot 2 + c\cdot 1) + 1\cdot(a\cdot 1 + b\, 1 + c\cdot 0) + (-3)(a\cdot 1 + b\cdot(-3) + c\cdot 0) =$
$= 2a + 4b + 2c + a + b - 3a + 9b = 0\cdot a + 14b + 2c$. Queremos $= 0$

$\Rightarrow$ Pelo método de ortogonalização de Gram-Schmidt, temos:

$\cdot \omega_1 = v_1 = (1,1,1)$

$\cdot \omega_2 = v_2 - \dfrac{\langle v_2, \omega_1 \rangle}{\langle \omega_1, \omega_1 \rangle}\omega_1 = (2,1,-3) - \dfrac{0}{3}(1,1,1) = (2,1,-3)$

(Pois φ_1 e φ_2 já ortogonais em x_1, x_2 e x_3)

$\cdot \omega_3 = v_3 - \dfrac{\langle v_3, \omega_1 \rangle}{\langle \omega_1, \omega_1 \rangle}\omega_1 - \dfrac{\langle v_3, \omega_2 \rangle}{\langle \omega_2, \omega_2 \rangle}\omega_2 =$

$= (a + 2b + c, a + b, a - 3b) - \dfrac{(3a+c)}{3}(1,1,1) - \dfrac{(14b+2c)}{14}\cdot$

$\cdot (2,1,-3) = \underline{(8c/21, -10c/21, 2c/21)}$

3. Mostre que os polinômios $L_0(x)=(x-1)(x-2)$, $L_1(x)=x(x-2)$ e $L_2(x)=x(x-1)$ formam uma família ortogonal em relação à tabela:

x	0	1	2
y	y_0	y_1	y_2

R.: Temos:
$$\begin{cases}(L_0(0),L_0(1),L_0(2))=(2,0,0)\\(L_1(0),L_1(1),L_1(2))=(0,-1,0)\\(L_2(0),L_2(1),L_2(2))=(0,0,2)\end{cases}$$

$$\left.\begin{array}{l}\cdot\langle L_0,L_1\rangle = 2\cdot 0+0\cdot(-1)+0\cdot 0=0\\ \cdot\langle L_0,L_2\rangle = 2\cdot 0+0\cdot 0+0\cdot 2=0\\ \cdot\langle L_1,L_2\rangle = 0\cdot 0+(-1)\cdot 0+0\cdot 2=0\\ \cdot\langle L_0,L_0\rangle = 2\cdot 2+0\cdot 0+0\cdot 0=4\neq 0\\ \cdot\langle L_1,L_1\rangle = 0\cdot 0+(-1)(-1)+0\cdot 0=1\neq 0\\ \cdot\langle L_2,L_2\rangle = 0\cdot 0+0\cdot 0+2\cdot 2=4\neq 0\end{array}\right\} \Rightarrow \{L_0,L_1,L_2\}$$ é ortogonal em relação à tabela acima.

4. Aproxime a função $h(x)=x^3$, por um polinômio de grau menor ou igual a dois, no intervalo: $[-1,1]$, com produto interno $\langle f,g\rangle=\int_{-1}^{1} f(x)g(x)dx$

R.: Para o caso contínuo do MMQ, com $h(x)=x^3$ e: $g_1(x)=1$, $g_2(x)=x$ e $g_3(x)=x^2$, levando a $a\,g_3(x)+b\,g_2(x)+c\,g_1(x)$, temos o seguinte sistema normal:

$$\begin{cases}c\langle g_1,g_1\rangle+b\langle g_1,g_2\rangle+a\langle g_1,g_3\rangle=\langle g_1,h\rangle\\ c\langle g_2,g_1\rangle+b\langle g_2,g_2\rangle+a\langle g_2,g_3\rangle=\langle g_2,h\rangle\\ c\langle g_3,g_1\rangle+b\langle g_3,g_2\rangle+a\langle g_3,g_3\rangle=\langle g_3,h\rangle\end{cases}$$

com:

$\cdot \langle g_1, g_1 \rangle = \int_{-1}^{1} 1 \cdot 1 \, dx = \int_{-1}^{1} 1 \, dx = x \Big|_{-1}^{1} = 1 + 1 = \underline{2},$

$\cdot \langle g_1, g_2 \rangle = \int_{-1}^{1} 1 \cdot x \, dx = \int_{-1}^{1} x \, dx = x^2/2 \Big|_{-1}^{1} = 1/2 - 1/2 = \underline{0},$

$\cdot \langle g_1, g_3 \rangle = \int_{-1}^{1} 1 \cdot x^2 \, dx = \int_{-1}^{1} x^2 \, dx = x^3/3 \Big|_{-1}^{1} = 1/3 + 1/3 = \underline{2/3},$

$\cdot \langle g_1, h \rangle = \int_{-1}^{1} 1 \cdot x^3 \, dx = \int_{-1}^{1} x^3 \, dx = x^4/4 \Big|_{-1}^{1} = 1/4 - 1/4 = \underline{0},$

$\cdot \langle g_2, g_2 \rangle = \int_{-1}^{1} x \cdot x \, dx = \int_{-1}^{1} x^2 \, dx = x^3/3 \Big|_{-1}^{1} = 1/3 + 1/3 = \underline{2/3},$

$\cdot \langle g_2, g_3 \rangle = \int_{-1}^{1} x \cdot x^2 \, dx = \int_{-1}^{1} x^3 \, dx = x^4/4 \Big|_{-1}^{1} = 1/4 - 1/4 = \underline{0},$

$\cdot \langle g_2, h \rangle = \int_{-1}^{1} x \cdot x^3 \, dx = \int_{-1}^{1} x^4 \, dx = x^5/5 \Big|_{-1}^{1} = 1/5 + 1/5 = \underline{2/5},$

$\cdot \langle g_3, g_3 \rangle = \int_{-1}^{1} x^2 \cdot x^2 \, dx = \int_{-1}^{1} x^4 \, dx = x^5/5 \Big|_{-1}^{1} = 1/5 + 1/5 = \underline{2/5},$

$\cdot \langle g_3, h \rangle = \int_{-1}^{1} x^2 \cdot x^3 \, dx = \int_{-1}^{1} x^5 \, dx = x^6/6 \Big|_{-1}^{1} = 1/6 - 1/6 = \underline{0},$

=> atualizando nosso sistema normal, temos:

$$\left[\begin{array}{ccc|c} 2 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 2/5 \\ 2/3 & 0 & 2/5 & 0 \end{array}\right] \xrightarrow{L_3' = L_3 - (\frac{2}{3} \cdot \overbrace{\frac{1}{2}) L_1}^{1/3}} \left[\begin{array}{ccc|c} 2 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 2/5 \\ 0 & -2/9 & 2/5 & -2/15 \end{array}\right] \Rightarrow$$

$$\xrightarrow{L_3'' = L_3' - (-\frac{2}{9} \cdot \overbrace{\frac{3}{2}) L_2}^{-1/3}} \left[\begin{array}{ccc|c} 2 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 2/5 \\ 0 & 0 & 2/5 & 0 \end{array}\right] \Rightarrow \begin{cases} a = 0 \cdot (5/2) = \underline{0}, \\ b = (2/5)(3/2) = \underline{3/5}, \\ c = (-(2/3) \cdot 0)/2 = \underline{0}, \end{cases}$$

=> $a \cdot g_3(x) + b \cdot g_2(x) + c \cdot g_1(x) = \underline{(3/5)\,x}$

5\. Em relação ao produto interno da questão anterior, os três primeiros polinômios ortogonais são $p_0(x) = 1$, $p_1(x) = x$ e $P_2(x) = x^2 - 1/3$. Ache o $P_3(x)$.

R.: Queremos $P_3(x) = x^3 + bx^2 + cx + d$ (assumindo-o como polinômio cúbico) onde: $\langle P_3(x), P_0(x) \rangle = 0$, $\langle P_3(x), P_1(x) \rangle = 0$ e $\langle P_3(x), P_2(x) \rangle = 0$.

Sabemos que: $P_K(x) = (x - a_K) P_{K-1}(x) - b_K P_{K-2}(x)$ onde $a_K = (\langle x P_{K-1}, P_{K-1} \rangle)/(\langle P_{K-1}, P_{K-1} \rangle)$ e $b_K = (\langle x P_{K-1}, P_{K-2} \rangle)/(\langle P_{K-2}, P_{K-2} \rangle)$

=> Para $K = 3$, temos:

$$a_3 = \frac{\langle x P_2, P_2 \rangle}{\langle P_2, P_2 \rangle} = \frac{\int_{-1}^{1} (x \cdot (x^2 - 1/3))(x^2 - 1/3)\,dx}{\int_{-1}^{1} (x^2 - 1/3)(x^2 - 1/3)\,dx} = \underline{0}$$

$$b_3 = \frac{\langle x P_2, P_1 \rangle}{\langle P_1, P_1 \rangle} = \frac{\int_{-1}^{1} (x \cdot (x^2 - 1/3))(x)\,dx}{\int_{-1}^{1} x \cdot x\,dx} = \frac{2/5 - 2/9}{2/3} = \frac{4}{15}$$

=> $P_3(x) = (x - 0) P_2(x) - (4/15) P_1(x) = x(x^2 - 1/3) - (4/15) \cdot x = \underline{x^3 - (3/5)\,x}$