18/06/2022 - Gabarito E3 - MAP0151

1. Encontre a reta que melhor se ajuste à tabela dada pelo MMQ. Qual é o resíduo quadrático?

x	0	1	3	5
y	1	3	2	1

R.: Para duas incógnitas, temos o caso geral:

$$\begin{cases} u_1 a + v_1 b = w_1 \\ \vdots \\ u_n a + v_n b = w_n \end{cases}, \text{ com } Q = \sum_{i=1}^{n} [w_i - (u_i a + v_i b)]^2 \text{ sendo o resíduo quadrático}$$

Com um sistema normal:

$$\begin{cases} a\langle u,u\rangle + b\langle u,v\rangle = \langle u,w\rangle \\ a\langle v,u\rangle + b\langle v,v\rangle = \langle v,w\rangle \end{cases}, \text{ com } u = (u_1, u_2, \dots, u_n), \ v = (v_1, \dots, v_n) \text{ e } W = (w_1, \dots, w_n)$$

=> Aplicando ao problema, temos:

$$\begin{cases} a \cdot 0 + b = 1 \\ a \cdot 1 + b = 3 \\ a \cdot 3 + b = 2 \\ a \cdot 5 + b = 1 \end{cases} \Rightarrow u = (0,1,3,5),\ v = (1,1,1,1) \text{ e } W = (1,3,2,1)$$

$$\Rightarrow \begin{cases} a\langle u,u\rangle + b\langle u,v\rangle = \langle u,w\rangle \\ a\langle v,u\rangle + b\langle v,v\rangle = \langle v,w\rangle \end{cases}$$

será:

$\cdot \langle u,u\rangle = \sum_{i=1}^{n} u_i^2 = 0^2+1^2+3^2+5^2 = 35$

$\cdot \langle u,v\rangle = \sum_{i=1}^{n} u_i \cdot v_i = 0\cdot1+1\cdot1+3\cdot1+5\cdot1 = 9$

$\cdot \langle v,v\rangle = \sum_{i=1}^{n} v_i^2 = 1^2+1^2+1^2+1^2 = 4$

$\cdot \langle u,w\rangle = \sum_{i=1}^{n} u_i \cdot w_i = 0\cdot1+1\cdot3+3\cdot2+5\cdot1 = 14$

$\cdot \langle v,w\rangle = \sum_{i=1}^{n} v_i \cdot w_i = 1\cdot1+1\cdot3+1\cdot2+1\cdot1 = 7$

$$\Rightarrow \begin{cases} a\cdot35 + b\cdot9 = 14 \\ a\cdot9 + b\cdot4 = 7 \end{cases} \Rightarrow \begin{bmatrix} 35 & 9 \\ 9 & 4 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 14 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 35 & 9 & 14 \\ 9 & 4 & 7 \end{array}\right] \xrightarrow{L_2' = L_2 - (9/35)L_1} \left[\begin{array}{cc|c} 35 & 9 & 14 \\ 0 & 59/35 & 17/5 \end{array}\right] \Rightarrow$$

$\Rightarrow b = (17/5)\cdot(35/59) = 17\cdot7/59 = \underline{119/59},$

$a = (14 - 9\cdot(119/59))/35 = \underline{-7/59},$

$\Rightarrow \underline{r: (-7/59)x + 119/59},$

Com o resíduo quadrático:

$Q = \sum_{i=1}^{n} [w_i - (u_i a + b)]^2 = (1-(0\cdot(-7/59)+119/59))^2 + (3-(1\cdot(-7/59)$
$+119/59))^2 + (2-(3\cdot(-7/59)+119/59))^2 + (1-(5\cdot(-7/59)+119/59))^2$
$= (-60/59)^2 + (65/59)^2 + (20/59)^2 + (25/59)^2 = \underline{150/59 \approx 2{,}5424},$

2. Encontre a parábola que melhor se ajuste aos pontos $A=(1,1)$, $B=(2,4)$, $C=(0,3)$ e $D=(-1,2)$

R.: · Tabela de pontos:

x	1	2	0	-1
y	1	4	3	2

· Função a ser adequada: $f(x) = ax^2+bx+c$

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$$\therefore \begin{cases} a\cdot 1^2 + b\cdot 1 + c = 1 \\ a\cdot 2^2 + b\cdot 2 + c = 4 \\ a\cdot 0^2 + b\cdot 0 + c = 3 \\ a\cdot(-1)^2 + b(-1) + c = 2 \end{cases} \Rightarrow \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\vec{a}} = \underbrace{\begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \end{bmatrix}}_{y}$$

$$\Rightarrow A^t \cdot A \cdot \vec{a} = A^t \cdot y \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} 1 & 4 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 1 \\ 1 & 2 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} 1\cdot 1 + 4\cdot 4 + 0\cdot 0 + 1\cdot 1 & 1 + 4\cdot 2 + 0\cdot 0 + 1\cdot(-1) & 6 \\ 1\cdot 1 + 2\cdot 4 + 0\cdot 0 + (-1)\cdot 1 & 1 + 2\cdot 2 + 0\cdot 0 + (-1)(-1) & 2 \\ 1\cdot 1 + 1\cdot 4 + 1\cdot 0 + 1\cdot 1 & 1 + 2 + 0 + (-1) & 4 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 + 4\cdot 4 + 0 + 2 \\ 1 + 2\cdot 4 + 0 - 2 \\ 1 + 4 + 3 + 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 19 \\ 7 \\ 10 \end{bmatrix} \quad \therefore \text{Por MMQ, temos:}$$

$$\left[\begin{array}{ccc|c} 18 & 8 & 6 & 19 \\ 8 & 6 & 2 & 7 \\ 6 & 2 & 4 & 10 \end{array}\right] \xrightarrow[\substack{L_3' = L_3 - (6/18)L_1 \\ \hookrightarrow 1/3}]{L_2' = L_2 - \overset{\nearrow 4/9}{(8/18)} L_1} \left[\begin{array}{ccc|c} 18 & 8 & 6 & 19 \\ 0 & 22/9 & -6/9 & -13/9 \\ 0 & -2/3 & 2 & 11/3 \end{array}\right] \xrightarrow{L_3' =}$$

$$\xrightarrow[\substack{\hookrightarrow \frac{-2}{3}\cdot\frac{9}{22} = \frac{-3}{11}}]{= L_3 - \overbrace{(-2/3/\tfrac{22}{9})}^{+3/11} L_2} \left[\begin{array}{ccc|c} 18 & 8 & 6 & 19 \\ 0 & 22/9 & -6/9 & -13/9 \\ 0 & 0 & 60/33 & 36/11 \end{array}\right] \begin{aligned} &\Rightarrow c = 36/11 \cdot 33/60 = 9/5 \\ &b = (-13/9 + 6/9 \cdot 9/5)\cdot 9/22 = \frac{-1}{10} \\ &a = (19 - 6\cdot 9/5 + 8\cdot 1/10)/18 = 1/2 \end{aligned}$$

$$\Rightarrow f(x) = (1/2)x^2 - x/10 + 9/5$$

3. Dada a tabela abaixo, encontrar o polinômio de grau menor ou igual a 3 que aproxima a tabela pelo MMQ.

x	0	1	2	3
y	-1	1	7	23

R.: $P(x) = ax^3 + bx^2 + cx + d$

$$\begin{cases} a \cdot 0^3 + b \cdot 0^2 + c \cdot 0 + d = -1 \\ a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + d = 1 \\ a \cdot 2^3 + b \cdot 2^2 + c \cdot 2 + d = 7 \\ a \cdot 3^3 + b \cdot 3^2 + c \cdot 3 + d = 23 \end{cases} \Rightarrow$$

$$\begin{cases} 0 \cdot a + 0 \cdot b + 0 \cdot c + d = -1 \\ 1 \cdot a + 1 \cdot b + 1 \cdot c + d = 1 \\ 8 \cdot a + 4 \cdot b + 2 \cdot c + d = 7 \\ 27 \cdot a + 9 \cdot b + 3 \cdot c + d = 23 \end{cases}$$, que é equivalente a:

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{\vec{a}} = \underbrace{\begin{bmatrix} -1 \\ 1 \\ 7 \\ 23 \end{bmatrix}}_{y}$$ $\Rightarrow A \cdot \vec{a} = y \Rightarrow$
$A^t \cdot A \cdot \vec{a} = A^t \cdot y \Leftrightarrow$

$$\begin{bmatrix} 794 & 276 & 98 & 36 \\ 276 & 98 & 36 & 14 \\ 98 & 36 & 14 & 6 \\ 36 & 14 & 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 678 \\ 236 \\ 84 \\ 30 \end{bmatrix}$$ $\Rightarrow$ Por MMQ:

$$\left[\begin{array}{cccc|c} 794 & 276 & 98 & 36 & 678 \\ 276 & 98 & 36 & 14 & 236 \\ 98 & 36 & 14 & 6 & 84 \\ 36 & 14 & 6 & 4 & 30 \end{array}\right]$$

$L_2' = L_2 - (276/794)L_1$

$L_3' = L_3 - (98/794)L_1$

$L_4' = L_4 - (36/794)L_1$

$$\left[\begin{array}{cccc|c} 794 & 276 & 98 & 36 & 678 \\ 0 & 818/397 & \frac{768}{397} & 590/397 & 128/397 \\ 0 & \frac{768}{397} & \frac{756}{397} & \frac{618}{397} & 126/397 \\ 0 & \frac{590}{397} & \frac{618}{397} & \frac{940}{397} & \frac{-294}{397} \end{array}\right]$$

$L_3' = L_3 - (768/818)L_2$

$L_4' = L_4 - (590/818)L_2$

$$\left[\begin{array}{cccc|c} 794 & 276 & 98 & 36 & 678 \\ 0 & \frac{818}{397} & \frac{768}{397} & \frac{590}{397} & 128/397 \\ 0 & 0 & 36/409 & \frac{66}{409} & 6/409 \\ 0 & 0 & 66/409 & \frac{530}{409} & -\frac{398}{409} \end{array}\right]$$

$L_4 \leftrightarrow L_3$

$L_4' = L_4 - (36/66)L_3$

$$\left[\begin{array}{cccc|c} 794 & 276 & 98 & 36 & 678 \\ 0 & \frac{818}{397} & \frac{768}{397} & \frac{590}{397} & \frac{128}{397} \\ 0 & 0 & \frac{66}{409} & \frac{530}{409} & -\frac{398}{409} \\ 0 & 0 & 0 & -\frac{6}{11} & 6/11 \end{array}\right] \Rightarrow d = \underline{-1},$$

$$\begin{cases} c = \left(-\frac{398}{409} + \frac{530}{409} \cdot 1\right) \cdot \frac{409}{66} = \underline{2}, \\ b = \cdots = \underline{-1}, \\ a = \cdots = \underline{1}, \end{cases}$$

$\Rightarrow \underline{P(x) = x^3 - x^2 + 2x - 1}$

4. Encontre um conjunto de polinômios que gerem os polinômios de grau menor ou igual a 2 e que sejam ortogonais em relação à tabela da questão 3.

R.: Pela tabela da questão 3, temos a família: $\{x^2, x, 1\}$ e os vetores $v_1 = (0, 1, 4, 9)$, $v_2 = (0, 1, 2, 3)$ e $v_3 = (1, 1, 1, 1)$ retirados dessa família, com os respectemos valores para x: 0, 1, 2 e 3

$\Rightarrow$ pelo processo de ortogonalização temos:

$\cdot\, \omega_1 = v_1 = (0,1,4,9)$

$\cdot\, \omega_2 = v_2 - \frac{\langle v_2, \omega_1 \rangle}{\langle \omega_1, \omega_1 \rangle}\omega_1 = (0,1,2,3) - \frac{36}{98}(0,1,4,9) =$

$= (0, 1 - 18/49, 2 - (18/49)\cdot 4, 3 - (18/49)\cdot 9) = \ldots$

$= (0, 80/49, 26/49, -15/49)$

$\cdot\, \omega_3 = v_3 - \frac{\langle v_3, \omega_1 \rangle}{\langle \omega_1, \omega_1 \rangle}\omega_1 - \frac{\langle v_3, \omega_2 \rangle}{\langle \omega_2, \omega_2 \rangle}\omega_2 =$

$= (1,1,1,1) - \frac{14}{98}(0,1,4,9) - \frac{91\cdot 49}{149}(0, 80/49, 26/49, -15/49) =$

$= (1,1,1,1) - (0, 14/98, 4/7, 9/7) - \left(0, \frac{7280}{149}, \frac{2366}{149}, \frac{-1365}{149}\right) =$

$= \left(1, \frac{-50066}{1043}, \frac{-16115}{1043}, \frac{9257}{1043}\right)$

$\Rightarrow g_1 = f_1 = x^2$

$g_2 = f_2 - \frac{36}{98} g_1 = x - \frac{36}{98}x^2$

$g_3 = f_3 - \frac{14}{98} g_1 - \frac{4459}{149} g_2 = 1 - \frac{14}{98}x^2 - \frac{4459}{149}\left(x - \frac{36}{98}x^2\right) =$

$= 1 + \frac{11317}{1043}x^2 - \frac{4459}{149}x$

(desenvolvida pelo método de Gram-Schmidt)

5. Sejam $\mathcal{F} = \{x^2, x, 1\}$ e:

x	0	1	2	3
y	y_1	y_2	y_3	y_4

Ache o sistema normal para o problema do MMQ. Se N é a matriz de coeficientes deste sistema normal, encontre uma decomposição $N = L\cdot D\cdot L^t$ de N, onde L é triangular inferior, com 1 na diagonal principal e D é matriz diagonal. Mostre que $\langle Nx, x\rangle > 0$, se $x \neq 0$.

R.:

X	0	1	2	3
Y	y_1	y_2	y_3	y_4

e $\mathcal{F} = \{x^2, x, 1\} \Rightarrow$

$$\Rightarrow \begin{cases} a\cdot 0^2 + b\cdot 0 + c\cdot 1 = y_1 \\ a\cdot 1^2 + b\cdot 1 + c\cdot 1 = y_2 \\ a\cdot 2^2 + b\cdot 2 + c\cdot 1 = y_3 \\ a\cdot 3^2 + b\cdot 3 + c\cdot 1 = y_4 \end{cases} \Leftrightarrow \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\vec{a}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}}_{y} \Rightarrow$$

$\Rightarrow$ Por MMQ geométrico: $A^t\cdot A\cdot \vec{a} = A^t\cdot y \Leftrightarrow$

$$\begin{bmatrix} 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 & 9 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \Leftrightarrow$$

$$\underbrace{\begin{bmatrix} 98 & 36 & 14 \\ 36 & 14 & 6 \\ 14 & 6 & 4 \end{bmatrix}}_{N} \cdot \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\vec{a}} = \underbrace{\begin{bmatrix} y_2 + 4y_3 + 9y_4 \\ y_2 + 2y_3 + 3y_4 \\ y_1 + y_2 + y_3 + y_4 \end{bmatrix}}_{A^t\cdot y}$$

· Usando a matriz N anterior usaremos a decomposição em L·U para a primeira parte desta etapa.

$N = L \cdot U \Rightarrow$

$$\begin{array}{cc} U & L \\ \left[\begin{array}{ccc|ccc} 98 & 36 & 14 & 1 & 0 & 0 \\ 36 & 14 & 6 & 0 & 1 & 0 \\ 14 & 6 & 4 & 0 & 0 & 1 \end{array}\right] \end{array} \Rightarrow \left\{\begin{array}{l} L: m_{21} = 36/98 \text{ e } m_{31} = 14/98 \\ U: L_2' = L_2 - (36/98)L_1 \\ \quad L_3' = L_3 - (14/98)L_1 \end{array}\right. \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 98 & 36 & 14 & 1 & 0 & 0 \\ 0 & \frac{38}{49} & \frac{5}{7} & \frac{18}{49} & 1 & 0 \\ 0 & \frac{6}{7} & 2 & \frac{1}{7} & 0 & 1 \end{array}\right] \Rightarrow \left\{\begin{array}{l} L: m_{32} = (6/7)\cdot(49/38) = 21/19 \\ \qquad\qquad\qquad \Rightarrow \\ U: L_3' = L_3 - (21/19)L_2 \end{array}\right.$$

$$\left[\begin{array}{ccc|ccc} 98 & 36 & 14 & 1 & 0 & 0 \\ 0 & \frac{38}{49} & 6/7 & \frac{18}{49} & 1 & 0 \\ 0 & 0 & 20/19 & 1/7 & 21/19 & 1 \end{array}\right]$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{18}{49} & 1 & 0 \\ 1/7 & 21/19 & 1 \end{bmatrix} \text{ e } U = \begin{bmatrix} 98 & 36 & 14 \\ 0 & \frac{38}{49} & 6/7 \\ 0 & 0 & \frac{20}{19} \end{bmatrix}$$

$\Rightarrow$ como $U = D \cdot L^t$:

$$\begin{bmatrix} 98 & 36 & 14 \\ 0 & \frac{38}{49} & \frac{6}{7} \\ 0 & 0 & \frac{20}{19} \end{bmatrix} = \begin{bmatrix} d_{11} & d_{11}\cdot\frac{18}{49} & d_{11}\cdot\frac{1}{7} \\ 0 & d_{22} & d_{22}\cdot\frac{21}{19} \\ 0 & 0 & d_{33} \end{bmatrix} \Rightarrow \left\{\begin{array}{l} d_{33} = \frac{20}{19} \\ d_{22} = 38/49 \\ d_{11} = 98 \end{array}\right.$$

$$\Rightarrow N = \begin{bmatrix} 1 & 0 & 0 \\ \frac{18}{49} & 1 & 0 \\ \frac{1}{7} & \frac{21}{19} & 1 \end{bmatrix} \cdot \begin{bmatrix} 98 & 0 & 0 \\ 0 & \frac{38}{49} & 0 \\ 0 & 0 & \frac{20}{19} \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{18}{49} & \frac{1}{7} \\ 0 & 1 & \frac{21}{19} \\ 0 & 0 & 1 \end{bmatrix}$$

$\cdot \langle Nx, x \rangle$:

$$\begin{bmatrix} 98 & 36 & 14 \\ 36 & 14 & 6 \\ 14 & 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 98x_1 + 36x_2 + 14x_3 \\ 36x_1 + 14x_2 + 6x_3 \\ 14x_1 + 6x_2 + 4x_3 \end{bmatrix} \Rightarrow$$

$\langle Nx, x \rangle = (98x_1 + 36x_2 + 14x_3)x_1 + (36x_1 + 14x_2 + 6x_3)$
$\cdot x_2 + (14x_1 + 6x_2 + 4x_3) \cdot x_3 =$
$= 98x_1^2 + 14x_2^2 + 4x_3^2 + 72x_1x_2 + 28x_1x_3 + 12x_2x_3,$

Como $f(x,y,z) = \dfrac{xy + xz + yz}{x^2 + y^2 + z^2} \Rightarrow$

$-\dfrac{1}{2} \leq \dfrac{xy + xz + yz}{x^2 + y^2 + z^2} \leq 1, \forall (x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. Então:

$\left(-\dfrac{1}{2}\right)(x^2 + y^2 + z^2) \leq xy + xz + yz \Rightarrow$

$\Rightarrow x^2 + y^2 + z^2 + \left(-\dfrac{1}{2}\right)(x^2 + y^2 + z^2) \leq xy + xz + yz + x^2 + y^2 + z^2 \Leftrightarrow$

$(1/2)(x^2 + y^2 + z^2) \leq xy + xz + yz + x^2 + y^2 + z^2$. Como $(1/2) \cdot$
$\cdot (x^2 + y^2 + z^2) > 0, \forall (x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\} \Rightarrow$

$0 < xy + xz + yz + x^2 + y^2 + z^2$, que é menor que $\langle Nx, x \rangle$.
Ou seja, $0 < xy + xz + yz + x^2 + y^2 + z^2 < \langle Nx, x \rangle \Rightarrow$
$\underline{\langle Nx, x \rangle > 0}$.

Resolução alternativa ao Ex. 5:

Do item anterior, temos: $N = L \cdot D L^t \Rightarrow$ como $\langle Nx, x\rangle = (Nx)^t \cdot x = (LDL^t x)^t x = \underbrace{x^t L}_{y} \cdot D^t \underbrace{L^t x}_{y}$

$$y = L^t x = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow$$

$\langle Nx, x\rangle = y^t \cdot D \cdot y = a_1 y_1^2 + a_2 y_2^2 + a_3 y_3^2$, que é soma de termos positivos □