Lecture Notes on Lie Algebras and Lie Groups

Luiz Agostinho Ferreira

Instituto de Física de São Carlos - IFSC/USP
Universidade de São Paulo
Caixa Postal 369, CEP 13560-970
São Carlos-SP, Brasil

May - 2022

.

Contents

Chapter 1

Elements of Group Theory

1.1 The concept of group

The idea of groups is one that has evolved from some very intuitive concepts we have acquired in our attempts of understanding Nature. One of these is the concept of mathematical structure. A set of elements can have a variety of degrees of structure. The set of the letters of the alphabet has some structure in it. They are ordered as $A < B < C ... < Z$. Although this order is fictitious, since it is a convenction, it endows the set with a structure that is very useful. Indeed, the relation between the letters can be extended to words such that a telephone directory can be written in an "ordered way". The set of natural numbers possesses a higher mathematical structure. In addition of being "naturally" ordered we can perform operations on it. We can do binary operations like adding or multiplying two elements and also unary operations like taking the square root of an element (in this case the result is not always in the set). The existence of an operation endows the set with a mathematical structure. In the case when this operation closes within the set, i.e. the composition of two elements is again an element of the set, the endowed structure has very nice properties. Let us consider some examples.

Example 1.1 *The set of integer numbers (positive and negative) is closed under the operations of addition, subtration and multiplication, but is not closed under, division. The set of natural numbers, on the other hand is not closed under subtraction and division but does close under addition and multiplication.*

Example 1.2 *Consider the set of all human beings living and dead and define a binary operation as follows: for any two persons take the latest common* *forefather. For the case of two brothers this would be their father; for two cousins their common grandfather; for a mother and his son, the mother's father, etc. This set is closed or not under such operation depending, of course, on how we understand everything has started.*

Example 1.3 *Take a rectangular box and imagine three mutually orthogonal axis, x, y and z, passing through the center of the box and each of them being orthogonal to two sides of the box. Consider the set of three rotations:*

$$x \equiv \textit{a half turn about the x-axis}$$
$$y \equiv \textit{a half turn about the y-axis}$$
$$z \equiv \textit{a half turn about the z-axis}$$

and let the operation on this set be the composition of rotations. So if we perform y and then x we get z, z then y we get x, and x then z we get y. However if we perform x then y and then z we get that the box gets back to its original position. Therefore the set is not closed. If we add to the set the operation (identity) I "leaves the box as it is", then we get a closed set of rotations.

For a set to be considered a group it has to have, in addition of a binary operation and closure, some other special structures. We now start discussing them by giving the formal definition of a group.

Definition 1.1 *An abstract group G is a set of elements furnished with a composition law (or product) defined for every pair of elements of G and that satisfies:*

a) If g_1 and g_2 are elements of G, then the product $g_1 g_2$ is also an element of G. (closure property)

b) The composition law is associative*, that is $(g_1 g_2) g_3 = g_1 (g_2 g_3)$ for every g_1, g_2 and $g_3 \in G$.*

c) There exists an unique element e in G , called identity element *such that $eg = ge = g$ for every $g \in G$.*

d) For every element g of G, there exists an unique inverse element*, denoted g^{-1} , such that $g^{-1} g = g g^{-1} = e$.*

There are some redundancies in these definition, and the axioms *c)* and *d)* could, in fact, be replaced by the weaker ones:

c′) There exists an element e in G, called left identity *such that $eg = g$ for every $g \in G$.*

d′) For every element g of G, there exists a left inverse*, denoted g^{-1} , such that $g^{-1}g = e$.*

These weaker axioms *c′)* and *d′)* together with the associativity property imply *c)* and *d)*. The proof is as follows:

Let g_2 be a left inverse of g_1, i.e. $(g_2g_1 = e)$, and g_3 be a left inverse of g_2, i.e. $(g_3g_2 = e)$. Then we have, since e is a left identity, that

$$\begin{array}{rll} e = ee & \\ g_2g_1 = (g_2g_1)e & \textit{since } g_2g_1 = e \\ g_3(g_2g_1) = g_3((g_2g_1)e) & \textit{multiplying both sides by } g_3 \\ (g_3g_2)g_1 = (g_3g_2)g_1e & \textit{using associativity} \\ eg_1 = eg_1e & \textit{since } g_3g_2 = e \\ g_1 = g_1e & \textit{using the fact } e \textit{ is a left identity.} \end{array}$$

Therefore e is also a right identity. We now want to show that a left inverse is also a right inverse. Since we know that e is both a left and right identity we have:

$$\begin{array}{rll} eg_2 = g_2e & \\ (g_2g_1)g_2 = g_2e & \textit{since } g_2 \textit{ is a left inverse of } g_1 \\ g_3((g_2g_1)g_2) = g_3(g_2e) & \textit{multiplying by } g_3 \textit{ where } g_3g_2 = e \\ (g_3g_2)(g_1g_2) = (g_3g_2)e & \textit{using associativity.} \\ e(g_1g_2) = ee & \textit{since } g_3g_2 = e. \\ g_1g_2 = e & \textit{since } e \textit{ is identity.} \end{array}$$

Therefore g_2 is also a right inverse of g_1 . Let us show the uniqueness of the identity and the inverses.

Any right and left identity is unique independently of the fact of the product being associative or not. Suppose there exist two identities e and e' such that $ge = eg = e'g = ge' = g$ for any $g \in G$. Then for $g = e$ we have $ee' = e$ and for $g = e'$ we have $ee' = e'$. Therefore $e = e'$ and the identity is unique.

Suppose that g has two right inverses g_1 and g_2 such that $gg_1 = gg_2 = e$ and suppose g_3 is a left inverse of g, i.e. $g_3g = e$. Then $g_3(gg_1) = g_3(gg_2)$ and using associativity we get $(g_3g)g_1 = (g_3g)g_2$ and so $eg_1 = eg_2$ and then $g_1 = g_2$. Therefore the right inverse is unique. A similar argument can be used to show the uniqueness of the left inverse. Now if g_3 and g_1 are respectively the left and right inverses of g, we have $g_3g = e = gg_1$ and then using associativity we get $(g_3g)g_1 = eg_1 = g_1 = g_3(gg_1) = g_3e = g_3$. So the left and right inverses are the the same.

We are very used to the fact that the inverse of the product of two elements (of a group, for instance) is the product of their inverses in the reversed order, i.e., the inverse of g_1g_2 is $g_2^{-1}g_1^{-1}$. However this result is true for products (or composition laws) which are associative. It may not be true for non associative products.

Example 1.4 *The subtraction of real numbers is not an associative operation, since* $(x-y)-z \neq x-(y-z)$ *, for* x, y *and* z *being real numbers. This operation possesses a right unity element, namely zero, but does not possess left unity since,* $x-0=x$ *but* $0-x \neq x$ *. The left and right inverses of* x *are equal and are* x *itself, since* $x-x=0$ *. Now the inverse of* $(x-y)$ *is not* $(y^{-1}-x^{-1})=(y-x)$ *. Since* $(x-y)-(y-x)=2(x-y) \neq 0$ *. This is an ilustration of the fact that for a non associative operation, the inverse of* $x*y$ *is not necessarily* $y^{-1}*x^{-1}$ *.*

The definition of abstract group given above is not the only possible one. There is an alternative definition that does not require inverse and identity. We could define a group as follows:

Definition 1.2 (alternative) *Take the definition of group given above (assuming it is a non empty set) and replace axioms c) and d) by: "For any given elements* $g_1, g_2 \in G$ *there exists a unique* g *satisfying* $g_1 g = g_2$ *and also a unique* g' *satisfying* $g' g_1 = g_2$ *".*

This definition is equivalent to the previous one since it implies that, given any two elements g_1 and g_2 there must exist unique elements e_1^L and e_2^L in G such that $e_1^L g_1 = g_1$ and $e_2^L g_2 = g_2$. But it also implies that there exists a unique g such that $g_1 g = g_2$. Therefore, using associativity, we get

$$(e_1^L g_1) g = g_1 g = g_2 = e_1^L (g_1 g) = e_1^L g_2 \tag{1.1}$$

From the uniquiness of e_2^L we conclude that $e_1^L = e_2^L$.Thus this alternative definition implies the existence of a unique left identity element e^L. On the other hand it also implies that for every $g \in G$ there exist an unique g_L^{-1} such that $g_L^{-1} g = e^L$. Consequently axioms c') and d') follows from the alternative axiom above.

Example 1.5 *The set of real numbers is a group under addition but it is not under multiplication, division, and subtraction. The last two operations are not associative and the element zero has no inverse under multiplication. The natural numbers under addition are not a group since there are no inverse elements.*

Example 1.6 *The set of all nonsingular* $n \times n$ *matrices is a group under matrix product. The set of* $p \times q$ *matrices is a group under matrix addition.*

Example 1.7 *The set of rotations of a box discussed in example 1.3 is a group under composition of rotations when the identity operation I is added to the set. In fact the set of all rotations of a body in 3 dimensions (or in any number of dimensions) is a group under the composition of rotations. This is called the rotation group and is denoted SO(3).*

Example 1.8 *The set of all human beings living and dead with the operation defined in example 1.2 is not a group. There are no unity and inverse elements and the operation is not associative. Indeed, consider the case of two brothers Fred and Tony and a cousin Paul. The father of Fred and Tone is Alex, and he is the brother of Mark, the father of Paul. Therefore, by operating Fred and Tony we get Alex, i.e. (Fred $\circ$ Tony)=Alex. Now, operating on Paul we get ((Fred $\circ$ Tony)$\circ$ Paul)=Alex $\circ$ Paul=father of Alex and Mark. However, if we do (Fred $\circ$ (Tony $\circ$ Paul))= grandfather of Alex and Mark. So, the operation is indeed non-associative.*

Example 1.9 *Consider the permutations of n elements which we shall represent graphically. In the case of three elements, for instance, the graph shown in figure 1.1 means the element* 1 *replaces* 3, 2 *replaces* 1 *and* 3 *replaces* 2. *We can compose permutations as shown in fig. 1.2. The set of all permutations of n elements forms a group under the composition of permutations. This is called the* symmetric group *of degree n, and it is generally denoted by S_n . The number of elements of this group is n!, since this is the number of distint permutations of n elements.*

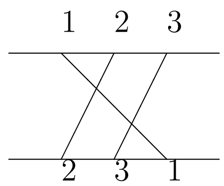

Figure 1.1: A permutation of three objects

Example 1.10 *The N^{th} roots of the unity form a group under multiplication. These roots are $exp(i2\pi m/N)$ with m=0,1,2..., N-1. The identity elements is* $1(m = 0)$ *and the inverse of $exp(i2\pi m/N)$ is $exp(i2\pi(N-m)/N)$. This group is called the* cyclic group of order N *and is denoted by Z_N .*

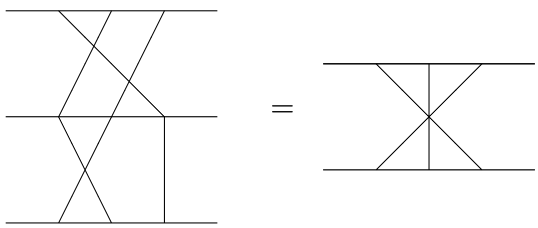

Figure 1.2: A composition of permutations

We say two elements, g_1 and g_2 , of a group commute with each other if their product is independent of the order, i.e., if $g_1 g_2 = g_2 g_1$. If all elements of a given group commute with one another then we say that this group is *abelian*. The real numbers under addition or multiplication (without zero) form an abelian group. The cyclic groups Z_n (see example 1.10) are abelian for any n. The symmetric group S_n (see example 1.9) is not abelian for $n > 2$, but it is abelian for $n = 2$.
Let us consider some groups of order two, i.e., with two elements. The elements 0 and 1 form a group under addition modulo 2. We have

$$0 + 0 = 0, \quad 0 + 1 = 1, \quad 1 + 0 = 1, \quad 1 + 1 = 0 \tag{1.2}$$

The elements 1 and -1 also form a group, but under multiplication. We have

$$1.1 = -1.(-1) = 1, \quad 1.(-1) = (-1).1 = -1 \tag{1.3}$$

The symmetric group of degree 2, S_2 , (see example 1.9) has two elements as shown in fig. 1.3.

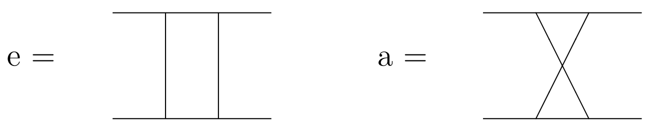

Figure 1.3: The elements of S_2

They satisfy

$$e.e = e, \quad e.a = a.e = a, \quad a.a = e \tag{1.4}$$

These three examples of groups are in fact different realizations of the same abstract group. If we make the identifications as shown in fig. 1.4 we see that the structure of these groups are the same. We say that these groups are isomorphic.

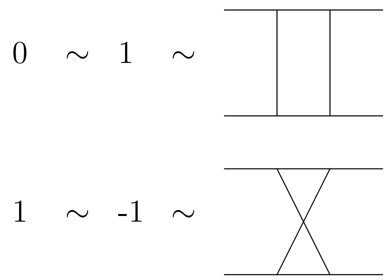

Figure 1.4: Isomorphism

Definition 1.3 *Two groups G and G' are* isomorphic *if their elements can be put into one-to-one correspondence which is preserved under the composition laws of the groups. The mapping between these two groups is called an* isomorphism.

If g_1, g_2 and g_3 are elements of a group G satisfying $g_1 g_2 = g_3$ and if G is isomorphic to another group G' , then the corresponding elements g_1', g_2' and g_3' in G' have to satisfy $g_1' g_2' = g_3'$.

There is the possibility of a group G being mapped into another group G' but not in a one-to-one manner, i.e. two or more elements of G are mapped into just one element of G'. If such mapping respects the product law of the groups we say they are *homomorphic.* The mapping is then called a *homomorphism* between G and G'.

Example 1.11 *Consider the cyclic groups Z_6 with elements e, a, a^2, ... a^5 and $a^6 = e$, and Z_2 with elements e' and b ($b^2 = e'$). The mapping $\sigma : Z_6 \rightarrow Z_2$ defined by*

$$\begin{aligned} \sigma(e) = \sigma(a^2) = \sigma(a^4) = e' \\ \sigma(a) = \sigma(a^3) = \sigma(a^5) = b \end{aligned} \tag{1.5}$$

is a homomorphism between Z_6 and Z_2.

Analogously one can define mappings of a given group G into itself, i.e., for each element $g \in G$ one associates another element g'. The one-to-one mappings which respect the product law of G are called *automorphisms* of G. In other words, an automorphism of G is an isomorphism of G onto itself.

Definition 1.4 *A one-to-one mapping $\sigma : G \to G$ is said to be an automorphism of G if it respects the product law in G, i.e., if $gg' = g''$ then $\sigma(g)\sigma(g') = \sigma(g'')$.*

Example 1.12 *Consider again the cyclic group Z_6 and the mapping $\sigma : Z_6 \to Z_6$ defined by*

$$\begin{array}{ccc} \sigma(e) = e & \sigma(a) = a^5 & \sigma(a^2) = a^4 \\ \sigma(a^3) = a^3 & \sigma(a^4) = a^2 & \sigma(a^5) = a \end{array} \tag{1.6}$$

This is an automorphism of Z_6.

In fact the above example is just a particular case of the automorphism of any abelian group where a given element is mapped into its inverse.

Notice that if σ and σ' are two automorphisms of a group G, then the composition of both $\sigma\sigma'$ is also an automorphism of G. Such composition is an associative operation. In addition, since automorphisms are one-to-one mappings, they are invertible. Therefore, if one considers the set of all automorphisms of a group G together with the identity mapping of G into G, one gets a group which is called the *automorphism group* of G.

Any element of G gives rise to an automorphism. Indeed, define the mapping $\sigma_{\bar{g}} : G \to G$

$$\sigma_{\bar{g}}(g) \equiv \bar{g}\, g\, \bar{g}^{-1} \qquad g, \bar{g} \in G \text{ and } \bar{g} \text{ fixed} \tag{1.7}$$

Then

$$\begin{aligned} \sigma_{\bar{g}}(gg') &= \bar{g}\, gg'\, \bar{g}^{-1} \\ &= \bar{g}\, g\bar{g}^{-1}\bar{g}g'\, \bar{g}^{-1} \\ &= \sigma_{\bar{g}}(g)\, \sigma_{\bar{g}}(g') \end{aligned} \tag{1.8}$$

and so it constitutes an automorphism of G. That is called an *inner automorphism*. The automorphism group that they generate is homomorphic to G, since

$$\sigma_{\bar{g}_1}(\sigma_{\bar{g}_2}(g)) = \bar{g}_1\bar{g}_2\, g\, \bar{g}_2^{-1}\bar{g}_1^{-1} = \sigma_{\bar{g}_1\bar{g}_2}(g) \tag{1.9}$$

All automorphisms which are not of such type are called *outer automorphisms*. In the cases where G has a trivial center (see the concept of center in Definition 1.9) then the automorphism group is isomorphic to G, and not only homomorphic.

1.2 Subgroups

A subset H of a group G which satisfies the group postulates under the same composition law used for G, is said to be a *subgroup* of G. The identity element and the whole group G itself are subgroups of G. They are called *improper subgroups*. All other subgroups of a group G are called *proper subgroups*. If H is a subgroup of G, and K a subgroup of H then K is a subgroup of G.
In order to find if a subset H of a group G is a subgroup we have to check only two of the four group postulates. We have to chek if the product of any two elements of H is in H (closure) and if the inverse of each element of H is in H. The associativity property is guaranteed since the composition law is the same as the one used for G. As G has an identity element it follows from the closure and inverse element properties of H that this identity element is also in H.

Example 1.13 *The real numbers form a group under addition.The integer numbers are a subset of the real numbers and also form a group under the addition. Therefore the integers are a subgroup of the reals under addition. However the reals without zero also form a group under multiplication, but the integers (with or without zero) do not. Consequently the integers are not a subgroup of the reals under multiplication.*

Example 1.14 *Take G to be the group of all integers under addition, H_1 to be all even integers under addition, H_2 all multiples of $2^2 = 4$ under addition, H_3 all multiples of $2^3 = 8$ under addition and son on. Then we have*

$$\begin{array}{rccccc} G: & ...-2, & -1, & 0, & 1, & 2... \\ H_1: & ...-4, & -2, & 0, & 2, & 4... \\ H_2: & ...-8, & -4, & 0, & 4, & 8... \\ H_3: & ...-16, & -8, & 0, & 8, & 16... \\ H_n: & ...-2.2^n, & -2^n, & 0, & 2^n, & 2.2^n... \end{array}$$

We see that each group is a subgroup of all groups above it, i.e.

$$G \supset H_1 \supset H_2 ... \supset H_n ... \tag{1.10}$$

Moreover there is a one to one correspondence between any two groups of this list such that the composition law is preserved. Therefore all these groups are isomorphic to one another

$$G \sim H_1 \sim H_2 ... \sim H_n ... \tag{1.11}$$

This shows that a group can be isomorphic to one of its proper subgroups. The same can not happen for finite groups.

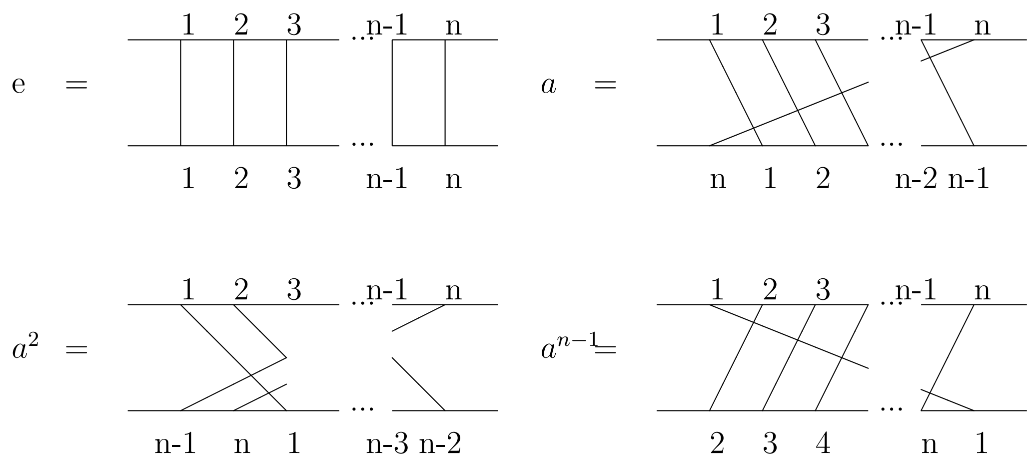

Figure 1.5: The cyclic permutations of n objects

Example 1.15 *The cyclic group Z_n , defined in example 1.10 , is a subgroup of the symmetric group S_n, defined in example 1.9 . In order to see this, consider the elements of S_n corresponding to cyclic permutations given in figure1.5. These elements form a subgroup of S_n which has the same structure as the group formed by the n^{th} roots of unity under ordinary multiplication of complex numbers, i.e., Z_n.*

This example is a particular case of a general theorem in the theory of finite groups, which we now state without proof. For the proof, see [HAM 62, chap 1] or [BUD 72, chap 9].

Theorem 1.1 (Cayley) *Every group G of order n is isomorphic to a subgroup of the symmetric group S_n.*

Definition 1.5 *The* order *of a finite group is the number of elements it has.*

Another important theorem about finite groups is the following.

Theorem 1.2 (Lagrange) *The order of a subgroup of a finite group is a divisor of the order of the group.*

Corollary 1.1 *If the order of a finite group is a prime number then it has no proper subgroups.*

The proof involves the concept of cosets and it is given in section 1.4. A finite group of prime order is necessarily a cyclic group and can be generated from any of its elements other than the identity element.

We say an element g of a group G is *conjugate* to an element $g' \in G$ if there exists $\bar{g} \in G$ such that

$$g = \bar{g} g' \bar{g}^{-1} \tag{1.12}$$

This concept of conjugate elements establishes an equivalence relation on the group. Indeed, g is conjugate to itself (just take $\bar{g} = e$), and if g is conjugate to g', so is g' conjugate to g (since $g' = \bar{g}^{-1} g \bar{g}$). In addition, if g is conjugate to g' and g' to g'', i.e. $g' = \tilde{g} g'' \tilde{g}^{-1}$, then g is conjugate to g'', since $g = \bar{g}\tilde{g} g'' \tilde{g}^{-1} \bar{g}^{-1}$. One can use such equivalence relation to divide the group G into classes.

Definition 1.6 *The set of elements of a group G which are conjugate to each other constitute a* conjugacy class *of G.*

Obviously different conjugacy classes have no common elements. The indentity element e constitute a conjugacy class by itself in any group. Indeed, if g' is conjugate to the identity e, $e = g g' g^{-1}$, then $g' = e$.

Given a subgroup H of a group G we can form the set of elements $g^{-1} H g$ where g is any fixed element of G and H stands for any element of the subgroup H. This set is also a subgroup of G and is said to be a *conjugate subgroup* of H in G. In fact the conjugate subgroups of H are all isomorphic to H, since if $h_1, h_2 \in H$ and $h_1 h_2 = h_3$ we have that $h_1' = g^{-1} h_1 g$ and $h_2' = g^{-1} h_2 g$ satisfy

$$h_1' h_2' = g^{-1} h_1 g g^{-1} h_2 g = g^{-1} h_1 h_2 g = g^{-1} h_3 g = h_3' \tag{1.13}$$

Notice that the images of two different elements of H, under conjugation by $g \in G$, can not be the same. Because if they were the same we would have

$$g^{-1} h_1 g = g^{-1} h_2 g \rightarrow g(g^{-1} h_1 g) g^{-1} = h_2 \rightarrow h_1 = h_2 \tag{1.14}$$

and that is a contradiction.
By choosing various elements $g \in$ G we can form different conjugate subgroups of H in G. However it may happen that for all $g \in G$ we have

$$g^{-1} H g = H \tag{1.15}$$

This means that all conjugate subgroups of H in G are not only isomorphic to H but are identical to H. In this case we say that the subgroup H is an *invariant subgroup* of G. This implies that, given an element $h_1 \in H$ we can find, for any element $g \in G$, an element $h_2 \in H$ such that

$$g^{-1} h_1 g = h_2 \rightarrow h_1 g = g h_2 \tag{1.16}$$

We can write this as

$$gH = Hg \tag{1.17}$$

and say that the invariant subgroup H, taken as an entity, commutes with all elements of G. The identity element and the group G itself are trivial examples of invariant subgroups of G. Any subgroup of an abelian group is an invariant subgroup.

Definition 1.7 *We say a group G is simple if its only invariant subgroups are the identity element and the group G itself. In other words, G is simple if it has no invariant proper subgroups. We say G is semisimple if none of its invariant subgroups is abelian.*

Example 1.16 *Consider the group of the non-singular real $n \times n$ matrices, which is generally denoted by $GL(n)$. The matrices of this group with unit determinant form a subgroup since if $detM = detN = 1$ we have $det(M.N) = 1$ and $detM^{-1} = detM = 1$. This subgroup of $GL(n)$ is denoted by $SL(n)$. If $g \in GL(n)$ and $M \in SL(n)$ we have that $g^{-1}Mg \in SL(n)$ since $det(g^{-1}Mg) = detM = 1$. Therefore $SL(n)$ is an invariant subgroup of $GL(n)$ and consequently the latter is not simple. Consider now the matrices of the form $R \equiv x\, \mathbb{1}_{n\times n}$, with x being a non-zero real number, and $\mathbb{1}_{n\times n}$ being the $n \times n$ identity matrix. Notice, that such set of matrices constitute a subgroup of $GL(n)$, since the identity belongs to it, the product of any two of them belongs to the set, and the inverse of $R \equiv x\, \mathbb{1}_{n\times n}$ is $R^{-1} \equiv (1/x)\, \mathbb{1}_{n\times n}$, which is also an element of the set. In addition, such subgroup is invariant since any matrix R commutes with any element of $GL(n)$ and so it is invariant under conjugation. Since that subgroup is abelian, it follows that $GL(n)$ is not semisimple.*

Definition 1.8 *Given an element g of a group G we can form the set of all elements of G which commute with g, i.e., all $x \in G$ such that $xg = gx$. This set is called the* centralizer *of g and it is a subgroup of G.*

In order to see it is a subgroup of G, take two elements x_1 and x_2 of the centralizer of g, i.e., $x_1g = gx_1$ and $x_2g = gx_2$. Then it follows that $(x_1x_2)g = x_1(x_2g) = x_1(gx_2) = g(x_1x_2)$. Therefore x_1x_2 is also in the centralizer. On the other hand, we have that

$$x_1^{-1}(x_1g)x_1^{-1} = x_1^{-1}(gx_1)x_1^{-1} \rightarrow gx_1^{-1} = x_1^{-1}g. \tag{1.18}$$

So the inverse of an element of the centralizer is also in the centralizer. Therefore the centralizer of an element $g \in G$ is a subgroup of G. Notice that although all elements of the centralizer commute with a given element g they do not have to commute among themselves and therefore it is not necessarily an abelian subgroup of G.

Definition 1.9 *The* center of a group G *is the set of all elements of* G *which commute with all elements of* G.

We could say that the center of G is the intersection of the centralizers of all elements of G. The center of a group G is a subgroup of G and it is abelian , since by definition its elements have to commute with one another. In addition, it is an (abelian) invariant subgroup.

Example 1.17 *The set of all unitary $n \times n$ matrices form a group, called $U(n)$, under matrix multiplication. That is because if U_1 and U_2 are unitary ($U_1^\dagger = U_1^{-1}$ and $U_2^\dagger = U_2^{-1}$) then $U_3 \equiv U_1 U_2$ is also unitary. In addition the inverse of U is just $U^\dagger$ and the identity is the unity $n \times n$ matrix. The unitary matrices with unity determinant constitute a subgroup, because the product of two of them, as well as their inverses, have unity determinant. That subgroup is denoted $SU(n)$. It is an invariant subgroup of $U(n)$ because the conjugation of a matrix of unity determinant by any unitary matrix gives a matrix of unity determinant, i.e.* $\det\left(UMU^\dagger\right) = \det M = 1$, *with $U \in U(n)$ and $M \in SU(n)$. Therefore, $U(n)$ is not simple. However, it is not semisimple either, because it has an abelian subgroup constituted by the matrices $R \equiv e^{i\theta}\, \mathbb{1}_{n\times n}$, with θ being real. Indeed, the multiplication of any two R's is again in the set of matrices R, the inverse of R is $R^{-1} = e^{-i\theta}\, \mathbb{1}_{n\times n}$, and so a matrix in the set. Notice the subgroup constituted by the matrices R is isomorphic to $U(1)$, the group of 1×1 unitary matrices, i.e. phases $e^{i\theta}$. Since, the matrices R commute with any unitary matrix, it follows they are invariant under conjugation by elements of $U(n)$. Therefore, the subgroup $U(1)$ is an abelian invariant subgroup of $U(n)$, and so $U(n)$ is not semisimple. The subgroup $U(1)$ is in fact the* center *of $U(n)$, i.e. the set of matrices commuting with all unitary matrices. Notice, that such $U(1)$ is not a subgroup of $SU(n)$, since their elements do not have unity deteminant. However, the discrete subset of matrices $e^{2\pi i m/n}\, \mathbb{1}_{n\times n}$ with $m = 0, 1, 2...(n-1)$ have unity determinant and belong to $SU(n)$. They certainly commute with all $n \times n$ matrices, and constitute the center of $SU(n)$. Those matrices form an abelian invariant subgroup of $SU(n)$, which is isomorphic to Z_n. Therefore, $SU(n)$ is not semisimple.*

Example 1.18 *Consider a 2×2 complex matrix U, and its inverse*

$$U = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}; \qquad U^{-1} = \frac{1}{(Z_1\, Z_4 - Z_2\, Z_3)} \begin{pmatrix} Z_4 & -Z_2 \\ -Z_3 & Z_1 \end{pmatrix} \tag{1.19}$$

with Z_a*,* $a = 1, 2, 3, 4$*, being complex numbers. By imposing that* $\det U = 1$ *and* $U^{-1} = U^{\dagger}$*, one gets that*

$$Z_4 = Z_1^*; \qquad Z_3 = -Z_2^*; \qquad \mid Z_1 \mid^2 + \mid Z_2 \mid^2 = 1 \tag{1.20}$$

Since the last relation above is the equation for a three dimensional sphere S^3*, one observes that the elements of the group* $SU(2)$ *can be put into a one-to-one correspondence with the point of such a sphere. So, the set of elements of the group* $SU(2)$ *constitute a manifold isomorphic to the three sphere. We can take the* 3 *real parameters of* $SU(2)$ *to be* 3 *angles, i.e.*

$$Z_1 = \cos\theta\, e^{i\varphi_1}; \quad Z_2 = \sin\theta\, e^{i\varphi_2}; \quad 0 \leq \theta \leq \frac{\pi}{2}; \quad 0 \leq \varphi_a \leq 2\pi; \quad a = 1, 2 \tag{1.21}$$

Therefore, the elements of $SU(2)$ *take the form*

$$U = \begin{pmatrix} \cos\theta\, e^{i\varphi_1} & \sin\theta\, e^{i\varphi_2} \\ -\sin\theta\, e^{-i\varphi_2} & \cos\theta\, e^{-i\varphi_1} \end{pmatrix} \tag{1.22}$$

Example 1.19 *The set of all orthogonal* $n \times n$ *matrices form a group, called* $O(n)$*, under matrix multiplication. If* M_1 *and* M_2 *are orthogonal, i.e.* $M_1\, M_1^T = M_1^T\, M_1 = \mathbb{1}$ *and* $M_2\, M_2^T = M_2^T\, M_2 = \mathbb{1}$*, the* $M_1\, M_2$ *is also orthogonal, since* $M_1\, M_2\, (M_1\, M_2)^T = M_1\, M_2\, M_2^T\, M_1^T = \mathbb{1}$*. Note that the determinant of an orthogonal matrix is either* 1 *or* -1*, i.e.* $\det M = \pm 1$*. The subset of orthogonal* $n \times n$ *matrices of unit determinant form a subgroup, and it is denoted* $SO(N)$*. Note that* $SO(N)$ *is an invariant subgroup of* $O(N)$*, since, if* $S \in SO(N)$*, and* $M \in O(N)$*, we have that* $\det(M\, S\, M^{-1}) = \det(S) = 1$*. In addition, the center of* $O(N)$ *has two elements, namely* $\mathbb{1}_{N\times N}$ *and* $-\mathbb{1}_{N\times N}$*. The center of* $SO(N)$*, for* N *even, has also two elements, namely* $\mathbb{1}_{N\times N}$ *and* $-\mathbb{1}_{N\times N}$*. The center of* $SO(N)$*, for* N *odd, has just the identity matrix* $\mathbb{1}_{N\times N}$*. For the case* $N = 2$*, the groups* $O(2)$ *and* $SO(2)$ *are abelian, and so the center, in both cases, is the whole group itself.*

1.3 Direct Products

We say a group G is the *direct product* of its subgroups $H_1, H_2 ... H_n$, denoted by $G = H_1 \otimes H_2 \otimes H_3 ... \otimes H_n$, if

1. the elements of different subgroups commute

2. Every element $g \in G$ can be expressed in one and only one way as

$$g = h_1 h_2 ... h_n \tag{1.23}$$

where h_i is an element of the subgroup H_i , $i = 1, 2, ..., n$.

From these requirements it follows that the subgroups H_i have only the identity e in common. Because if $f \neq e$ is a common element to H_2 and H_5 say, then the element $g = h_1 f h_3 h_4 f^{-1} h_6 ... h_n$ could be also written as $g = h_1 f^{-1} h_3 h_4 f h_6 ... h_n$. Every subgroup H_i is an invariant subgroup of G, because if $h'_i \in H_i$ then

$$g^{-1} h'_i g = (h_1 h_2 ... h_n)^{-1} h'_i (h_1 h_2 ... h_n) = h_i^{-1} h'_i h_i \in H_i \tag{1.24}$$

Example 1.20 *Consider the cyclic group Z_6 with elements e, a, a^2, a^3, a^4 and a^5 (and $a^6 = e$). It can be written as the direct product of its subgroups $H_1 = \{e, a^2, a^4\}$ and and $H_2 = \{e, a^3\}$ since*

$$e = ee, a = a^4 a^3, a^2 = a^2 e, a^3 = ea^3, a^4 = a^4 e, a^5 = a^2 a^3 \tag{1.25}$$

Therefore we write $Z_6 = H_1 \otimes H_2$ (or $Z_6 = Z_3 \otimes Z_2$).

Given two groups G and G' we can construct another group by taking the direct product of G and G' as follows: the elements of $G'' = G \otimes G'$ are formed by the pairs (g, g') where $g \in G$ and $g' \in G'$. The composition law for G'' is defined by

$$(g_1, g'_1)(g_2, g'_2) = (g_1 g_2, g'_1 g'_2) \tag{1.26}$$

where $g_1 g_2$, $(g'_1 g'_2)$ is the product of g_1 by g_2, (g'_1 by g'_2) according to the composition law of G (G'). If e and e' are respectively the identity elements of G and G', then the sets $G \otimes 1 = \{(g, e') \mid g \in G\}$ and $1 \otimes G' = \{(e, g') \mid g' \in G'\}$ are subgroups of $G'' = G \otimes G'$ and are isomorphic respectively to G and G'. Obviously $G \otimes 1$ and $1 \otimes G'$ are invariant subgroups of $G'' = G \otimes G'$.

1.4 Cosets

Given a group G and a subgroup H of G we can divide the group G into disjoint sets such that any two elements of a given set differ by an element of H multiplied from the right. That is, we construct the sets

$gH \equiv \{$ all elements gh of G such that h is any element of H and g is a fixed element of $G\}$

If $g = e$ the set eH is the subgroup H itself. All elements in a set gH are different, because if $gh_1 = gh_2$ then $h_1 = h_2$. Therefore the numbers of elements of a given set gH is the same as the number of elements of the subgroup H. Also an element of a set gH is not contained by any other set $g'H$ with $g' \neq g$. Because if $gh_1 = g'h_2$ then $g = g'h_2h_1^{-1}$ and therefore g would be contained in $g'H$ and consequently $gH \equiv g'H$[1]. Thus we have split the group G into disjoint sets, each with the same number of elements, and a given element $g \in G$ belongs to one and only one of these sets.

Proof of Lagrange's theorem(section 1.2).
From the considerations above we see that for a finite group G of order m with a proper subgroup H of order n, we can write

$$m = kn \tag{1.27}$$

where k is the number of disjoint sets gH.□

The set of elements gH are called *left cosets* of H in G. They are certainly not subgroups of G since they do not contain the identity element, except for the set $eH = H$.

Analogously we could have split G into sets Hg which are formed by elements of G which differ by an element of H multiplied from the left. The same results would be true for these sets. They are called *right cosets* of H in G.

The set of left cosets of H in G is denoted by G/H and is called the *left coset space*. An element of G/H is a set of elements of G, namely gH. Analogously the set of right cosets of H in G is denoted by $H \setminus G$ and it is called the *right coset space*.

If the subgroup H of G is an invariant subgroup then the left and right cosets are the same since $g^{-1}Hg = H$ implies $gH = Hg$. In addition, the coset space G/H, for the case in which H is invariant, has the structure of a group and it is called the *factor group or the quocient group*. In order to show this we consider the product of two elements of two different cosets. We get

$$gh_1g'h_2 = gg'g'^{-1}h_1g'h_2 = gg'h_3h_2 \tag{1.28}$$

where we have used the fact that H is invariant, and therefore there exists $h_3 \in H$ such that $g'^{-1}h_1g' = h_3$. Thus we have obtained an element of a third coset, namely $gg'H$. If we had taken any other elements of the cosets gH and $g'H$, their product would produce an element of the same coset $gg'H$.

[1] Notice that two sets gH and $g'H$ may coincide for $g' \neq g$. However, in that case g and g' differ by an element of H, i.e. $g' = gh$.

Consequently we can introduce, in a well defined way, the product of elements of the coset space G/H, namely

$$gHg'H \equiv gg'H \tag{1.29}$$

The invariant subgroup H plays the role of the identity element since

$$(gH)H = H(gH) = gH \tag{1.30}$$

The inverse element is $g^{-1}H$ since

$$g^{-1}HgH = g^{-1}gH = H = gHg^{-1}H \tag{1.31}$$

The associativity is guaranteed by the associativity of the composition law of the group G. Therefore the coset space $G/H \equiv H \setminus G$ is a group in the case where H is an invariant subgroup. Notice that such group is not necessarily a subgroup of G or H.

Example 1.21 *The real numbers without the zero, $R-0$, form a group under multiplication. The positive real numbers, R^+, close under multiplication and the inverse of a positive real number x is also positive $(1/x)$. Therefore R^+ is a subgroup of $R-0$. In addition we have that the conjugation of a real x by another real y is equal to x , $(y^{-1}xy = x)$. Therefore R^+ is an invariant subgroup of $R-0$. The coset space $(R-0)/R^+$ has two elements, namely R^+ and R^- (the negative real numbers). This coset space is a group and it is isomorphic to the cyclic group of order 2, Z_2 (see example 1.10), since its elements satisfy $R^+.R^+ \subset R^+$, $R^+.R^- \subset R^-$, $R^-.R^- \subset R^+$.*

Example 1.22 *Any subgroup of an abelian group is an invariant subgroup.*

Example 1.23 *Consider the cyclic group Z_6 with elements e, a, a^2, ... a^5 and $a^6 = e$ and the subgroup Z_2 with elements e and a^3. Then the cosets are given by*

$$c_0 = \{e, a^3\} \ , \ c_1 = \{a, a^4\} \ , \ c_2 = \{a^2, a^5\} \tag{1.32}$$

Since Z_2 is an invariant subgroup of Z_6 the coset space Z_6/Z_2 is a group. Following the definition of the product law on the coset given above one easily sees it is isomorphic to Z_3 since

$$\begin{aligned} c_0.c_0 = c_0 \ , \ c_0.c_1 = c_1 \ , \ c_0.c_2 = c_2 \\ c_1.c_1 = c_2 \ , \ c_1.c_2 = c_0 \ , \ c_2.c_2 = c_1 \end{aligned} \tag{1.33}$$

If we now take the subgroup Z_3 of Z_6 with elements e, a^2 and a^4 we get the cosets

$$d_0 = \{e, a^2, a^4\} \ , \ d_1 = \{a, a^3, a^5\} \tag{1.34}$$

Again the coset space Z_6/Z_3 is a group and it is isomorphic to Z_2 since

$$d_0.d_0 = d_0 \ , \ d_0.d_1 = d_1 \ , \ d_1.d_1 = d_0 \tag{1.35}$$

1.5 Representations

The concept of abstract groups we have been discussing plays an important role in Physics. However, its importance only appears when some quantities in the physical theory realize, in a concrete way, the structure of the abstract group. Here comes the concept of *representation of an abstract group.*

Suppose we have a set of operators D_1 , D_2... acting on a vector space V

$$D_i \mid v\rangle =\mid v'\rangle \quad ; \quad \mid v\rangle, \mid v'\rangle \in V \tag{1.36}$$

We can define the product of these operators by the composition of their action, i.e., an operator D_3 is the product of two other operators D_1 and D_2 if

$$D_1(D_2 \mid v\rangle) = D_1 \mid v'\rangle = D_3 \mid v\rangle \tag{1.37}$$

for all $\mid v\rangle \in V$. We then write

$$D_1.D_2 = D_3. \tag{1.38}$$

Suppose that these operators form a group under this product law. We call it an *operator group* or *group of transformations.*

If we can associate to each element g of an abstract group G an operator, which we shall denote by $D(g)$, such that the group structure of G is preserved, i.e., if for $g, g' \in G$ we have

$$D(g)D(g') = D(gg') \tag{1.39}$$

then we say that such set of operators is a representation of the abstract group G in the *representation space* V. In fact, the mapping between the operator group D and the abstract group G is a homomorphism. In addition to eq.(1.39) one also has that

$$\begin{aligned} D(g^{-1}) &= D^{-1}(g) \\ D(e) &= 1 \end{aligned} \tag{1.40}$$

where 1 stands for the unit operator in D.

Definition 1.10 *The* dimension of the representation *is the dimension of the representation space.*

Notice that we can associate the same operator to two or more elements of G, but we can not do the converse. In the case where there is a one-to-one correspondence between the elements of the abstract group and the set of operators, i.e., to one operator D there is only one element g associated, we say that we have a *faithful representation* .

Example 1.24 *The unit matrix of any order is a trivial representation of any group. Indeed, if we associate all elements of a given group to the operator* 1 *we have that the relation* $1.1 = 1$ *reproduces the composition law of the group* $g.g' = g''$. *This is an example of an extremely non faithful representation.*

When the operators D are linear operators, i.e.,

$$\begin{aligned} D(\mid v\rangle + \mid v'\rangle) &= D \mid v\rangle + D \mid v'\rangle \\ D(a \mid v\rangle) &= aD \mid v\rangle \end{aligned} \tag{1.41}$$

with $\mid v\rangle, \mid v'\rangle \in V$ and a being a c-number, we say they form a *linear representation* of G.

Given a basis $\mid v_i\rangle$ $(i = 1, 2...n)$ of the vector space V (of dimension n) we can construct the matrix representatives of the operators D of a given representation. The action of an operator D on an element $\mid v_i\rangle$ of the basis produces an element of the vector space which can be written as a linear combination of the basis

$$D \mid v_i\rangle = \mid v_j\rangle D_{ji} \tag{1.42}$$

The coefficients D_{ji} of this expansion constitute the matrix representatives of the operator D. Indeed, we have

$$D'(D \mid v_i\rangle) = D' \mid v_j\rangle D_{ji} = \mid v_k\rangle D'_{kj} D_{ji} = \mid v_k\rangle (D'D)_{ki} \tag{1.43}$$

So, we can now associate to the matrix D_{ij}, the element of the abstract group that is associated to the operator D. We have then what is called a *matrix representation* of the abstract group. Notice that the matrices in each representation have to be non singular because of the existence of the inverse element. In addition the unit element e is always represented by the unit matrix, i.e., $D_{ij}(e) = \delta_{ij}$.

Example 1.25 *In example 1.9 we have defined the group* S_n . *We can construct a representation for this group in terms of* $n \times n$ *matrices as follows: take a vector space* V_n *and let* $\mid v_i\rangle$, $i = 1, 2, ...n$, *be a basis of* V_n. *One can define* $n!$ *operators that acting on the basis permute them, reproducing the* $n!$ *permutations of* n *elements. Using (1.42) one then obtains the matrices. For instance, in the case of* S_3, *consider the matrices*

$$D(a_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad D(a_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$D(a_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \qquad D(a_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$D(a_4) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}; \qquad D(a_5) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.44}$$

where a_m*,* $m = 0, 1, 2, 3, 4, 5$*, are the* 6 *elements of* S_3*. One can check that the action*

$$D(a_m) \mid v_i\rangle = \mid v_j\rangle D_{ji}(a_m) \tag{1.45}$$

gives the 6 *permutations of the three basis vectors* $\mid v_i\rangle$*,* $i = 1, 2, 3$*, of* V_3*. In addition the product of these matrices reproduces the composition law of permutations in* S_3*.*

By considering V_3 *as the space of column vectors* 3×1 *, and taking the canonical basis*

$$\mid v_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \ \mid v_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \ \mid v_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.46}$$

one can check that the matrices given above play the role of the operators permuting the basis too

$$D_{ij}(a_m) \mid v_k\rangle_j = \mid v_l\rangle_i D_{lk}(a_m) \tag{1.47}$$

Example 1.26 *As an example of a non-linear representation consider the transformation on the complex plane* z *as*

$$z \to z' = \frac{a_1\, z + a_2}{a_3\, z + a_4}; \qquad z\,, a_i \in C\,; \qquad a_1\, a_4 - a_2\, a_3 \neq 0 \tag{1.48}$$

Now consider a second transformation composed with the first ($b_i \in C$)

$$\begin{aligned} z' \;\to\; z'' &= \frac{b_1\, z' + b_2}{b_3\, z' + b_4} = \frac{b_1\, \frac{a_1\, z + a_2}{a_3\, z + a_4} + b_2}{b_3\, \frac{a_1\, z + a_2}{a_3\, z + a_4} + b_4} \\ &= \frac{(b_1\, a_1 + b_2\, a_3)\; z + (b_1\, a_2 + b_2\, a_4)}{(b_3\, a_1 + b_4\, a_3)\; z + (b_3\, a_2 + b_4\, a_4)} \end{aligned} \tag{1.49}$$

Note that such transformations compose in the same way as the product of 2×2 *complex matrices, i.e.*

$$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} b_1\, a_1 + b_2\, a_3 & b_1\, a_2 + b_2\, a_4 \\ b_3\, a_1 + b_4\, a_3 & b_3\, a_2 + b_4\, a_4 \end{pmatrix} \tag{1.50}$$

Since the determinant of the matrices are not zero, the transformations (1.48) constitute a group isomorphic to $GL(2\,,C)$*. By imposing* $\mid a_1 \mid^2 + \mid a_2 \mid^2 = \mid a_3 \mid^2 + \mid a_4 \mid^2 = 1$*, and* $(a_1\, a_3^* + a_2\, a_4^*) = 0$*, one gets a group isomorphic to* $U(2)$*. If in addition one imposes* $(a_1\, a_4 - a_2\, a_3) = 1$*, one gets a group isomorphic to* $SU(2)$*. The transformations (1.48) are called the Möbius transformations of the complex plane.*

In a non faithful representation of a group G, the set of elements which are mapped on the unit operator constitute an invariant subgroup of G. Indeed, if the representatives of the elements h and h' of G are the unit operator, i.e., $D(h) = D(h') = 1$, then $D(hh') = D(h)D(h') = 1$. In addition one has that $D(h^{-1}) = 1$ since $D(h)D(h^{-1}) = D(e) = 1 = 1D(h^{-1})$. So, such subset of G is indeed a subgroup. To see it is invariant one uses eq. (1.40) to get

$$D(g^{-1}hg) = D(g)^{-1}D(h)D(g) = D^{-1}(g)1D(g) = 1 \tag{1.51}$$

Denoting by H this invariant subgroup, we see that all elements in a given coset gH of the coset space G/H are mapped on the same matrix $D(g)$ since

$$D(gh) = D(g)D(h) = D(g)1 = D(g) \quad ; \quad h \in H \tag{1.52}$$

Therefore the representation D of G constitute a faithful representation of the factor group G/H.

Two representations D and D' of an abstract group G are said to be *equivalent representations* if there exists an operator C such that

$$D'(g) = CD(g)C^{-1} \tag{1.53}$$

with C being the same for every $g \in G$. Such thing happens, for instance, when one changes the basis of the representation

$$\mid v_i' \rangle = \mid v_j \rangle \Lambda_{ji} \tag{1.54}$$

Then

$$\begin{aligned} D(g) \mid v_i' \rangle &\equiv \mid v_j' \rangle D_{ji}'(g) \\ &= \mid v_k \rangle D_{kl}(g)\Lambda_{li} \\ &= \mid v_n \rangle \Lambda_{nj}\Lambda_{jk}^{-1} D_{kl}(g)\Lambda_{li} \\ &= \mid v_j' \rangle \Lambda_{jk}^{-1} D_{kl}(g)\Lambda_{li} \end{aligned} \tag{1.55}$$

Therefore the new matrix representatives are

$$D_{ji}'(g) = \Lambda_{jk}^{-1} D_{kl}(g)\Lambda_{li} \tag{1.56}$$

So, the matrix representatives change as in (1.53) with $C = \Lambda^{-1}$. Although the structure of the representation does not change the matrices look different.

As we have said before the operators of a given representation act on the representation space V as a group of transformations. In the case where a subspace V_1 of V is left invariant by all transformations, we say the representation is *reducible* . By invariant subspace we mean that

$$D(g) \mid v_1\rangle \in V_1 \qquad \text{for any } \mid v_1\rangle \in V_1 \quad \text{and} \quad g \in G \tag{1.57}$$

Therefore, since (1.39) holds for any element of V, it follows that V_1 itself constitutes a representation of G.

If one orders the basis of V such that the first m elements constitute a basis of V_1, with $m = \dim V_1$, then the corresponding matrix representation can be written in the form

$$D(g) = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \tag{1.58}$$

where A, B and C are respectively $m \times m$, $n \times n$ and $m \times n$ matrices. The dimension of the representation is $m + n$. The subspace V_1 of V generated by the first m elements of the basis is left invariant, since

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = \begin{pmatrix} Av_1 \\ 0 \end{pmatrix} \tag{1.59}$$

i.e., V_1 does not mix with the rest of V. The subspace V_2 of V generated by the last n elements of the basis is not invariant since

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = \begin{pmatrix} Cv_2 \\ Bv_2 \end{pmatrix} \tag{1.60}$$

When both subspaces V_1 and V_2 are invariant we say the representation is *completely reducible*. In this case the matrices take the form

$$D(g) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{1.61}$$

Lemma 1.1 (Schur) *Any matrix which commutes with all matrices of a given irreducible representation of a group G must be a multiple of the unit matrix.*

Proof Let A be a matrix that commutes will all matrices $D(g)$ of a given irreducible representation of G, i.e.

$$AD(g) = D(g)A \tag{1.62}$$

for any $g \in G$. Consider the eigenvalue equation

$$A \mid v\rangle = \lambda \mid v\rangle \tag{1.63}$$

where $\mid v\rangle$ is some vector in the representation space V. Notice that, if v is an eigenvector with eigenvalue λ, then $D(g) \mid v\rangle$ has also eigenvalue λ since

$$AD(g) \mid v\rangle = D(g)A \mid v\rangle = \lambda D(g) \mid v\rangle. \tag{1.64}$$

Therefore the subspace of V generated by all eigenvectors of A with eigenvalue λ is an invariant subspace of V. But if the representation is irreducible that means this subspace is the zero vector or is the entire V. In the first case we get that $A = 0$, and in the second we get that A has only one eigenvalue and therefore $A = \lambda 1$. □

Corollary 1.2 *Every irreducible representation of an abelian group is one dimensional.*

Proof Since the group is abelian any matrix has to commute with all other matrices of the representation. According to Schur's lemma they have to be proportional to the identity matrix. So, any vector of the representation space V generates an invariant subspace. Therefore V has to be unidimensional if the representation is irreducible. □

Note that in order to calculate the eigenvalues and the eigenvectors in (1.63) one has to solve algebraic equations. Therefore, Schur's lemma has to be considered on representations where the representation space is over an algebraically closed field. The real numbers are not algebraically closed but the complex numbers are. That is exactly what happens in the Example 1.27 for the two dimensional real representation of the rotation group on the plane. The eigenvalues of the matrices $R(\theta)$ are complex, and so to diagonalize them one has to consider the representation as being complex.

Definition 1.11 *A representation D is said to be* unitary *if the matrices D_{ij} of the operators are unitary, i.e. $D^\dagger = D^{-1}$.*

An important result in the theory of finite groups is the following theorem

Theorem 1.3 *Any representation of a finite group is equivalent to a unitary representation*

Proof Let G be a finite group of order N, and D be a representation of G of dimension d. We introduce a hermitian matrix H ($H^\dagger = H$) by

$$H \equiv \frac{1}{N} \sum_{g \in G} D^\dagger(g) D(g) \tag{1.65}$$

For any $g' \in G$

$$D^\dagger(g') H D(g') = \frac{1}{N} \sum_{g \in G} D^\dagger(gg') D(gg') = H \tag{1.66}$$

by redefining the sum (remember that if $g_1 g' = g_2 g'$ then $g_1 = g_2$). Since H is hermitian it can be diagonalized by a unitary matrix, i.e. $H' \equiv U^\dagger H U$ is diagonal. For any non zero colunm vector v (with complex entries), the quantity

$$v^\dagger H v = \frac{1}{N} \sum_{g \in G} \mid D(g) v \mid^2 \tag{1.67}$$

is real and positive. But, introducing $v' \equiv U^\dagger v$

$$\begin{aligned} v^\dagger H v &= v'^\dagger H' v' \\ &= \sum_{i=1}^{d} H'_{ii} \mid v'_i \mid^2 \end{aligned} \tag{1.68}$$

where v'_i are the components of v'. Since v'_i are arbitrary we conclude that each entry H'_{ii} of H' is real and positive. We then define a diagonal real matrix h with entries $h_{ii} = \sqrt{H'_{ii}}$, i.e. $H' = hh$. Therefore

$$H = U H' U^\dagger = U h h U^\dagger \equiv S S \tag{1.69}$$

where we have defined $S = U h U^\dagger$. Notice that S is hermitian, since h is real and diagonal.

Defining the representation of G given by the matrices

$$D'(g) \equiv S D(g) S^{-1} \tag{1.70}$$

we then get from eq. (1.66)

$$\left(S^{-1} D'(g) S\right)^\dagger (SS) \left(S^{-1} D'(g) S\right) = SS \tag{1.71}$$

and so

$$D'^{\dagger}(g)D'(g) = \mathbb{1} \tag{1.72}$$

Therefore the representation $D(g)$ is equivalent to the unitary representation $D'(g)$.This result, as we will discuss later, is also true for compact Lie groups.□

Definition 1.12 *Given two representations D and D' of a given group G, one can construct what is called the* tensor product representation *of D and D'. Denoting by $\mid v_i\rangle$, $i = 1, 2, \ldots \dim D$, and $\mid v'_l\rangle$, $l = 1, 2, \ldots \dim D'$, the basis of D and D' respectively, one constructs the basis of $D \otimes D'$ as*

$$\mid w_{il}\rangle = \mid v_i\rangle \otimes \mid v'_l\rangle \tag{1.73}$$

The operators representing the group elements act as

$$D^{\otimes}(g) \mid w_{il}\rangle = D(g) \otimes D'(g) \mid w_{il}\rangle = D(g) \mid v_i\rangle \otimes D'(g) \mid v'_l\rangle \tag{1.74}$$

The dimension of the representation $D \otimes D'$ is the product of the dimensions of D and D', i.e. $\dim D \otimes D' = \dim D \dim D'$.

The matrices representing a given group element in two equivalent representations may look quite different one from the other. That means the matrices contain a lot of redundant information. Much of the relevant properties of a representation can be encoded in the *character*.

Definition 1.13 *In a given representation D of a group G we define the character $\chi^D(g)$ of a group element $g \in G$ as the trace of the matrix representing it, i.e.*

$$\chi^D(g) \equiv Tr(D(g)) = \sum_{i=1}^{dimD} D_{ii}(g) \tag{1.75}$$

Obviously, the characters of a given group element in two equivalent representations are the same, since from (1.53)

$$Tr(D'(g)) = Tr(CD(g)C^{-1}) = Tr(D(g)) \to \chi^D(g) = \chi^{D'}(g) \tag{1.76}$$

Analogously, the elements of a given conjugacy class have the same character. Indeed, from definition 1.6, if two elements g' and g'' are conjugate, $g' = gg''g^{-1}$, then in any representation D one has $Tr(D(g')) = Tr(D(g''))$. Nothing prevents however, the elements of two different conjugacy class of having the same character in some particular representation. In fact, this happens in the representation discussed in example 1.24.

We have seen that the identity element e of a group G is always represented by the unity matrix. Therefore the character of e gives the dimension of the representation

$$\chi^D(e) = dim\ D \tag{1.77}$$

We now state, without proof, some theorems concerning characters. For the proofs see, for instance, [COR 84].

Theorem 1.4 *Let D and D' be two irreducible representations of a finite group G and χ^D and $\chi^{D'}$ the corresponding characters. Then*

$$\frac{1}{N(G)}\sum_{g\in G}(\chi^D(g))^*\chi^{D'}(g) = \delta_{DD'} \tag{1.78}$$

where $N(G)$ is the order of G, $\delta_{DD'} = 1$ if D and D' are equivalent representations and $\delta_{DD'} = 0$ otherwise.

Theorem 1.5 *A sufficient conditions for two representations of a finite group G to be equivalent is the equality of their character systems.*

Theorem 1.6 *The number of times n_D that an irreducible representation D appears in a given reducible representation D' of a finite group G is given by*

$$n_D = \frac{1}{N(G)}\sum_{g\in G}\chi^{D'}(g)(\chi^D(g))^* \tag{1.79}$$

where χ^D and $\chi^{D'}$ are the characters of D and D' respectively, and $N(G)$ is the order of G.

Theorem 1.7 *A necessary and suficient condition for a representation D of a finite group G to be irreducible is*

$$\frac{1}{N(G)}\sum_{g\in G}\mid\chi^D(g)\mid^2 = 1 \tag{1.80}$$

where χ^D are the characters of D and $N(G)$ the order of G.

All these four theorems are also true for compact Lie groups (see definition in chapter 2) with the replacement of the sum $\frac{1}{N(G)}\sum_{g\in G}$ by the invariant integration $\int_G \mathcal{D}g$ on the group manifold.

Characters are also used to prove theorems about the number of inequivalent irreducible representations of a finite group.

Theorem 1.8 *The sum of the squares of the dimensions of the inequivalent irreducible representations of a finite group G is equal to the order of G.*

Theorem 1.9 *The number of inequivalent irreducible representations of a finite group G is equal to the number of conjugacy classes of G.*

For the proofs see [COR 84].

Definition 1.14 *If all the matrices of a representation are real the representation is said to be* real.

Notice that if D is a matrix representation of a group G, then the matrices $D^*(g)$, $g \in G$, also constitute a representation of G of the same dimension as D, since

$$D(g)D(g') = D(gg') \to D^*(g)D^*(g') = D^*(gg') \qquad (1.81)$$

If D is equivalent to a real representation D_R, then D is equivalent to D^*. The reason is that there exists a matrix C such that

$$D_R(g) = CD(g)C^{-1} \qquad (1.82)$$

and so

$$D_R(g) = C^*D^*(g)(C^*)^{-1} \qquad (1.83)$$

Therefore

$$D^*(g) = (C^{-1}C^*)^{-1}D(g)C^{-1}C^* \qquad (1.84)$$

and D is equivalent to D^*. However the converse is not always true, i.e. , if D is equivalent to D^* it does not means D is equivalent to a real representation. So we classify the representations into three classes regarding the relation between D and D^*.

Definition 1.15

1. *If D is equivalent to a real representation it is said* potentially real.
2. *If D is equivalent to D^* but not equivalent to a real representation it is said* pseudo real.
3. *If D is not equivalent to D^* then it is said* essentially complex.

Notice that if D is potentially real or pseudo real then its characters are real.

Example 1.27 *The rotation group on the plane, denoted $SO(2)$, can be represented by the matrices*

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{1.85}$$

such that

$$R(\theta) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix} \tag{1.86}$$

One can easily check that $R(\theta)R(\varphi) = R(\theta + \varphi)$. This group is abelian and according to corollary 1.2 such representation is reducible, if the vector space of the representation is taken over the field of the complex numbers. Indeed, one gets

$$MR(\theta)M^{-1} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \tag{1.87}$$

where

$$M = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{1.88}$$

The vectors of the representation space are then transformed as

$$M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + iy \\ ix + y \end{pmatrix} \tag{1.89}$$

The characters of these equivalent representations are

$$\chi(\theta) = 2\cos\theta \tag{1.90}$$

Example 1.28 *In example 1.25 we have discussed a 3-dimensional matrix representation of S_3. From the definition 1.13 one can easily evaluate the characters in such representation*

$$\begin{aligned} \chi^D(a_0) &= 3 \\ \chi^D(a_1) &= \chi^D(a_2) = \chi^D(a_3) = 1 \\ \chi^D(a_4) &= \chi^D(a_5) = 0 \end{aligned} \tag{1.91}$$

Therefore

$$\frac{1}{6}\sum_{i=0}^{5} \mid \chi^D(a_i) \mid^2 = 2 \tag{1.92}$$

From theorem 1.7 one sees that such 3-dimensional representation is not irreducible. Indeed, the one dimensional subspace generated by the vector

$$\mid w_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1.93}$$

is an invariant subspace. The basis of the orthogonal complement of such subspace can be taken as

$$\mid w_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}; \quad \mid w_2\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \tag{1.94}$$

Such a basis is related to the canonical basis defined in (1.46) by

$$\mid w_i\rangle = \mid v_j\rangle \Lambda_{ji} \tag{1.95}$$

where $i, j = 1, 2, 3$ *and*

$$\Lambda = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \tag{1.96}$$

According to (1.56) the matrix representatives of the elements of S_3 *change as*

$$D'(a_m) = \Lambda^{-1} D(a_m)\Lambda \tag{1.97}$$

where $m = 0, 1, 2, 3, 4, 5$ *and* $\Lambda^{-1} = \Lambda^{\top}$*. One can easily check that*

$$D'(a_m) = \begin{pmatrix} D''(a_m) & 0 \\ 0 & 1 \end{pmatrix} \tag{1.98}$$

where $D''(a_m)$ *is a 2-dimensional representation of* S_3 *given by*

$$\begin{aligned} D''(a_0) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; & D''(a_1) &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \\ D''(a_2) &= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}; & D''(a_3) &= \begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}; \\ D''(a_4) &= \begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}; & D''(a_5) &= \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix} \end{aligned} \tag{1.99}$$

The characters in the representation D'' are given by

$$\begin{aligned}
\chi^{D''}(a_0) &= 2 \\
\chi^{D''}(a_1) &= \chi^{D''}(a_2) = \chi^{D''}(a_3) = 0 \\
\chi^{D''}(a_4) &= \chi^{D''}(a_5) = -1
\end{aligned} \tag{1.100}$$

Therefore

$$\frac{1}{6}\sum_{i=0}^{5} \mid \chi^{D''}(a_i) \mid^2 = 1 \tag{1.101}$$

According to theorem 1.7 the representation D'' is irreducible. Consequentely the 3-dimensional representation D defined in (1.44) is completely reducible. It decomposes into the irreducible 2-dimensional representation D'' and the 1-dimensional representation given by 1.

We have seen so far that S_3 has two irreducible representations, the two dimensional representation D'', and the scalar representation (one dimensional) where all elements are represented by the number 1. Since, $2^2 + 1^2 = 5$ and since the order of S_3 is 6, we observe from theorem 1.8 that it is missing one irreducible representation of dimension 1. That is easy to construct, and in fact any S_n group has it. It is the representation where the permutations made of an even number of simple permutations is representated by 1, and those with an odd number by -1. Since the composition of permutations add up the number of simple permutations it follows it is indeed a representation. Therefore, the missing one dimensional irreducible representation of S_3 is given by

$$\begin{aligned}
D'''(a_0) &= D'''(a_4) = D'''(a_5) = 1 \\
D'''(a_1) &= D'''(a_2) = D'''(a_3) = -1
\end{aligned} \tag{1.102}$$

Chapter 2

Lie Groups and Lie Algebras

2.1 Lie groups

So far we have been looking at groups as set of elements satisfying certain postulates. However we can take a more geometrical point of view and look at the elements of a group as being points of a space. The groups S_n and Z_n , discussed in examples 1.9 and 1.10, have a finite number of elements and therefore their corresponding spaces are discrete spaces. Groups like these ones are called *finite discrete groups*. The group formed by the integer numbers under addition is also discrete but has an infinite number of elements. It constitutes a one dimensional regular lattice. These type of groups are called *infinite discrete groups*. The interesting geometrical properties of groups appear when their elements correspond to the points of a continuous space. We have then what is called a *continuous group*. The real numbers under addition constitute a continuous group since its elements can be seen as the points of an infinite line. The group of rotations on a two dimensional plane is also a continuous group. Its elements can be parametrized by an angle varying from 0 to 2π and therefore they define a space which is a circle. In this sense the real numbers under addition constitute a *non compact group* and the rotations on the plane a *compact group*.

Given a group G we can parametrize its elements by a set of parameters x_1 , x_2, ... x_n . If the group is continuous these parameters are continuous and can be taken to be real parameters. The elements of the group can then be denoted as $g = g(x_1, x_2...x_n)$. A set of continuous parameters x_1, x_2, ... x_n is said to be *essential* if one can not find a set of continuous parameters y_1, y_2, ... y_m , with $m < n$, which suffices to label the elements of the group. When we take the product of two elements of a group

$$g(x)g(x') = g(x'') \tag{2.1}$$

the parameters of the resulting element is a function of the parameters of the other two elements.

$$x'' = F(x, x') \tag{2.2}$$

Analogously the parameters of the inverse element of a given $g \in G$ is a function of the parameters of g and vice-versa. If

$$g(x)g(x') = e = g(x')g(x) \tag{2.3}$$

then

$$x' = f(x) \tag{2.4}$$

If the elements of a group G form a topological space and if the functions $F(x, x')$ and $f(x)$ are continuous functions of its arguments then we say that G is a *topological group*. Notice that in a topological group we have to have some compatibility between the algebraic and the topological structures.

When the elements of a group G constitute a manifold and when the functions $F(x, x')$ and $f(x)$, discussed above, possess derivatives of all orders with respect to its arguments, i.e., are analytic functions, we say the group G is a *Lie group* . This definition can be given in a formal way.

Definition 2.1 *A Lie group is an analytic manifold which is also a group such that the analytic structure is compatible with the group structure, i.e. the operation $G \times G \to G$ is an analytic mapping.*

For more details about the geometrical concepts involved here see [HEL 78, CBW 82, ALD 86, FLA 63].

Example 2.1 *The real numbers under addition constitute a Lie group. Indeed, we can use a real variable x to parametrize the group elements. Therefore for two elements with parameters x and x' the function in (2.2) is given by*

$$x'' = F(x, x') = x + x' \tag{2.5}$$

The function given in (2.4) is just

$$f(x) = -x \tag{2.6}$$

These two functions are obviously analytic functions of the parameters.

Example 2.2 *The group of rotations on the plane, discussed in example 1.27, is a Lie group. In fact the groups of rotations on* $\mathbb{R}^n$ *, denoted by* $SO(n)$*, are Lie groups. These are the groups of orthogonal* $n \times n$ *real matrices* O *with unit determinant (*$O^\top O = 1\!\!1$*,* $det O = 1$*)*

Example 2.3 *The groups* $GL(n)$ *and* $SL(n)$ *discussed in example 1.16 are Lie groups, as well as the group* $SU(n)$ *discussed in example 1.17*

Example 2.4 *The groups* S_n *and* Z_n *discussed in examples 1.9 and 1.10 are not Lie groups.*

2.2 Lie Algebras

The fact that Lie groups are differentiable manifolds has very important consequences. Manifolds are locally Euclidean spaces. Using the differentiable structure we can approximate the neighborhood of any point of a Lie group G by an Euclidean space which is the tangent space to the Lie group at that particular point. This approximation is some sort of local linearization of the Lie group and it is the approach we are going to use in our study of the algebraic structure of Lie groups. Obviously this approach does not tell us much about the global properties of the Lie groups.

Let us begin by making some comments about tangent planes and tangent vectors. A convenient way of describing tangent vectors is through linear operators acting on functions. Consider a differentiable curve on a manifold M and let the coordinates x^i , $i = 1, 2, ...\dim M$, of its points be parametrized by a continuous variable t varying, let us say, from -1 to 1. Let f be any differentiable function defined on a neighbourhood of the point p of the curve corresponding to $t = 0$. The vector V_p tangent to the curve at the point p is defined by

$$V_p(f) = \frac{dx^i(t)}{dt} \mid_{t=0} \frac{\partial f}{\partial x^i} \tag{2.7}$$

Since the function f is arbitrary the tangent vector is independent of it. The vector V_p is a *tangent vector* to M at the point p.

The tangent vectors at p to all differentiable curves passing through p form the *tangent space* T_pM of the manifold M at the point p. This space is a vector space since the sum of tangent vectors is again a tangent vector and the muliplication of a tangent vector by a scalar (real or complex number) is also a tangent vector.

Given a set of local coordinates x^i , $i = 1, 2, ...\dim M$ in a neighbourhood of a point p of M we have that the operators $\frac{\partial}{\partial x^i}$ are linearly independent and constitute a basis for the tangent space T_pM. Then, any tangent vector V_p on T_pM can be written as a linear combination of this basis

$$V_p = V_p^i \frac{\partial}{\partial x^i} \tag{2.8}$$

Now suppose that we vary the point p along a differentiable curve. As we do that we obtain vectors tangent to the curve at each of its points. These tangent vectors are continuously and differentiably related. If we choose a tangent vector on T_pM for each point p of the manifold M such that this set of vectors are differentiably related in the manner described above we obtain what is called a *vector field* . Given a set of local coordinates on M we can write a vector field V, in that coordinate neighbourhood, in terms of the basis $\frac{\partial}{\partial x^i}$, and its components V^i are differentiable functions of these coordinates.

$$V = V^i(x) \frac{\partial}{\partial x^i} \tag{2.9}$$

Given two vector fields V and W in a coordinate neighbourhood we can evaluate their composite action on a function f. We have

$$W(Vf) = W^j \frac{\partial V^i}{\partial x^j} \frac{\partial f}{\partial x^i} + W^j V^i \frac{\partial^2 f}{\partial x^j \partial x^i} \tag{2.10}$$

Due to the second term on the r.h.s of (2.10) the operator WV is not a vector field and therefore the ordinary composition of vector fields is not a vector field. However if we take the commutator of the linear operators V and W we get

$$[V, W] = \left(V^i \frac{\partial W^j}{\partial x^i} - W^i \frac{\partial V^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \tag{2.11}$$

and this is again a vector field. So, the set of vector fields close under the operation of commutation and they form what is called a *Lie algebra.*

Definition 2.2 *A Lie algebra $\mathcal{G}$ is a vector space over a field k with a bilinear composition law*

$$\begin{aligned} (x, y) &\rightarrow [x, y] \\ [x, ay + bz] &= a[x, y] + b[x, z] \end{aligned} \tag{2.12}$$

with x, y, $z \in L$ and a, $b \in k$, and such that

1. $[x, x] = 0$
2. $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$; *(Jacobi identity)*

Notice that (2.12) implies that $[x, y] = -[y, x]$, since

$$\begin{aligned} [x + y, x + y] &= 0 \\ &= [x, y] + [y, x] \end{aligned} \tag{2.13}$$

Definition 2.3 *A field is a set k together with two operations*

$$(a, b) \to a + b \tag{2.14}$$

and

$$(a, b) \to ab \tag{2.15}$$

called respectively addition and multiplication such that

1. *k is an abelian group under addition*
2. *k without the identity element of addition is an abelian group under multiplication*
3. *multiplication is distributive with respect to addition, i.e.*

$$\begin{aligned} a\,(b + c) &= ab + ac \\ (a + b)\,c &= ac + bc \end{aligned}$$

The real and complex numbers are fields.

2.3 The Lie algebra of a Lie group

We have seen that vector fields on a manifold form a Lie algebra. We now want to show that the Lie algebra of some special vector fields on a Lie group is related to its group structure.

If we take a fixed element g of a Lie group G and multiply it from the left by every element of G, we obtain a transformation of G onto G which is called *left translation* on G by g. In a similar way we can define *right translations* on G. Under a left translation by g, an element g', which is parametrized by the coordinates x'^i ($i = 1, 2, \ldots$ dim G), is mapped into the element $g'' = gg'$, and the parameters x''^i of g'' are analytic functions of x'^i . This mapping of G onto G induces a mapping between the tangent spaces of G as follows: let V be a vector field on G which corresponds to the tangent vectors $V_{g'}$ and $V_{g''}$ on the tangent spaces to G at g' and g'' respectively. Let f be an arbitrary function of the parameters x''^i of g''. We define a tangent vector $W_{g''}$ on $T_{g''}G$ (the tangent plane to G at g'') by

$$W_{g''} f \equiv V_{g'}(f \circ x'') = V^i_{g'} \frac{\partial}{\partial x'^i} f(x'') = V^i_{g'} \frac{\partial x''^j}{\partial x'^i} \frac{\partial f}{\partial x''^j} \tag{2.16}$$

This defines a mapping between the tangent spaces of G since, given $V_{g'}$ in $T_{g'}G$, we have associated a tangent vector $W_{g''}$ in $T_{g''}G$. The vector $W_{g''}$ does not have necessarily to coincide with the value of the vector field V at $T_{g''}G$, namely $V_{g''}$. However, when that happens we say that the vector field V is a *left invariant vector field* on G, since that transformation was induced by left translations on G.

The commutator of two left invariant vector fields, V and $\bar{V}$, is again a left invariant vector field. To check this consider the commutator of this vector fields at group element g'. According to (2.11)

$$\tilde{V}_{g'} \equiv [V_{g'} \, , \, \bar{V}_{g'}] = \left(V^i_{g'} \frac{\partial \bar{V}^j_{g'}}{\partial x'^i} - \bar{V}^i_{g'} \frac{\partial V^j_{g'}}{\partial x'^i} \right) \frac{\partial}{\partial x'^j} \tag{2.17}$$

Since V and $\bar{V}$ are left invariant, at the group element $g'' = gg'$ we have, according to (2.16), that

$$\begin{aligned} \tilde{V}_{g''} &\equiv [V_{g''} \, , \, \bar{V}_{g''}] \\ &= \left(V^i_{g''} \frac{\partial \bar{V}^j_{g''}}{\partial x''^i} - \bar{V}^i_{g''} \frac{\partial V^j_{g''}}{\partial x''^i} \right) \frac{\partial}{\partial x''^j} \end{aligned}$$

$$
\begin{aligned}
&= \left(V^k_{g'} \frac{\partial x''^i}{\partial x'^k} \frac{\partial}{\partial x''^i} \left(\bar{V}^l_{g'} \frac{\partial x''^j}{\partial x'^l} \right) - \bar{V}^k_{g'} \frac{\partial x''^i}{\partial x'^k} \frac{\partial}{\partial x''^i} \left(V^l_{g'} \frac{\partial x''^j}{\partial x'^l} \right) \right) \frac{\partial}{\partial x''^j} \\
&= \left(V^i_{g'} \frac{\partial \bar{V}^j_{g'}}{\partial x'^i} - \bar{V}^i_{g'} \frac{\partial V^j_{g'}}{\partial x'^i} \right) \frac{\partial x''^k}{\partial x'^j} \frac{\partial}{\partial x'^k} \\
&= \tilde{V}^j_{g'} \frac{\partial x''^k}{\partial x'^j} \frac{\partial}{\partial x'^k}
\end{aligned}
\tag{2.18}
$$

So, $\tilde{V}$ is also left invariant. Therefore the set of left invariant vector fields form a Lie algebra. They constitute in fact a Lie subalgebra of the Lie algebra of all vector fields on G.

Definition 2.4 *A vector subspace $\mathcal{H}$ of a Lie algebra $\mathcal{G}$ is said to be a* Lie subalgebra *of $\mathcal{G}$ if it closes under the Lie bracket, i.e.*

$$[\mathcal{H} \, , \, \mathcal{H}] \subset \mathcal{H} \tag{2.19}$$

and if $\mathcal{H}$ itself is a Lie algebra.

One should notice that a left invariant vector field is completely determined by its value at any particular point of G. In particular it is determined by its value at the group identity e . An important consequence of this is that the Lie algebra of the left invariant vector fields at any point of G is completely determined by the Lie algebra of these fields at the identity element of G.

Definition 2.5 *The Lie algebra of the left invariant vector fields on a Lie group is the Lie algebra of this Lie group.*

Notice that the Lie algebra of a Lie group G is a subalgebra of the Lie algebra of all vector fields on G. The Lie algebra of right invariant vector fields is isomorphic to the Lie algebra of left invariant vector fields. Therefore the definition above could also be given in terms of right invariant vector fields.

For any Lie group G it is always possible to find a number of linearly independent left-invariant vector fields which is equal to the dimension of G. These vector fields, which we shall denote by T_a $(a = 1, 2, ...\text{dim } G)$, constitute a basis of the tangent plane to G at any particular point, and they satisfy

$$[T_a \, , \, T_b] = i f^c_{ab} T_c \tag{2.20}$$

If we move from one point of G to another, this relation remains unchanged, and therefore the quantities f^c_{ab} are point independent. For this reason they are called the *structure constants* of the Lie algebra of G. Later we will see that these constants contain all the information about the Lie algebra of G. Since the relation above is point independent we are going to fix the tangent plane to G at the identity element, T_eG, as the vector space of the Lie algebra of G. We could have defined right invariant vector fields in a similar way. Their Lie algebra is isomorphic to the Lie algebra of the left-invariant fields.

A *one parameter subgroup* of a Lie group G is a differentiable curve, i.e., a differentiable mapping from the real numbers onto G, $t \rightarrow g(t)$ such that

$$\begin{aligned} g(t)g(s) &= g(t+s) \\ g(0) &= e \end{aligned} \tag{2.21}$$

If we take a fixed element g' of G, we obtain that the mapping $t \rightarrow g'g(t)$ is a differentiable curve on G. However this curve is not a one parameter subgroup, since $g'g(t)g'g(s) \neq g'g(t+s)$. If we let g' to vary over G we obtain a family of curves which completely covers G. There are several curves of this family passing through at a given point of G. However, one can show (see [AUM 77]) that all curves of the family passing through a point have the same tangent vector at that point. Therefore the family of curves $g'g(t)$ can be used to define a vector field on G. One can also show that this is a left-invariant vector field. Consequently to each one parameter subgroup of G we have associated a left invariant vector field.

If T is the tangent vector at the identity element to a differentiable curve $g(t)$ which is a one parameter subgroup, then it is possible to show that

$$g(t) = exp(tT) \tag{2.22}$$

This means that the straight line on the tangent plane to G at the identity element, T_eG, is mapped onto the one parameter subgroup of G, $g(t)$. This is called the *exponential mapping* of the Lie algebra of G (T_eG) onto G. In fact, it is possible to prove that in general, the exponential mapping is an analytic mapping of T_eG onto G and that it maps a neighbourhood of the zero element of T_eG in a one to one manner onto a neighbourhood of the identity element of G. In several cases this mapping can be extended globally on G.

For more details about the exponential mapping and other geometrical concepts involved here see [HEL 78, ALD 86, CBW 82, AUM 77].

2.4 Basic notions on Lie algebras

In the last section we have seen that the Lie algebra, $\mathcal{G}$,of a Lie group G possesses a basis T_a , $a = 1, 2, ...$ dim G (= dim $\mathcal{G}$, satisfying

$$[T_a\, ,\, T_b] = i f_{ab}^c T_c \qquad (2.23)$$

where the quantities f_{ab}^c are called the *structure constants* of the algebra. We have introduced the imaginary unity i on the r.h.s of (2.23) because if the generators T_a are hermitian, $T_a^\dagger = T_a$, then the structure constants are real. Notice that $f_{ab}^c = -f_{ba}^c$. From the definition of Lie algebra given in section 2.2 we have that the generators T_a satisfy the Jacobi identity

$$[T_a, [T_b, T_c]] + [T_c, [T_a, T_b]] + [T_b, [T_c, T_a]] = 0 \qquad (2.24)$$

and consequently the structure constants have to satisfy

$$f_{ad}^e f_{bc}^d + f_{cd}^e f_{ab}^d + f_{bd}^e f_{ca}^d = 0 \qquad (2.25)$$

with sum over repeated indices. We have also seen that the elements g of G close to the identity element can be written, using the exponential mapping, as

$$g = \exp\left(i\zeta^a T_a\right) \qquad (2.26)$$

where ζ^a are the parameters of the Lie group. Under certain circunstances this relation is also true for elements quite away from the identity element (which corresponds to $\zeta^a = 0$).

If we conjugate elements of the Lie algebra by elements of the Lie group we obtain elements of the Lie algebra again. Indeed, if L and T are elements of the algebra one gets

$$\exp\left(L\right) T \exp\left(-L\right) = T + [L, T] + \frac{1}{2!}[L, [L, T]] + \frac{1}{3!}[L, [L, [L, T]]] + ... \qquad (2.27)$$

In order to prove that relation consider que quantity

$$f\left(\lambda\right) \equiv \exp\left(\lambda L\right) T \exp\left(-\lambda L\right) \qquad (2.28)$$

then

$$\begin{aligned} f' &= \exp\left(\lambda L\right) [\, L\, ,\, T\,] \exp\left(-\lambda L\right) \\ f'' &= \exp\left(\lambda L\right) [\, L\, ,\, [\, L\, ,\, T\,]\,] \exp\left(-\lambda L\right) \\ \ldots &= \ldots \\ f^{(n)} &= \exp\left(\lambda L\right) [\, L\, ,\, \ldots [\, L\, ,\, [\, L\, ,\, T\,]\,]\,] \exp\left(-\lambda L\right) \end{aligned} \qquad (2.29)$$

Then using Taylor expansion around $\lambda = 0$ one gets

$$f(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \, \mathrm{ad}_L^n T \tag{2.30}$$

where we have denoted $\mathrm{ad}_L T \equiv [\, L \, , \, T \,]$. Taking $\lambda = 1$ one gets (2.27).

The r.h.s. of (2.27) is and element of the algebra, and therefore the conjugation gTg^{-1} defines a transformation on the algebra. In addition if $g'' = g'g$ we see that the composition of the transformations associated to g' and g give the transformation associated to g''. Consequently, according to the concepts discussed in section 1.5, these transformations define a representation of the group G on a representation space which is the Lie algebra of G. Such representation is called the *adjoint representation* of the Lie group G . The matrices $d(g)$ representing the elements $g \in G$ in this representation are given by

$$gT_a g^{-1} = T_b d^b_{\ a}(g) \tag{2.31}$$

One can easily check that the $n \times n$ matrices $d^b_{\ a}(g)$, $n = \dim G$, form a representation of G, since if we take the element $g_1 g_2$ we get

$$\begin{aligned} g_1 g_2 T_a (g_1 g_2)^{-1} &= T_b d^b_{\ a}(g_1 g_2) \\ &= g_1 (g_2 T_a g_2^{-1}) g_1^{-1} \\ &= g_1 T_c g_1^{-1} d^c_{\ a}(g_2) \\ &= T_b d^b_{\ c}(g_1) d^c_{\ a}(g_2) \end{aligned} \tag{2.32}$$

Since the generators T_a are linearly independent we have

$$d(g_1 g_2) = d(g_1) d(g_2) \tag{2.33}$$

From the definition (2.31) we see that the dimension of the adjoint representation $d(g)$ of G is equal to the dimension of G. It is a real representation in the sense that the entries of the matrices $d(g)$ are real.

Notice that the conjugation defines a mapping of the Lie algebra $\mathcal{G}$ into itself which respects the commutation relations. Defining $\sigma : \mathcal{G} \to \mathcal{G}$

$$\sigma(T) \equiv gTg^{-1} \tag{2.34}$$

for a fixed $g \in G$ and any $T \in \mathcal{G}$, one has

$$\begin{aligned} [\sigma(T), \sigma(T')] &= [gTg^{-1}, gT'g^{-1}] \\ &= g[T, T']g^{-1} \\ &= \sigma([T, T']) \end{aligned} \tag{2.35}$$

Such mapping is called an *automorphism* of the Lie algebra.

Definition 2.6 *A mapping σ of a Lie algebra $\mathcal{G}$ into itself is an automorphism if it preserves the Lie bracket of the algebra, i.e.*

$$[\sigma(T), \sigma(T')] = \sigma([T, T']) \tag{2.36}$$

for any $T, T' \in \mathcal{G}$.

The mapping (2.34) in particular, is called an *inner automorphism*. All other automorphism which are not conjugations are called *outer automorphism*.

If g is an element of G infinitesimally close to the identity, its parameters in (2.26) are very small and we can write

$$g = 1 + i\varepsilon^a T_a \tag{2.37}$$

with ε^a infinitesimally small. From (2.31) we have

$$\begin{aligned}(1 + i\varepsilon^a T_a)T_b(1 - i\varepsilon^c T_c) &= T_c d^c_b(1 + i\varepsilon^a T_a) \\ &= T_c(\delta^c_b + i\varepsilon^a d^c_b(T_a)) \\ &= T_b + i\varepsilon^a [T_a, T_b] \\ &= T_b - \varepsilon^a f^c_{ab} T_c \end{aligned} \tag{2.38}$$

Since the infinitesimal parameters are arbitrary we get

$$d^c_b(T_a) = i f^c_{ab} \tag{2.39}$$

Therefore in the adjoint representation the matrices representing the generators are given by the structure constants of the algebra. This defines a matrix representation of the Lie algebra. In fact, whenever one has a matrix representation of a Lie group one gets, through the exponential mapping, a matrix representation of the corresponding Lie algebra.

The concept of representation of a Lie algebra is basically the same as the one we discussed in section 1.5 for the case of groups. The representation theory of Lie algebras will be discussed in more details later, but here we give the formal definition.

Definition 2.7 *If one can associate to every element T of a Lie algebra $\mathcal{G}$ a $n \times n$ matrix $D(t)$ such that*

1. $D(T + T') = D(T) + D(T')$
2. $D(aT) = aD(T)$

3. $D([T,T']) = [D(T), D(T')]$

for $T, T' \in \mathcal{G}$ *and* a *being a c-number. Then we say that the matrices* D *define a* n*-dimensional matrix representation of* $\mathcal{G}$*.*

Notice that given an element T of a Lie algebra $\mathcal{G}$, one can define a transformation in $\mathcal{G}$ as

$$T: \mathcal{G} \rightarrow \mathcal{G}' = [\, T \, , \, \mathcal{G} \,] \tag{2.40}$$

Using the Jacobi identity one can easily verify that the commutator of the composition of two of such transformations reproduces the Lie bracket operation on $\mathcal{G}$, i.e.

$$[\, T \, , \, [\, T' \, , \, \mathcal{G} \,]\,] - [\, T' \, , \, [\, T \, , \, \mathcal{G} \,]\,] = [\,[\, T \, , \, T' \,] \, , \, \mathcal{G} \,] \tag{2.41}$$

Therefore such transformations define a representation of $\mathcal{G}$ on $\mathcal{G}$, which is called the adjoint representation of $\mathcal{G}$. Obviously, it has the same dimension as $\mathcal{G}$. Introducing the coeeficients $d_a^b(T)$ as

$$[\, T \, , \, T_a \,] \equiv T_b d_a^b(T) \tag{2.42}$$

where T_a's constitute a basis for $\mathcal{G}$, one then gets (2.41)

$$\begin{aligned} [\, T \, , \, [\, T' \, , \, T_a \,]\,] - [\, T' \, , \, [\, T \, , \, T_a \,]\,] &= T_c \, d_b^c(T) d_a^b(T') - T_c \, d_b^c(T') d_a^b(T) \\ &= [\,[\, T \, , \, T' \,] \, , \, T_a \,] \\ &= T_c \, d_a^c([\, T \, , \, T' \,]) \end{aligned} \tag{2.43}$$

and so

$$[\, d(T) \, , \, d(T') \,] = d([\, T \, , \, T' \,]) \tag{2.44}$$

Therefore, the matrices defined in (2.42) constitute a matrix representation of $\mathcal{G}$, which is the adjoint representation $\mathcal{G}$. Using (2.23) and (2.42) one gets that $d_b^c(T_a)$ is indeed equal to $i f_{ab}^c$, as obtained in (2.39).

In a given finite dimensional representation D of a Lie algebra we define the quantity

$$\eta^D(T, T') \equiv Tr\,(D(T)D(T')) \tag{2.45}$$

which is symmetric and bilinear

1. $\eta^D(T, T') = \eta^D(T', T)$
2. $\eta^D(T, xT' + yT'') = x\eta^D(T, T') + y\eta^D(T, T'')$

It satisfies

$$\eta^D([T, T'], T'') + \eta^D(T, [T'', T'] = 0 \tag{2.46}$$

since using the cyclic property of the trace

$$Tr([D(T), D(T')]D(T'')) = Tr(D(T)[D(T'), D(T'')]) \tag{2.47}$$

Eq. (2.46) is an invariance property of $\eta^D(T, T')$. Indeed from (2.45) we see that

$$\eta^D(T, T') = \eta^D(gTg^{-1}, gT'g^{-1}) \tag{2.48}$$

and taking g to be of the form (2.37) we obtain (2.46) as the first order approximation in ε of (2.48). So η^D is a symmetric rank two tensor invariant under the adjoint representation.

The quantity $\eta^D(T, T')$ is called an *invariant bilinear trace form* for the Lie algebra $\mathcal{G}$. In the adjoint representation it is called the *Killing form*. From (2.39) and (2.45) we have that the Killing form is given by

$$\eta_{ab} \equiv \eta(T_a, T_b) \equiv Tr(d(T_a)d(T_b)) = -f_{ac}^d f_{bd}^c \tag{2.49}$$

Definition 2.8 *A Lie algebra is said to be* abelian *if all its elements commute with one another.*

In this case all the structure constants vanish and consequently the Killing form is zero. However there might exist some representation D of an abelian algebra for which the bilinear form (2.45) is not zero.

Definition 2.9 *A subalgebra $\mathcal{H}$ of $\mathcal{G}$ is said to be an* invariant subalgebra *(or ideal) if*

$$[\mathcal{H}, \mathcal{G}] \subset \mathcal{H} \tag{2.50}$$

From (2.27) we see the Lie algebra of an invariant subgroup of a group G is an invariant subalgebra of the Lie algebra of G.

Definition 2.10 *We say a Lie algebra $\mathcal{G}$ is* simple *if it has no invariant subalgebras, except zero and itself, and it is* semisimple *if it has no invariant abelian subalgebras.*

Notice that if $\mathcal{G}$ has an invariant subalgebra $\mathcal{H}$, i.e. $[\,\mathcal{G}\,,\,\mathcal{H}\,] \subset \mathcal{H}$, then from (2.41) one observes that the vector space of $\mathcal{H}$ defines a representation of $\mathcal{G}$, which is in fact an invariant subspace of the adjoint representation. Therefore, for non-simple Lie algebras, the adjoint representation is not irreducible.

Theorem 2.1 (Cartan) *A Lie algebra $\mathcal{G}$ is semisimple if and only if its Killing form is non degenerated, i.e.*

$$det \mid Tr(d(T_a)d(T_b)) \mid \neq 0. \tag{2.51}$$

or in other words, there is no $T \in \mathcal{G}$ such that

$$Tr(d(T)d(T')) = 0 \tag{2.52}$$

for every $T' \in \mathcal{G}$.

For the proof see chap. III of [JAC 79] or sec. 6 of appendix E of [COR 84].

Definition 2.11 *We say a semisimple Lie algebra is* compact *if its Killing form is positive definite.*

The Lie algebra of a compact semisimple Lie group is a compact semisimple Lie algebra. By choosing a suitable basis T_a we can put the Killing form of a compact semisimple Lie algebra in the form .

$$\eta_{ab} = \delta_{ab} \tag{2.53}$$

In order to see that consider a basis T'_a where the Killing form is not diagonal, i.e.

$$\eta'_{ab} = \mathrm{Tr}\,(T'_a\,T'_b) \tag{2.54}$$

But η'_{ab} is a real and symmetric matrix, and so can be diagonalised by an orthogonal transformation,

$$\eta' = \Lambda\,\eta^{\text{diag.}}\,\Lambda^T; \qquad \text{with} \qquad \eta^{\text{diag.}}_{ab} = \lambda_a\,\delta_{ab} \tag{2.55}$$

If we now change the basis as

$$T''_a = T'_b\,\Lambda_{ba} \tag{2.56}$$

we get that

$$\eta''_{ab} = \mathrm{Tr}\,(T''_a\,T''_b) = \Lambda^T_{ac}\mathrm{Tr}\,(T'_c\,T'_d)\,\Lambda_{db} = \lambda_a\,\delta_{ab} \tag{2.57}$$

But if the algebra is compact, all the eigenvalues λ_a of the Killing form are strictly positive, and so $\sqrt{\lambda_a}$ is real and positive. Then we can define

$$T_a = \frac{1}{\sqrt{\lambda_a}}\,T''_a; \qquad \text{and so} \qquad \mathrm{Tr}\,(T_a\,T_b) = \delta_{ab} \tag{2.58}$$

and so we get the result (2.53).

Let us define the quantity

$$f_{abc} \equiv f_{ab}^{d} \eta_{dc} \tag{2.59}$$

From (2.49) we have

$$f_{abc} = f_{ab}^{d} Tr(d(T_d)d(T_c)) = -iTr(d([T_a, T_b]T_c)) \tag{2.60}$$

Using the cyclic property of the trace one sees that f_{abc} is antisymmetric with respect to all its three indices. Notice that, in general, f_{abc} is not a structure constant.

For a compact semisimple Lie algebra we have from (2.53) that $f_{ab}^{c} = f_{abc}$, and therefore the commutation relations (2.23) can be written as

$$[T_a, T_b] = if_{abc} T_c \tag{2.61}$$

Therefore the structure constants of a compact semisimple Lie algebra can be put in a completely antisymmetric form.

2.5 su(2) and sl(2): Lie algebra prototypes

As we have seen the group $SU(2)$ is defined as the group of 2×2 complex unitary matrices with unity determinant. If an element of such group is written as $g = \exp iT$, then the matrix T has to be hermitian and traceless. Therefore the basis of the algebra $su(2)$ of this group can be taken to be (half of) the Pauli matrices ($T_i \equiv \frac{1}{2}\sigma_i$)

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \; T_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \; T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.62}$$

They satisfy the following commutation relations

$$[T_i \, , \, T_j] = i\epsilon_{ijk} T_k \tag{2.63}$$

The matrices (2.62) define what is called the spinor (2-dimensional) representation of the algebra $su(2)$.

From (2.39) we obtain the adjoint representation (3-dimensional) of $su(2)$

$$d_{ij}(T_k) = i\epsilon_{kji} = i\epsilon_{ikj} \tag{2.64}$$

and so

$$d(T_1) = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad ; \quad d(T_2) = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} ;$$

$$d(T_3) = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.65}$$

One can easily check that they satisfy (2.63).

As we have seen the group of rotations in three dimensions $SO(3)$ is defined as the group of 3×3 real orthogonal matrices. Its elements close to the identity can be written as $g = \exp iT$, and therefore the Lie algebra $so(3)$ of this group is given by 3×3 pure imaginary, antisymmetric and traceless matrices. But the matrices (2.65) constitute a basis for such algebra. Thefore the Lie algebras $su(2)$ and $so(3)$ are isomorphic, although the Lie groups $SU(2)$ and $SO(3)$ are just homomorphic (in fact $SO(3) \sim SU(2)/Z_2$).

The Killing form of this algebra, according to (2.49), is given by

$$\eta_{ij} = Tr(d(T_i T_j)) = 2\delta_{ij} \tag{2.66}$$

So, it is non degenerate. This is in agreement with theorem 2.1, since this algebra is simple. According to the definition 2.11 this is a compact algebra.

The trace form (2.45) in the spinor representation is given by

$$\eta^s_{ij} = Tr(D(T_i T_j)) = \frac{1}{2}\delta_{ij} \tag{2.67}$$

So, it is proportional to the Killing form, $\eta^s = \frac{1}{4}\eta$. This is a particular example of a general theorem we will prove later: the trace form in any representation of a simple Lie algebra is proportional to the Killing form.

Notice that the matrices in these representations discussed above are hermitian and therefore the matrices representing the elements of the group are unitary ($g = \exp iT$). In fact this is a result which constitute a generalization of theorem 1.3 to the case of compact Lie groups: any finite dimensional representation of a compact Lie group is equivalent to a unitary representation. Since the generators are hermitian we can always choose one of them to be diagonal. Traditionally one takes T_3 to be diagonal and defines (in the spinor rep. T_3 is already diagonal)

$$T_\pm = T_1 \pm iT_2 \tag{2.68}$$

Notice that formally, these are not elements of the algebra $su(2)$ since we have taken complex linear combination of the generators. These are elements of the complex algebra denoted by A_1.

Using (2.63) one finds

$$\begin{aligned} [T_3, T_\pm] &= \pm T_\pm \\ [T_+, T_-] &= 2T_3 \end{aligned} \tag{2.69}$$

Therefore the generators of A_1 are written as eigenvectors of T_3 . The eigenvalues ± 1 are called the roots of $su(2)$. We will show later that all Lie algebras can be put in a similar form. In any representation one can check that the operator

$$C = T_1^2 + T_2^2 + T_3^2 \tag{2.70}$$

commutes with all generators of $su(2)$. It is called the *quadractic Casimir operator*. The basis of the representation space can always be chosen to be eigenstates of the operators T_3 and C simultaneously. These states can be labelled by the spin j and the weight m

$$T_3 \mid j, m\rangle = m \mid j, m\rangle \tag{2.71}$$

The operators $T_\pm$ raise and lower the eigenvalue of T_3 since using (2.69)

$$\begin{aligned} T_3 T_\pm \mid j, m\rangle &= ([T_3, T_\pm] + T_\pm T_3) \mid j, m\rangle \\ &= (m \pm 1)\, T_\pm \mid j, m\rangle \end{aligned} \tag{2.72}$$

We are interested in finite representations and therefore there can only exists a finite number of eigenvalues m in a given representation. Consequently there must exist a state which possess the highest eigenvalue of T_3 which we denote j

$$T_+ \mid j, j\rangle = 0 \tag{2.73}$$

The other states of the representation are obtained from $\mid j, j\rangle$ by applying T_- successively on it. Again, since the representation is finite there must exist a positive integer l such that

$$(T_-)^{l+1} \mid j, j\rangle = 0 \tag{2.74}$$

Using (2.68) one can write the Casimir operator (2.70) as

$$C = T_3^2 + \frac{1}{2}\,(T_+ T_- + T_- T_+) \tag{2.75}$$

So, using (2.69), (2.71) and (2.73)

$$\begin{aligned} C \mid j,j\rangle &= \left(T_3^2 + \frac{1}{2}[T_+,T_-] + T_-T_+\right) \mid j,j\rangle \\ &= j\,(j+1) \mid j,j\rangle \end{aligned} \tag{2.76}$$

Since C commutes with all generators of the algebra, any state of the representation is an eigenstate of C with the same eigenvalue

$$C \mid j,m\rangle = j\,(j+1) \mid j,m\rangle \tag{2.77}$$

where $\mid j,m\rangle = (T_-)^n \mid j,j\rangle$ for $m = j - n$ and $n \leq l$. From Schur's lemma (see lemma1.1), in a irreducible representation, the Casimir operator has to be proportional to the unity matrix and so

$$C = j(j+1)\mathbb{1} \tag{2.78}$$

Using (2.75) one can write

$$T_+T_- = C - T_3^2 + T_3 \tag{2.79}$$

Therefore applying T_+ on both sides of (2.74)

$$\begin{aligned} T_+T_-(T_-)^l \mid j,j\rangle &= 0 \\ &= \left(j(j+1) - (j-l)^2 + (j-l)\right)(T_-)^l \mid j,j\rangle \end{aligned} \tag{2.80}$$

Since, by assumption the state $(T_-)^l \mid j,j\rangle$ does exist, one must have

$$j(j+1) - (j-l)^2 + (j-l) = (2j-l)(l+1) = 0 \tag{2.81}$$

Since l is a positive integer, the only possible solution is $l = 2j$. Therefore we conclude that

1. The lowest eigenvalue of T_3 is $-j$
2. The eigenvalues of T_3 can only be integers or half integers and in a given representation they vary from j to $-j$ in integral steps.

The group $SL(2)$, as defined in example 1.16, is the group of 2×2 real matrices with unity determinant. If one writes the elements close to the identity as $g = \exp L$ (without the i factor), then L is a real traceless 2×2 matrix. So the basis of the algebra $sl(2)$ can be taken as

$$L_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\; L_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};\; L_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.82}$$

This defines a 2-dimensional representation of $sl(2)$ which differ from the spinor representation of $su(2)$, given in (2.62), by a factor i in L_2. One can check the they satisfy

$$[L_1, L_2] = -L_3; \quad [L_1, L_3] = -L_2; \quad [L_2, L_3] = -L_1 \tag{2.83}$$

From these commutation relations one can obtain the adjoint representation of $sl(2)$, using (2.39)

$$d(L_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \quad ; \quad d(L_2) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$d(L_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.84}$$

According to (2.49), the Killing form of $sl(2)$ is given by

$$\eta_{ij} = Tr(d(L_i L_j)) = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.85}$$

$sl(2)$ is a simple algebra and we see that its Killing form is indeed non-degenerate (see theorem 2.1). From definition 2.11 we conclude $sl(2)$ is a non-compact Lie algebra.

The trace form (2.45) in the 2-dimensional representation (2.82) of $sl(2)$ is

$$\eta_{ij}^{2-dim} = Tr(L_i L_j) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.86}$$

Similarly to the case of $su(2)$, this trace form is proportional to the Killing form, $\eta^{2-dim} = \frac{1}{4}\eta$.

The operators

$$L_\pm \equiv L_1 \pm L_2 \tag{2.87}$$

according to (2.83), satisfy commutation relations identical to (2.69)

$$[L_3, L_\pm] = \pm L_\pm; \quad [L_+, L_-] = 2L_3 \tag{2.88}$$

The quadratic Casimir operator of $sl(2)$ is

$$C = L_1^2 - L_2^2 + L_3^2 = L_3^2 + \frac{1}{2}\left(L_+ L_- + L_- L_+\right) \tag{2.89}$$

The analysis we did for $su(2)$, from eqs. (2.71) to (2.81), applies also to $sl(2)$ and the conclusions are the same, i.e. , in a finite dimensional representation of $sl(2)$ with highest eigenvalue j of L_3 the lowest eigenvalue is $-j$. In addition the eigenvalues of L_3 can only be integers or half integers varying from j to $-j$ in integral steps. The striking difference however, is that the finite representations of $sl(2)$ (where these results hold) are not unitary. On the contrary, the finite dimensional representations of $su(2)$ are all equivalent to unitary representations. Indeed, the exponentiation of the matrices (2.62) and (2.65) (with the i factor) provide unitary matrices while the exponentiation of (2.82) and (2.84) do not. All unitary representations of $sl(2)$ are necessarily infinite dimensional. In fact this is true for any non compact Lie algebra.

The structures discussed in this section for the cases of $su(2)$ and $sl(2)$ are in fact the basic structures underlying all simple Lie algebras. The rest of this course will be dedicated to this study.

2.6 The structure of semisimple Lie algebras

We now start the study of the features which are common to all semisimple Lie algebras. These features are in fact a generalization of the properties of the algebra of angular momentum discussed in section 2.5. We will be mainly interested in compact semisimple algebras although several results also apply to the case of non-compact Lie algebras.

Theorem 2.2 *Given a subalgebra $\mathcal{H}$ of a compact semisimple Lie algebra $\mathcal{G}$ we can write*

$$\mathcal{G} = \mathcal{H} + \mathcal{P} \tag{2.90}$$

where

$$[\mathcal{H}, \mathcal{P}] \subset \mathcal{P} \tag{2.91}$$

where $\mathcal{P}$ is the orthogonal complement of $\mathcal{H}$ in $\mathcal{G}$ w.r.t. a trace form in a given representation, i.e.

$$Tr(\mathcal{P}\mathcal{H}) = 0 \tag{2.92}$$

Proof $\mathcal{P}$ does not contain any element of $\mathcal{H}$ and contains all elements of $\mathcal{G}$ which are not in $\mathcal{H}$. Using the cyclic property of the trace

$$Tr(\mathcal{H}[\mathcal{H}, \mathcal{P}]) = Tr([\mathcal{H}, \mathcal{H}]\mathcal{P}) = Tr(\mathcal{H}\mathcal{P}) = 0 \tag{2.93}$$

Therefore

$$[\mathcal{H}, \mathcal{P}] \subset \mathcal{P}. \tag{2.94}$$

□

This theorem does not apply to non compact algebras because the trace form does not provide an Euclidean type metric, i.e. there can exist null vectors which are orthogonal to themselves. As an example consider $sl(2)$.

Example 2.5 *Consider the subalgebra $\mathcal{H}$ of $sl(2)$ generated by $(L_1 + L_2)$ (see section 2.5). Its complement $\mathcal{P}$ is generated by $(L_1 - L_2)$ and L_3. However this is not an orthogonal complement since, using (2.85)*

$$Tr((L_1 + L_2)(L_1 - L_2)) = 4 \tag{2.95}$$

In addition $(L_1 \pm L_2)$ are null vectors, since

$$Tr(L_1 + L_2)^2 = Tr(L_1 - L_2)^2 = 0 \tag{2.96}$$

Using (2.83) one can check (2.91) is not satisfied. Indeed

$$\begin{aligned} [L_1 + L_2, L_1 - L_2] &= 2L_3 \\ [L_1 + L_2, L_3] &= -(L_1 + L_2) \end{aligned} \tag{2.97}$$

So

$$[\mathcal{H}, \mathcal{P}] \subset \mathcal{H} + \mathcal{P} \tag{2.98}$$

Notice $\mathcal{P}$ is a subalgebra too

$$[L_3, L_1 - L_2] = -(L_1 - L_2) \tag{2.99}$$

Theorem 2.3 *A compact semisimple Lie algebra is a direct sum of simple algebras that commute among themselves.*

Proof If $\mathcal{G}$ is not simple then it has an invariant subalgebra $\mathcal{H}$ such that

$$[\mathcal{H}, \mathcal{G}] \subset \mathcal{H} \tag{2.100}$$

But from theorem 2.2 we have that

$$[\mathcal{H}, \mathcal{P}] \subset \mathcal{P} \tag{2.101}$$

and therefore, since $\mathcal{P} \cap \mathcal{H} = 0$, we must have

$$[\mathcal{H}, \mathcal{P}] = 0 \tag{2.102}$$

But $\mathcal{P}$, in this case, is a subalgebra since

$$Tr([\mathcal{P}, \mathcal{P}]\mathcal{H}) = Tr(\mathcal{P}[\mathcal{P}, \mathcal{H}]) = 0 \tag{2.103}$$

and from theorem 2.2 again

$$[\mathcal{P}, \mathcal{P}] \subset \mathcal{P} \tag{2.104}$$

If $\mathcal{P}$ and $\mathcal{H}$ are not simple we repeat the process. □

Theorem 2.4 *For a simple Lie algebra the invariant bilinear trace form defined in eq. (2.45) is the same in all representations up to an overall constant. Consequentely they are all proportional to the Killing form.*

Proof Using the definition (2.31) of the adjoint representation and the invariance property (2.48) of $\eta^D(T, T')$ we have

$$\begin{aligned} \eta^D(T_a, T_b) &= Tr(D(gT_ag^{-1}gT_bg^{-1})) \\ &= Tr(D(T_c d^c_a(g) T_d d^d_b(g))) \\ &= (d^\top)^c_a(g)\eta^D(T_c, T_d) d^d_b(g) \\ &= (d^\top \eta^D d)_{ab} \end{aligned} \tag{2.105}$$

Therefore η^D is an invariant tensor under the adjoint representation. This is true for any representation D, in particular the adjoint itself. So, the Killing form defined in (2.49) also satisfies (2.105). From theorem 2.1 we have that for a semisimple Lie algebra, $\det\eta \neq 0$ and therefore η has an inverse. Then multiplying both sides of (2.105) by η^{-1} and using the fact that $\eta^{-1} = (d^\top \eta d)^{-1}$ we get

$$\eta^{-1}\eta^D = (d^\top \eta d)^{-1}(d^\top \eta^D d) = d^{-1}\eta^{-1}\eta^D d \tag{2.106}$$

and so

$$d(g)\eta^{-1}\eta^D = \eta^{-1}\eta^D d(g) \tag{2.107}$$

For a simple Lie algebra the adjoint representation is irreducible. Therefore using Schur's lemma (see lemma 1.1) we get

$$\eta^{-1}\eta^D = \lambda \mathbb{1} \rightarrow \eta^D = \lambda\eta \tag{2.108}$$

So, the theorem is proven. □

The constant λ is representation dependent and is called the *Dynkin index* of the representation D.

We will now show that it is possible to find a set of commuting generators such that all other generators are written as eigenstates of them (under the commutator). These commuting generators are the generalization of T_3 in $su(2)$ and they generate what is called the Cartan subalgebra.

Definition 2.12 *For a semisimple Lie algebra $\mathcal{G}$, the* Cartan subalgebra *is the maximal set of commuting elements of $\mathcal{G}$ which can be diagonalized simultaneously.*

The formal definition of the Cartan subalgebra of a Lie algebra (semisimple or not) is a little bit more sophisticated and involves two concepts which we now discuss. The *normalizer* of a subalgebra $\mathcal{K}$ of $\mathcal{G}$ is defined by the set

$$N(\mathcal{K}) \equiv \{x \in \mathcal{G} \mid [x, \mathcal{K}] \subset \mathcal{K}\} \tag{2.109}$$

Using the Jacobi identity we have

$$[[x, x'], \mathcal{K}] \subset \mathcal{K} \qquad (2.110)$$

with $x, x' \in N(\mathcal{K})$. Therefore the normalizer $N(\mathcal{K})$ is a subalgebra of $\mathcal{G}$ and $\mathcal{K}$ is an invariant subalgebra of $N(\mathcal{K})$. So we can say that the normalizer of $\mathcal{K}$ in $\mathcal{G}$ is the largest subalgebra of $\mathcal{G}$ which contains $\mathcal{K}$ as an invariant subalgebra.

Consider the sequence of subspaces of $\mathcal{G}$

$$\mathcal{G}_0 = \mathcal{G};\;\; \mathcal{G}_1 = [\mathcal{G}, \mathcal{G}];\;\; \mathcal{G}_2 = [\mathcal{G}, \mathcal{G}_1];\; ...\, \mathcal{G}_i = [\mathcal{G}, \mathcal{G}_{i-1}] \qquad (2.111)$$

We have that $\mathcal{G}_0 \supset \mathcal{G}_1 \supset \mathcal{G}_2 \supset ... \supset \mathcal{G}_i$ and each $\mathcal{G}_i$ is a invariant subalgebra of $\mathcal{G}$. We say $\mathcal{G}$ is a *nilpotent algebra* if $\mathcal{G}_n = 0$ for some n. Nilpotent algebras are not semisimple.

Similarly we can define the derived series

$$\mathcal{G}_{(0)} = \mathcal{G};\;\; \mathcal{G}_{(1)} = [\mathcal{G}, \mathcal{G}];\;\; \mathcal{G}_{(2)} = [\mathcal{G}_{(1)}, \mathcal{G}_{(1)}];\; ...\, \mathcal{G}_{(i)} = [\mathcal{G}_{(i-1)}, \mathcal{G}_{(i-1)}] \qquad (2.112)$$

If $\mathcal{G}_{(n)} = 0$ for some n then we say $\mathcal{G}$ is a *solvable algebra* . All nilpotent algebras are solvable, but the converse is not true.

Definition 2.13 *A* Cartan subalgebra *of a Lie algebra $\mathcal{G}$ is a nilpotent subalgebra which is equal to its normalizer in $\mathcal{G}$.*

Lemma 2.1 *If $\mathcal{G}$ is semisimple then a Cartan subalgebra of $\mathcal{G}$ is a maximal abelian subalgebra of $\mathcal{G}$ such that its generators can be diagonalized simultaneously.*

Definition 2.14 *The dimension of the Cartan subalgebra of $\mathcal{G}$ is the rank of $\mathcal{G}$.*

Notice that if H_1 , H_2 ... H_r are the generators of the Cartan subalgebra then $g^{-1}H_1 g$, $g^{-1}H_2 g$ ... $g^{-1}H_r g$ ($g \in G$) generates an abelian subalgebra of $\mathcal{G}$ with the same dimension as that one generated by H_i, $i = 1, 2, ...r$. This is also a Cartan subalgebra. Therefore there are an infinite number of Cartan subalgebras in $\mathcal{G}$ and they are all related by conjugation by elements of the group G which algebra is $\mathcal{G}$.

By choosing suitable linear combinations one can make the basis of the Cartan subalgebra to be orthonormal with respect to the Killing form of $\mathcal{G}$, i.e.[1]

$$Tr(H_i H_j) = \delta_{ij} \qquad (2.113)$$

[1]As we have shown, up to an overall constant, the trace form of a simple Lie algebra is the same in all representations. We will simplify the notation from now on, and write $Tr(TT')$ instead of $\eta^D(T, T')$. We shall specify the representation where the trace is being evaluated only when that is relevant.

with $i, j = 1, 2, ...$ rank $\mathcal{G}$. From the definition of Cartan subalgebra we see that these generators can be diagonalized simultaneously.

We now want to construct the generalization of the operators $T_{\pm} = T_1 + iT_2$ of $su(2)$, discussed in section 2.5, for the case of any compact semisimple Lie algebra. They are called *step operators* and their number is dim $\mathcal{G}$ - rank $\mathcal{G}$. According to theorem 2.2 they constitute the orthogonal complement of the Cartan subalgebra and therefore

$$Tr(H_i T_m) = 0 \tag{2.114}$$

with $i = 1, 2...$ rank $\mathcal{G}$, $m = 1, 2...$ (dim $\mathcal{G}$ - rank $\mathcal{G}$). In addition, since a compact semisimple Lie algebra is an Euclidean space we can make the basis T_m orthonormal, i.e.

$$Tr(T_m T_n) = \delta_{mn} \tag{2.115}$$

Again from theorem 2.2 we have that the commutator of an element of the Cartan subalgebra with T_m is an element of the subspace generated by the basis T_m . Then, since the algebra is compact we can put its structure constants in a completely antisymmetric form, and write

$$[H_i, T_m] = i f_{imn} T_n \tag{2.116}$$

or

$$[H_i, T_m] = (h_i)_{mn} T_n \tag{2.117}$$

where we have defined the matrices

$$(h_i)_{mn} = i f_{imn} \tag{2.118}$$

of dimension (dim $\mathcal{G}$ - rank $\mathcal{G}$) and which are hermitian

$$(h_i)^{\dagger}_{mn} = (h_i)^*_{nm} = -i f_{inm} = i f_{imn} = (h_i)_{mn} \tag{2.119}$$

Therefore we can find a unitary transformation that diagonalizes the matrices h_i without affecting the Cartan subalgebra generators H_i .

$$\begin{aligned} T_m &\rightarrow U_{mn} T_n \\ (h_i)_{mn} &\rightarrow (U h_i U^{\dagger})_{mn} \end{aligned} \tag{2.120}$$

with $U^{\dagger} = U^{-1}$. We shall denote by E_{α} the new basis of the subspace orthogonal to the Cartan subalgebra. The indices stand for the eigenvalues of the matrix h_i (or of the generators H_i). The commutation relations (2.117) can now be written as

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{2.121}$$

The eigenvalues α_i are the components of a vector of dimension rank $\mathcal{G}$ and they are called the *roots* of the algebra $\mathcal{G}$. The generators E_α are called *step operators* and they are complex linear combinations of the hermitian generators T_m. Notice that the roots α are real since they are the eigenvalues of the hermitian matrices h_i.

From (2.118) we see that the matrices h_i are antisymmetric, and their off diagonal elements are purely imaginary. So

$$h_i^\dagger = h_i; \quad h_i^* = -h_i \tag{2.122}$$

Therefore if v is an eigenstate of the matrix h_i then since the eigenvalue α_i is real we have

$$h_i v = \alpha_i v \tag{2.123}$$

and then

$$h_i^* v^* = -h_i v^* = \alpha_i v^* \tag{2.124}$$

Consenquently if α is a root its negative ($-\alpha$) is also a root. Thus the roots always occur in pairs.

We have shown that we can decompose a compact semisimple algebra L as

$$\mathcal{G} = \mathcal{H} + \sum_\alpha \mathcal{G}_\alpha \tag{2.125}$$

where $\mathcal{H}$ is generated by the commuting generators H_i and constitute the Cartan subalgebra of $\mathcal{G}$. The subspace $\mathcal{G}_\alpha$ is generated by the step operators E_α. This is called the *root space decomposition* of $\mathcal{G}$.In addition one can show that for a semisimple Lie algebra

$$dim\ \mathcal{G}_\alpha = 1; \quad for\ any\ root\ \alpha \tag{2.126}$$

and consequently the roots are not degenerated. So, there are not two step operators E_α and E'_α corresponding to the same root α. Therefore for a semisimple Lie algebra one has

$$\text{dim } \mathcal{G} \text{ - rank } \mathcal{G} = \textstyle\sum_\alpha \text{dim } \mathcal{G}_\alpha = \text{number of roots} = \text{even number}$$

Using the Jacobi identity and the commutation relations (2.121) we have that if α and β are roots then

$$\begin{aligned} [H_i, [E_\alpha, E_\beta]] &= -[E_\alpha, [E_\beta, H_i]] - [E_\beta, [H_i, E_\alpha]] \\ &= (\alpha_i + \beta_i) [E_\alpha, E_\beta] \end{aligned} \tag{2.127}$$

Since the algebra is closed under the commutator we have that $[E_\alpha, E_\beta]$ must be an element of the algebra. We have then three possibilities

1. $\alpha + \beta$ is a root of the algebra and then $[E_\alpha, E_\beta] \sim E_{\alpha+\beta}$
2. $\alpha + \beta$ is not a root and then $[E_\alpha, E_\beta] = 0$
3. $\alpha + \beta = 0$ and consequently $[E_\alpha, E_\beta]$ must be an element of the Cartan subalgebra since it commutes with all H_i .

Since in a semisimple Lie algebra the roots are not degenerated (see (2.126)), we conclude from (2.127) that 2α is never a root.

We then see that the knowlegde of the roots of the algebra provides all the information about the commutation relations and consequently about the structure of the algebra. From what we have learned so far, we can write the commutation relations of a semisimple Lie algebra $\mathcal{G}$ as

$$[H_i, H_j] = 0 \tag{2.128}$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{2.129}$$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ H_\alpha & \text{if } \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.130}$$

where $H_\alpha \equiv 2\alpha.H/\alpha^2$, $i, j = 1, 2, ...$ rank $\mathcal{G}$ (see discussion leading to (2.134) and (2.135)). The structure constants $N_{\alpha\beta}$ will be determined later. The basis $\{H_i, E_\alpha\}$ is called the *Weyl-Cartan basis* of a semisimple Lie algebra.

Using the cyclic property of the trace (2.47) (or equivalently, the invariance property (2.46)) we get that, in a given representation

$$Tr([H_i, E_\alpha]E_\beta) = Tr(E_\alpha[E_\beta, H_i]) \tag{2.131}$$

and so

$$(\alpha_i + \beta_i)Tr(E_\alpha E_\beta) = 0 \tag{2.132}$$

The step operators are orthogonal unless they have equal and opposite roots. In particular E_α is orthogonal to itself. If it was orthogonal to all others, the Killing form would have vanishing determinant and the algebra would not be semisimple. Therefore for semisimple algebras if α is a root then $-\alpha$ must also be a root, and $Tr(E_\alpha E_{-\alpha}) \neq 0$. The value of $Tr(E_\alpha E_{-\alpha})$ is connected to the structure constant of the second relation in (2.130). We know that $[E_\alpha, E_{-\alpha}]$ must be an element of the Cartan subalgebra. Therefore we write

$$[E_\alpha, E_{-\alpha}] = x_i H_i \tag{2.133}$$

Using (2.113) and the cyclic property of the trace we get

$$\begin{aligned} Tr(x_i H_i H_j) &= x_j \\ &= Tr([E_\alpha, E_{-\alpha}]H_j) \\ &= Tr([H_j, E_\alpha]E_{-\alpha}) \\ &= \alpha_j Tr(E_\alpha E_{-\alpha}) \end{aligned} \qquad (2.134)$$

Consequently $[E_\alpha, E_{-\alpha}]$ must be proportional to $\alpha.H$. Normalizing the step operators such that

$$Tr(E_\alpha E_{-\alpha}) = \frac{2}{\alpha^2} \qquad (2.135)$$

we obtain the second relation in (2.130).

Again using the invariance property (2.46) we have that

$$Tr([H_i, E_\alpha]H_j) = Tr([H_j, H_i]E_\alpha) \qquad (2.136)$$

and so

$$\alpha_i Tr(H_j E_\alpha) = 0 \qquad (2.137)$$

Since by assumption α is a root and therefore different from zero we get

$$Tr(H_i E_\alpha) = 0 \qquad (2.138)$$

From the above results and (2.113) we see that we can normalize the Cartan subalgebra generators H_i and the step operator E_α such that the Killing form becomes

$$\begin{aligned} Tr(H_i H_j) &= \delta_{ij} \; ; \quad i,j = 1,2,...rank\, \mathcal{G} \\ Tr(H_i E_\alpha) &= 0 \\ Tr(E_\alpha E_\beta) &= \frac{2}{\alpha^2}\delta_{\alpha+\beta,0} \end{aligned} \qquad (2.139)$$

This is the usual normalization of the *Weyl-Cartan basis*.

Notice that linear combinations $(E_\alpha \pm E_{-\alpha})$ diagonalizes the Killing form (2.139). However, by taking real linear combinations of H_i, $(E_\alpha + E_{-\alpha})$ and $i(E_\alpha - E_{-\alpha})$ one obtains a compact algebra since the eigenvalues of the Killing form are all of the same sign. On the hand, if one takes real linear combinations of H_i, $(E_\alpha + E_{-\alpha})$ and $(E_\alpha - E_{-\alpha})$ one obtains a non compact algebra.

Example 2.6 *In section 2.5 we have discussed the algebra of the group $SU(2)$. In that case the Cartan subalgebra is generated by T_3 only. The step operators are T_+ and T_- corresponding to the roots $+1$ and -1 respectively . So the rank of $SU(2)$ is one. We can represent these roots by the diagram 2.1*

Figure 2.1: The root diagram of A_1 ($su(2)$,so(3) or $sl(2)$)

2.7 The algebra $su(3)$

In example 1.17 we defined the groups $SU(N)$. We now discuss in more detail the algebra of the group $SU(3)$. As we have seen this is defined as the group of all 3×3 unitary matrices with unity determinant. If we write an element of this group as $g = \exp(iT)$ we see that T has to be hermitian in order g to be unitary. In addition using the fact that $det(\exp A) = \exp(TrA)$ we see that $TrT = 0$ in order to $detg = 1$. So the Lie algebra of $SU(3)$ is generated by 3×3 hermitian and traceless matrices. Its dimension is $2.3^2 - 3^2 - 1 = 8$. The Cartan subalgebra is generated by the diagonal matrices. Since they have to be traceless we have only two linearly independent diagonal matrices. Therefore the rank of $SU(3)$ is two, and consequently it has six roots. The usual basis of the algebra $su(3)$ is given by the Gell-Mann matrices which are a generalizition of the Pauli matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ;$$
$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \qquad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \; ;$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \; ; \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \; ;$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \; ; \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{2.140}$$

The trace form in such matrix representation is given by

$$Tr(\lambda_i \lambda_j) = 2\delta_{ij} \tag{2.141}$$

with $i, j = 1, 2, ...8$. The algebra $su(3)$ is simple and therefore according to theorem 2.4 the Killing form is proportinal to (2.141). Therefore, according to the definition 2.11 we see $su(3)$ is a compact algebra.

The matrices (2.140) satisfy the commutation relations

$$[\lambda_i, \lambda_j] = i f_{ijk} \lambda_k \tag{2.142}$$

where the structure constants f_{ijk} are completly antisymmetric (see (2.61)) and are given in table 2.1. The diagonal matrices λ_3 and λ_8 are the generators

i	j	k	f_{ijk}
1	2	3	2
1	4	7	1
1	5	6	-1
2	4	6	1
2	5	7	1
3	4	5	1
3	6	7	-1
4	5	8	$\sqrt{3}$
6	7	8	$\sqrt{3}$

Table 2.1: Structure constants of $su(3)$

of the Cartan subalgebra. One can easily check that they satisfy the conditions of the definition 2.13. We see that the remaining matrices play the role of T_m in (2.117). Therefore we can construct the step operators as linear combination of them. However, like the $su(2)$ case, these are complex linear combination and the step operators are not really generators of $su(3)$. Doing that, and normalizing the generators conveniently, we obtain the Weyl-Cartan basis for for such algebra

$$\begin{aligned} H_1 &= \frac{1}{\sqrt{2}}\lambda_3 \; ; & H_2 &= \frac{1}{\sqrt{2}}\lambda_8 \; ; \\ E_{\pm\alpha_1} &= \frac{1}{2}(\lambda_1 \pm i\lambda_2) \; ; & E_{\pm\alpha_2} &= \frac{1}{2}(\lambda_6 \pm i\lambda_7) \\ E_{\pm\alpha_3} &= \frac{1}{2}(\lambda_4 \pm i\lambda_5) \end{aligned} \tag{2.143}$$

So they satisfy

$$Tr(H_i H_j) = \delta_{ij} \; ; \quad Tr(E_{\alpha_m} E_{-\alpha_n}) = \delta_{mn} \tag{2.144}$$

with $i, j = 1, 2$ and $m, n = 1, 2, 3$. One can check that in such basis the commutation relations read

$$\begin{aligned} [H_1, E_{\pm\alpha_1}] &= \pm\sqrt{2}E_{\pm\alpha_1} \; ; & [H_2, E_{\pm\alpha_1}] &= 0 \; ; \\ [H_1, E_{\pm\alpha_2}] &= \mp\frac{\sqrt{2}}{2}E_{\pm\alpha_2} \; ; & [H_2, E_{\pm\alpha_2}] &= \pm\sqrt{\frac{3}{2}}E_{\pm\alpha_2} \; ; \\ [H_1, E_{\pm\alpha_3}] &= \pm\frac{\sqrt{2}}{2}E_{\pm\alpha_3} \; ; & [H_2, E_{\pm\alpha_3}] &= \pm\sqrt{\frac{3}{2}}E_{\pm\alpha_3} \end{aligned} \tag{2.145}$$

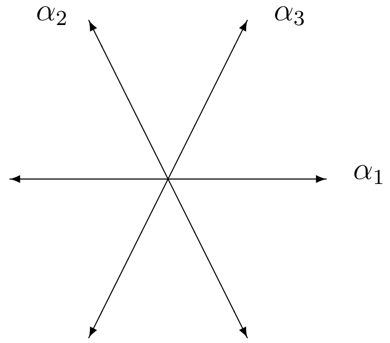

Figure 2.2: The root diagram of A_2 ($SU(3)$ or $SL(3)$)

Therefore the roots of $su(3)$ are

$$\alpha_1 = (\sqrt{2}, 0) \; ; \;\; \alpha_2 = (-\frac{\sqrt{2}}{2}, \sqrt{\frac{3}{2}}) \; ; \;\; \alpha_3 = (\frac{\sqrt{2}}{2}, \sqrt{\frac{3}{2}}) \tag{2.146}$$

and the corresponding negative ones.

Notice that all roots have the same lenght ($\alpha^2 = 2$) and the angle between any two of them is a multiple of $\frac{\pi}{3}$. The six roots of $su(3)$ form a regular diagram shown in figure 2.2. This is called the *root diagram* for $su(3)$. The root diagram of a Lie algebra lives in an Euclidean space of the same dimension as the Cartan subalgebra, i.e., the rank of the algebra. The root diagram is very useful in understanding the structure of the algebra. For instance, from (2.130) and the diagram 2.2 one sees that

$$\begin{aligned} [E_{\alpha_1}, E_{\alpha_3}] = [E_{\alpha_3}, E_{\alpha_2}] = [E_{\alpha_2}, E_{-\alpha_1}] = 0 \\ [E_{-\alpha_1}, E_{-\alpha_3}] = [E_{-\alpha_3}, E_{-\alpha_2}] = [E_{-\alpha_2}, E_{\alpha_1}] = 0 \end{aligned} \tag{2.147}$$

and also

$$\begin{aligned} [E_{\alpha_1}, E_{-\alpha_1}] &= \sqrt{2} H_1 \\ [E_{\alpha_2}, E_{-\alpha_2}] &= -\frac{\sqrt{2}}{2} H_1 + \sqrt{\frac{3}{2}} H_2 \\ [E_{\alpha_3}, E_{-\alpha_3}] &= \frac{\sqrt{2}}{2} H_1 + \sqrt{\frac{3}{2}} H_2 \end{aligned} \tag{2.148}$$

Whenever the sum of two roots is a root of the diagram we know, from (2.130), that the corresponding step operators do not commute. One can check that the non vanishing commutators between step operators are

$$\begin{aligned} [E_{\alpha_1}, E_{\alpha_2}] &= E_{\alpha_3} \; ; & [E_{-\alpha_1}, E_{-\alpha_2}] &= -E_{-\alpha_3} \; ; \\ [E_{\alpha_1}, E_{-\alpha_3}] &= -E_{-\alpha_2} \; ; & [E_{-\alpha_1}, E_{\alpha_3}] &= E_{\alpha_2} \; ; \\ [E_{\alpha_3}, E_{-\alpha_2}] &= E_{\alpha_1} \; ; & [E_{-\alpha_3}, E_{\alpha_2}] &= -E_{-\alpha_1} \end{aligned} \tag{2.149}$$

We have seen that the algebra $su(3)$ is generated by real linear combination of the Gell-Mann matrices (2.140), or equivalently of the matrices H_i, $i = 1, 2$, $(E_{\alpha_m} + E_{-\alpha_m})$ and $-i(E_{\alpha_m} - E_{-\alpha_m})$, $m = 1, 2, 3$. These are hermitian matrices. If one takes real linear combinations of H_i, $(E_{\alpha_m} + E_{-\alpha_m})$ and $(E_{\alpha_m} - E_{-\alpha_m})$ instead, one obtains the algebra $sl(3)$ which is not compact. This is very similar to the relation between $su(2)$ and $sl(2)$ which we saw in section 2.5. This generalizes in fact, to all $su(N)$ and $sl(N)$.

2.8 The Properties of roots

We have seen that for a semisimple Lie algebra $\mathcal{G}$, if α is a root then, $-\alpha$ is also a root. This means that for each step operator E_α there exists a corresponding step operator E_α. Together with $H_\alpha = 2\alpha.H/\alpha^2$ they constitute a $sl(2)$ subalgebra of $\mathcal{G}$, since from (2.129) and (2.130) one gets

$$\begin{aligned} [H_\alpha, E_{\pm\alpha}] &= \pm 2 E_{\pm\alpha} \\ [E_\alpha, E_{-\alpha}] &= H_\alpha \end{aligned} \tag{2.150}$$

This subalgebra is isomorphic to $sl(2)$ since H_α plays the role of $2T_3$, E_α and $E_{-\alpha}$ play the role of T_+ and T_- respectively (see section 2.5). Therefore to each pair of roots α and $-\alpha$ we can construct a $sl(2)$ subalgebra. These subalgebras, however, do not have to commute among themselves.

We have learned in section 2.5 that T_3 , the third component of the angular momentum, has half integer eigenvalues, and consenquently H_α ($\equiv 2T_3$) must have integer eigenvalues. From (2.129) we have

$$[H_\alpha, E_\beta] = \frac{2\alpha.\beta}{\alpha^2} E_\beta \tag{2.151}$$

Therefore if $\mid m\rangle$ is an eigenstate of H_α with an integer eigenvalue m them the state $E_\beta \mid m\rangle$ has eigenvalue $m + \frac{2\alpha.\beta}{\alpha^2}$ since

$$\begin{aligned} H_\alpha E_\beta \mid m\rangle &= (E_\beta H_\alpha + [H_\alpha, E_\beta]) \mid m\rangle \\ &= \left(m + \frac{2\alpha.\beta}{\alpha^2} \right) E_\beta \mid m\rangle \end{aligned} \tag{2.152}$$

$\frac{2\alpha.\beta}{\alpha^2}$	$\frac{2\alpha.\beta}{\beta^2}$	θ	$\frac{\alpha^2}{\beta^2}$
0	0	$\frac{\pi}{2}$	undetermined
1	1	$\frac{\pi}{3}$	1
-1	-1	$\frac{2\pi}{3}$	1
1	2	$\frac{\pi}{4}$	2
-1	-2	$\frac{3\pi}{4}$	2
1	3	$\frac{\pi}{6}$	3
-1	-3	$\frac{5\pi}{6}$	3

Table 2.2: The possible scalar products, angles and ratios of squared lenght for the roots

This implies that

$$\frac{2\alpha.\beta}{\alpha^2} = integer \tag{2.153}$$

for any roots α and β. This result is crucial in the study of the structure of semisimple Lie algebras. In order to satisfy this condition the roots must have some very special properties. From Schwartz inequality we get (The roots live in a Euclidean space since they inherit the scalar product from the Killing form of $\mathcal{G}$ restricted to the Cartan subalgebra by $\alpha.\beta \equiv Tr(\alpha.H\beta.H) = \sum_{i=1}^{rank\mathcal{G}} \alpha_i\beta_i$)

$$\alpha.\beta = \mid \alpha \mid\mid \beta \mid \cos\theta \leq \mid \alpha \mid\mid \beta \mid \tag{2.154}$$

where θ is the angle between α and β. Consenquently

$$\frac{2\alpha.\beta}{\alpha^2}\frac{2\alpha.\beta}{\beta^2} = mn = 4(\cos\theta)^2 \leq 4 \tag{2.155}$$

where m and n are integers according to (2.153), and so

$$0 \leq mn \leq 4 \tag{2.156}$$

This condition is very restrictive and from it we get that the possible values of scalar products, angles and ratio of squared lenghts between any two roots are those given in table 2.2. For the case of α being parallel or anti-parallel to β we have $\cos\theta = \pm 1$ and consequently $mn = 4$. In this case the possible values of m and n are

1. $\frac{2\alpha.\beta}{\alpha^2} = \pm 2$ and $\frac{2\alpha.\beta}{\beta^2} = \pm 2$

2. $\frac{2\alpha.\beta}{\alpha^2} = \pm 1$ and $\frac{2\alpha.\beta}{\beta^2} = \pm 4$

3. $\frac{2\alpha.\beta}{\alpha^2} = \pm 4$ and $\frac{2\alpha.\beta}{\beta^2} = \pm 1$

In case 1 we have that $\alpha = \beta$, which is trivial, or $\alpha = -\beta$ which is a fact discussed earlier, i.e., to every root α there corresponds a root $-\alpha$ in a semisimple Lie algebra. In case 2 we have $\alpha = \pm 2\beta$ which is impossible to occur in a semisimple Lie algebra. In (2.126) we have seen that dim $\mathcal{G} = 1$ and therefore there exist only one step operator corresponding to a root α. From (2.127) we see that 2α or -2α can not be roots since $[E_\alpha, E_\alpha] = [E_{-\alpha}, E_{-\alpha}] = 0$. The case 3 is similar to 2. Therefore in a semisimple Lie algebra *the only roots which are multiples of* α *are* $\pm\alpha$.

Notice that there are only three possible values for the ratio of lenghts of roots, namely 1, 2 and 3 (there are five if one considers the reciprocals $\frac{1}{2}$ and $\frac{1}{3}$). However for a given simple Lie algebra, where there are no disjoint, mutually orthogonal set of roots, there can occur only two different lenght of roots. The reason is that if α, β, and γ are roots of a simple Lie algebra and $\frac{\alpha^2}{\beta^2} = 2$ and $\frac{\alpha^2}{\gamma^2} = 3$ then it follows that $\frac{\gamma^2}{\beta^2} = \frac{2}{3}$ and this is not an allowed value for the ratio of two roots (see table 2.2).

2.9 The Weyl group

In the section 2.8 we have shown that to each pair of roots α and $-\alpha$ of a semisimple Lie algebra we can construct a $sl(2)$ (or $su(2)$) subalgebra generated by the operators H_α , E_α and $E_{-\alpha}$ (see eq. (2.150)). We now define the hermitian operators:

$$T_1(\alpha) = \frac{1}{2}(E_\alpha + E_{-\alpha})$$
$$T_2(\alpha) = \frac{1}{2i}(E_\alpha - E_{-\alpha}) \tag{2.157}$$

which satisfy the commutation relations

$$\begin{aligned} [H_i, T_1(\alpha)] &= i\alpha_i T_2(\alpha) \\ [H_i, T_2(\alpha)] &= -i\alpha_i T_1(\alpha) \\ [T_1(\alpha), T_2(\alpha)] &= \frac{i}{2} H_\alpha \end{aligned} \tag{2.158}$$

The operator $T_2(\alpha)$ is the generator of rotations about the 2-axis, and a rotation by π is generated by the element

$$S_\alpha = \exp(i\pi T_2(\alpha)) \tag{2.159}$$

Using (2.27) and (2.158) one can check that

$$\begin{aligned} S_\alpha(x.H)S_\alpha^{-1} &= x.H + x.\alpha T_1(\alpha)\sin\pi + \frac{x.\alpha}{\alpha^2}\alpha.H(\cos\pi - 1) \\ &= \left(x_i - 2\frac{x.\alpha}{\alpha^2}\alpha_i\right) H_i \\ &= \sigma_\alpha(x).H \end{aligned} \tag{2.160}$$

where we have defined the operator σ_α, acting on the root space, by

$$\sigma_\alpha(x) \equiv x - 2\frac{x.\alpha}{\alpha^2}\alpha \tag{2.161}$$

This operator corresponds to a reflection w.r.t the plane perpendicular to α. Indeed, if θ is the angle between x and α then $\frac{x.\alpha}{\alpha^2}\alpha = \mid x \mid \cos\theta \frac{\alpha}{|\alpha|}$. Therefore $\sigma_\alpha(x)$ is obtained from x by subtracting a vector parallel (or anti-parallel) to α and with lenght twice the projection of x in the direction of α. These reflections are called *Weyl reflections* on the root space.

We now want to show that if α and β are roots of a given Lie algebra $\mathcal{G}$, then $\sigma_\alpha(\beta)$ is also a root. Let us introduce the operator

$$\tilde{E}_\beta \equiv S_\alpha E_\beta S_\alpha^{-1} \tag{2.162}$$

where E_β is a step operator of the algebra and S_α is defined in (2.159). From the fact that (see (2.129))

$$[x.H, E_\beta] = x.\beta E_\beta \tag{2.163}$$

we get, using (2.160) that

$$\begin{aligned} S_\alpha [x.H, E_\beta] S_\alpha^{-1} &= [S_\alpha x.H S_\alpha^{-1}, S_\alpha E_\beta S_\alpha^{-1}] \\ &= [\sigma_\alpha(x).H, \tilde{E}_\beta] \qquad (2.164) \\ &= x.\beta S_\alpha E_\beta S_\alpha^{-1} \qquad (2.165) \\ &= x.\beta \tilde{E}_\beta \qquad (2.166) \end{aligned}$$

and so

$$[\sigma_\alpha(x).H, \tilde{E}_\beta] = x.\beta \tilde{E}_\beta \tag{2.167}$$

However, if we perform a reflection twice we get back to where we started, i.e., $\sigma^2 = 1$. Therefore denoting $\sigma_\alpha(x)$ by y we get that $\sigma_\alpha(y) = x$, and then from (2.167)

$$[y.H, \tilde{E}_\beta] = \sigma_\alpha(y).\beta \tilde{E}_\beta \tag{2.168}$$

and so

$$[H_i, \tilde{E}_\beta] = \sigma_\alpha(\beta)_i \tilde{E}_\beta \tag{2.169}$$

Therefore $\tilde{E}_\beta$, defined in (2.162), is a step operator corresponding to the root $\sigma_\alpha(\beta)$. Consequently if α and β are roots, $\sigma_\alpha(\beta)$ is necessarily a root (similarly $\sigma_\beta(\alpha)$).

Example 2.7 *In section 2.7 we have discussed the algebra of the group $SU(3)$. The root diagram with the planes perpendicular to the roots is given in figure 2.3. One can sees that the root diagram is invariant under Weyl reflections. We have*

$$\begin{array}{llll} \sigma_1 : & \alpha_1 \leftrightarrow -\alpha_1 & \alpha_2 \leftrightarrow \alpha_3 & -\alpha_2 \leftrightarrow -\alpha_3 \\ \sigma_2 : & \alpha_1 \leftrightarrow \alpha_3 & \alpha_2 \leftrightarrow -\alpha_2 & -\alpha_1 \leftrightarrow -\alpha_3 \\ \sigma_3 : & \alpha_1 \leftrightarrow -\alpha_2 & \alpha_2 \leftrightarrow -\alpha_1 & \alpha_3 \leftrightarrow -\alpha_3 \end{array}$$

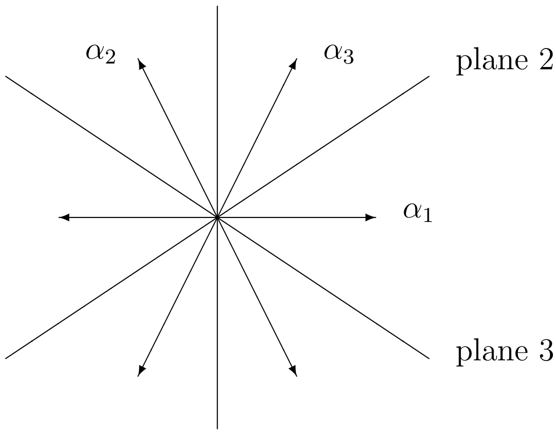

Figure 2.3: The planes orthogonal to the roots of A_2 ($SU(3)$ or $SL(3)$)

$$\sigma_1\sigma_2 : \begin{cases} \alpha_1 \to \alpha_2 & \alpha_2 \to -\alpha_3 & \alpha_3 \to -\alpha_1 \\ -\alpha_1 \to -\alpha_2 & -\alpha_2 \to \alpha_3 & -\alpha_3 \to \alpha_1 \end{cases}$$
$$\sigma_2\sigma_1 : \begin{cases} \alpha_1 \to -\alpha_3 & \alpha_2 \to \alpha_1 & \alpha_3 \to -\alpha_2 \\ -\alpha_1 \to \alpha_3 & -\alpha_2 \to -\alpha_1 & -\alpha_3 \to \alpha_2 \end{cases} \tag{2.170}$$

Notice that the composition of Weyl reflections is not necessarily a reflection and that reflections do not commute. In this particular case the operation $\sigma_2\sigma_1$ is a rotation by an angle of $\frac{2\pi}{3}$ and $\sigma_1\sigma_2$ is its inverse. The set of a Weyl reflexions and the composition of two or more of them form a group called the Weyl group. *It leaves the root diagram of $su(3)$ invariant. This group is isomorphic to S_3 , and in fact the Weyl group of $su(N)$ is S_N , the group of permutations of N elements.*

Definition 2.15 *The* Weyl group *of a Lie algebra, or of its root system, is the finite discrete group generated by the Weyl reflections.*

From the considerations above we see that the Weyl group leaves invariant the root system. However it does not contain all the symmetries of the root system. The inversion $\alpha \leftrightarrow -\alpha$ is certainly a symmetry of the root system of any semisimple Lie algebra but, in general, it is not an element of Weyl group. In the case of $su(3)$ discussed in example 2.7 the inversion can not be written in terms of reflections. In addition, the root diagram of $su(3)$ is invarint under rotations of $\frac{\pi}{3}$, and this operation is not an element of the Weyl group of $su(3)$.

As we have seen the conjugation by the group element S_α defined in (2.159) maps $x.H$ into $\sigma_\alpha(x).H$ and E_β into $E_{\sigma_\alpha}(\beta)$. Therefore, such mapping imitates, in the algebra, the Weyl reflections of the roots. According to (2.34) this is an inner automorphism of the algebra. Consequently any transformation of the Weyl group can be elevated to an inner automorphism of the corresponding algebra. In fact, any symmetry of the root diagram can be used to construct an automorphism of the algebra. However those symmetries which do not belong to the Weyl group give rise to outer automorphisms. We will see later that the mapping $H_i \rightarrow -H_i$, $E_\alpha \rightarrow -E_{-\alpha}$ and $E_{-\alpha} \rightarrow -E_\alpha$ is an automorphism of any semisimple Lie algebra. It is a consequence of the invariance of the root diagram under the inversion $\alpha \leftrightarrow -\alpha$. It will be an inner (outer) automorphism if the inversion is (is not) an element of the Weyl group.

We can summarize all the results about roots we have obtained so far in the form of four postulates.

Definition 2.16 *A set* Φ *of vectors in a Euclidean space is the* root system *or* root diagram *of a semisimple Lie algebra* $\mathcal{G}$ *if*

1. Φ *does not contain zero, spans an Euclidean space of the same dimension as the rank of the Lie algebra* $\mathcal{G}$ *and the number of elements of* Φ *is equal to dim* $\mathcal{G}$ *- rank* $\mathcal{G}$.

2. *If* $\alpha \in \Phi$ *then the only multiples of* α *in* Φ *are* $\pm\alpha$

3. *If* $\alpha, \beta \in \Phi$, *then* $\frac{2\alpha.\beta}{\alpha^2}$ *is an integer*

4. *If* $\alpha, \beta \in \Phi$, *then* $\sigma_\alpha(\beta) \in \Phi$, *i.e., the Weyl group leaves* Φ *invariant.*

Notice that if the root diagram decomposes into two or more disjoint and mutually orthogonal subdiagrams then the corresponding Lie algebra is not simple. Suppose the rank of the algebra is r and that the diagram decomposes into two orthogonal subdiagrams of dimensions m and n such that $m + n = r$. By taking basis v_i $(i = 1, 2...m)$ and u_k $(k = 1, 2...n)$ in each subdiagram we can split the generators of the Cartan subalgebra into two subsets of the form $H_v \equiv v.H$ and $H_u = u.H$. From (2.163) we see that the generatorsa H_v commute with all step operators corresponding to roots in the subdiagram generated by u_k , and vice versa. In addition, since the sum of a root of one subdiagram with a root of the other is not a root, we conclude that the corresponding step operators commute. Therefore each subdiagram corresponds to an invariant subalgebra of the Lie algebra which root diagram is their union.

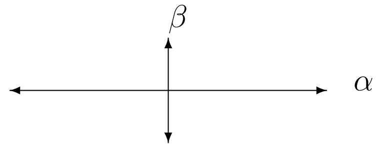

Figure 2.4: The root diagram of $su(2) \oplus su(2)$

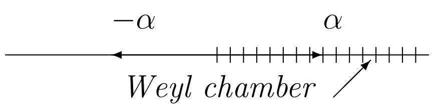

Figure 2.5: The Weyl chambers of A_1 ($su(2)$,so(3) or $sl(2)$)

Example 2.8 *The root diagram shown in figure 2.4 is made of two ortoghonal diagrams. Since each one is the diagram of an $su(2)$ algebra we conclude, from the discussion above, that it corresponds to the algebra $su(2) \oplus su(2)$. Remember that the ratio of the squared lenght of the ortoghonal roots are undetermined in this case (see table 2.2).*

2.10 Weyl Chambers and simple roots

The hyperplanes perpendicular to the roots, defined in section 2.9 partition the root space into finitely many regions. These connected regions (without the hyperplanes) are called *Weyl Chambers* . Due to the regularity of the root systems all the Weyl chambers have the same form and are equivalent.

Example 2.9 *In the case of $su(2)$ (or $so(3)$ and $sl(2)$) there are only two Weyl chambers, each one corresponding to a half line. These are shown in figure 2.5. In the case of $su(3)$ there are 6 Weyl chambers. They are shown in figure 2.6.*

Notice that under a Weyl reflection, all points of a Weyl chamber are mapped into the same Weyl chamber, and therefore the Weyl group takes one Weyl Chamber into another. In fact the Weyl group acts transitively on Weyl Chambers and its order is the number of Weyl Chambers. In general the number of roots is bigger than the number of Weyl Chambers.

Since the Weyl Chambers are equivalent one to another, we will choose one of them and call it the *Fundamental Weyl Chamber*. Consider now a vector x inside this particular chamber. The scalar product of x with any root α is always different from zero, since if it was zero x would be on the hyperplane

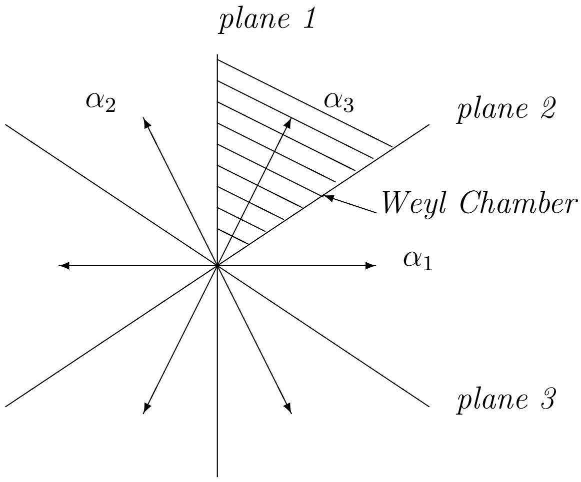

Figure 2.6: The Weyl chambers of A_2 ($SU(3)$ or $SL(3)$)

perpendicular to α and therefore not inside a Weyl chamber. As we move x within the chamber the sign of $\alpha.x$ does not change, since in order to change $\alpha.x$ would have to vanish and therefore x would have to cross a hyperplane. Therefore the scalar product of a root with any vector inside a Weyl Chamber has a definite sign.

Definition 2.17 *Let x be any vector inside the Fundamental Weyl chamber. We say α is a* positive root *if $\alpha.x > 0$ and a* negative root *if $\alpha.x < 0$.*

Definition 2.18 *We say a positive root is a* simple root *if it can not be written as the sum of two positive roots.*

Example 2.10 *In the case of $su(3)$, if we choose the Fundamental Weyl chamber to be the one shown in figure 2.6, then the positive roots are α_1, α_2 and α_3. We see that α_1 and α_2 are simple, but α_3 is not since $\alpha_3 = \alpha_1 + \alpha_2$.*

Theorem 2.5 *Let α and β be non proportional roots. Then*

1. *if $\alpha.\beta > 0$, $\alpha - \beta$ is a root*
2. *if $\alpha.\beta < 0$, $\alpha + \beta$ is a root*

Proof If $\alpha.\beta > 0$ we see from table 2.2 that either $\frac{2\alpha.\beta}{\alpha^2}$ or $\frac{2\alpha.\beta}{\beta^2}$ is equal to 1. Without loss of generality we can take $\frac{2\alpha.\beta}{\alpha^2} = 1$. Therefore

$$\sigma_\alpha(\beta) = \beta - \frac{2\alpha.\beta}{\alpha^2}\alpha = \beta - \alpha \qquad (2.171)$$

So, from the invariance of the root system under the Weyl group, $\beta - \alpha$ is also a root, as well as $\alpha - \beta$. The proof for the case $\alpha.\beta < 0$ is similar. □

Theorem 2.6 *Let α and β be distinct simple roots. Then $\alpha - \beta$ is not a root and $\alpha.\beta \leq 0$.*

Proof Suppose $\alpha - \beta \equiv \gamma$ is a root. If γ is positive we write $\alpha = \gamma + \beta$, and if it is negative we write $\beta = \alpha + (-\gamma)$. In both cases we get a contradiction to the fact α and β are simple. Therefore $\alpha - \beta$ can not be a root. From theorem 2.5 we conclude $\alpha.\beta$can not be positive. □

Theorem 2.7 *Let α_1, α_2,... α_r be the set of all simple roots of a semisimple Lie algebra $\mathcal{G}$. Then $r =$ rank $\mathcal{G}$ and each root α of $\mathcal{G}$ can be written as*

$$\alpha = \sum_{a=1}^{r} n_a \alpha_a \tag{2.172}$$

where n_a are integers, and they are positive or zero if α is a positive root and negative or zero if α is a negative root.

Proof Suppose the simple roots are linear dependent. Denote by x_a and $-y_a$ the positive and negative coefficients, respectively, of a vanishing linear combination of the simple roots. Then write

$$\sum_{a=1}^{s} x_a \alpha_a = \sum_{b=s+1}^{r} y_b \alpha_b \equiv v \tag{2.173}$$

with each α_a being different from each α_b. Therefore

$$v^2 = \sum_{ab} x_a y_b \alpha_a . \alpha_b \leq 0 \tag{2.174}$$

Since v is a vector on an Euclidean space it follows that that the only possibility is $v^2 = 0$, and so $v = 0$. But this implies $x_a = y_b = 0$ and consequently the simple roots must be linear independent. Now let α be a positve root. If it is not simple then $\alpha = \beta + \gamma$ with β and γ both positive. If β and/or γ are not simple we can write them as the sum of two positive roots. Notice that α can not appear in the expansion of β and/or γ in terms of two positive roots, since if x is a vector of the Fundamental Weyl Chamber we have $x.\alpha = x.\beta + x.\gamma$. Since they are all positive roots we have $x.\alpha > x.\beta$ and $x.\alpha > a.\gamma$. Therefore β or γ can not be written as $\alpha + \delta$ with δ a positive root. For the same reason β and γ will not appear in the expansion of any further root appearing in this process. Thus, we can continue such process until α is written as a sum of simple roots, i.e. $\alpha = \sum_{a=1}^{r} n_a \alpha_a$ with each n_a being zero or a positive integer. Since, for semisimple Lie algebras, the roots come in pairs (α and $-\alpha$) it follows that the negative roots are written in terms of the simple roots in the same way, with n_a being zero or negative integers. We then see that the set of simple roots span the root space. Since they are linear independent, they form a basis and consequently $r = \text{rank}\ \mathcal{G}$. $\Box$

2.11 Cartan matrix and Dynkin diagrams

In order to define positive and negative roots and then simple roots we have chosen one particular Weyl Chamber to play a special role. This was called the Fundamental Weyl Chamber. However any Weyl Chamber can play such role since they are all equivalent. As we have seen the Weyl group transforms one Weyl Chamber into another. In fact, one can show (see pag. 51 of [HUM 72]) that there exists one and only one element of the Weyl group which takes one Weyl Chamber into any other.

By changing the choice of the fundamental Weyl Chamber one changes the set of simple roots. This implies that the choices of simple roots are related by Weyl reflections. From the figure 2.6 we see that in the case of $SU(3)$ any of the pairs of roots (α_1, α_2), $(\alpha_3, -\alpha_1)$, $(\alpha_2, -\alpha_3)$, $(-\alpha_1, -\alpha_2)$, $(-\alpha_3, \alpha_1)$, $(-\alpha_2, \alpha_3)$, could be taken as the simple roots. The common features in these pairs are the angle between the roots and the ratio of their lenghts. (in the case of $SU(3)$ this is trivial since all roots have the same length, but in other cases it is not).

Therefore the important information about the simple roots can be encoded into their scalar products. For this reason we introduce an $r \times r$ matrix ($r =$ rank $\mathcal{G}$) as

$$K_{ab} \equiv \frac{2\alpha_a . \alpha_b}{\alpha_b^2} \tag{2.175}$$

$(a, b = 1, 2, ...$ rank $\mathcal{G})$ which is called the *Cartan matrix* of the Lie algebra. As we will see it contains all the relevant information about the structure of the algebra $\mathcal{G}$. Let us see some of its properties:

1. It provides the angle between any two simple roots since

$$K_{ab} K_{ba} = 4 \frac{\alpha_a . \alpha_b}{\alpha_b^2} \frac{\alpha_a . \alpha_b}{\alpha_a^2} \tag{2.176}$$

with no summation on a or b, and so

$$\cos\theta = -\frac{1}{2}\sqrt{K_{ab} K_{ba}} \tag{2.177}$$

where θ is the angle between α_a and α_b. We take the minus sign because, according to theorem 2.6, the simple roots always form obtuse angles.

2. The Cartan matrix gives the ratio of the lenghts of any two simple roots since

$$\frac{K_{ab}}{K_{ba}} = \frac{\alpha_a^2}{\alpha_b^2} \tag{2.178}$$

3. $K_{aa} = 2$. The diagonal elements do not give any information.

4. From the properties of the roots discussed in section 2.8 we see that

$$K_{ab}K_{ba} = 4\,(\cos\theta)^2 = 0, 1, 2, 3 \qquad (2.179)$$

we do not get 4 because we are taking $a \neq b$. But from theorem 2.6 we have $\alpha_a.\alpha_b \leq 0$ and so the off diagonal elements of the Cartan matrix can take the values

$$K_{ab} = 0, -1, -2, -3 \qquad (2.180)$$

with $a \neq b$. From the table 2.2 we see that if $K_{ab} = -2$ or -3 then we necessarily have $K_{ba} = -1$.

5. If α_a and α_b are orthogonal, obviously $K_{ab} = K_{ba} = 0$. At the end of section 2.9 we have shown that if the root diagram decomponses into two or more mutually orthogonal subdiagrams then the corresponding algebra is not simple. As a consequence of that if follows that the Cartan matrix of a Lie algebra, which is not simple, necessarily has a block-diagonal form.

6. The Cartan matrix is symmetric only when all roots have the same lenght.

Example 2.11 *The algebra of $SO(3)$ or $SU(2)$ has only one simple root and therefore its Cartan matrix is trivial, i.e., $K = 2$.*

Example 2.12 *The algebra of $SO(4)$ is not simple. It is isomorphic to $su(2) \oplus su(2)$. Its root diagram is given in figure 2.4. The simple roots are α and β (for instance) and the ratio of their lenght is not determined. The Cartan matrix is*

$$K = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad (2.181)$$

Example 2.13 *From figure 2.6 we see that the Cartan matrix of A_2 ($su(3)$ or $sl(3)$) is*

$$K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad (2.182)$$

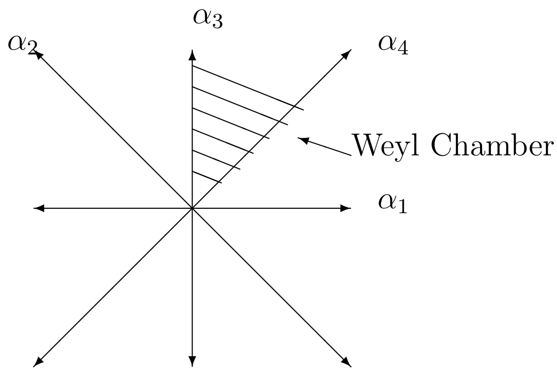

Figure 2.7: The root diagram and Fundamental Weyl chamber of $so(5)$ (or $sp(2)$)

Example 2.14 *The algebra of $SO(5)$ has dimension 10 and rank 2. So it has 8 roots. It root diagram is shown in figure 2.7. The Fundamental Weyl Chamber is the shaded region. Notice that all roots lie on the hyperplanes perpendicular to the roots. The positive roots are α_1, α_2, α_3 and α_4 as shown on the diagram. All the others are negative. The simple roots are α_1 and α_2, and the ratio of their squared lenghts is 2. The angle between them is $\frac{3\pi}{4}$. The Cartan matrix of $so(5)$ is*

$$K = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \tag{2.183}$$

Example 2.15 *The last simple Lie algebra of rank 2 is the exceptional algebra G_2 . Its root diagram is shown in figure 2.8. It has 12 roots and therefore dimension 14. The Fundamental Weyl Chamber is the shaded region. The positive roots are the ones labelled from 1 to 6 on the diagram. The simple roots are α_1 and α_2. The Cartan matrix is given by*

$$K = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \tag{2.184}$$

We have seen that the relevant information contained in the Cartan matrix is given by its off-diagonal elements. We have also seen that if $K_{ab} \neq 0$ then one of K_{ab} or K_{ba} is necessarily equal to -1. Therefore the information of the off-diagonal elements can be given by the positive integers $K_{ab}K_{ba}$ (no sum in

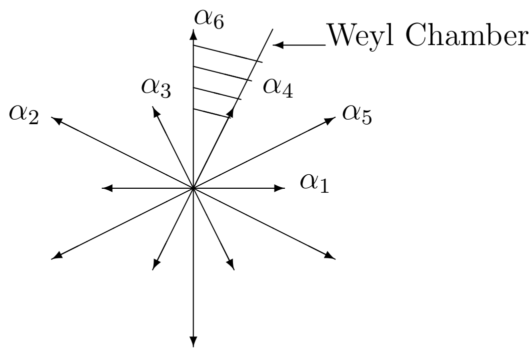

Figure 2.8: The root diagram and Fundamental Weyl Chamber of G_2

a and b). These integers can be encoded in a diagram called *Dynkin diagram* which is constructed in the following way:

1. Draw r points, each corresponding to one of the r simple roots of the algebra (r is the rank of the algebra).

2. Join the point a to the point b by $K_{ab}K_{ba}$ lines. Remember that the number of lines can be 0, 1, 2 or 3.

3. If the number of lines joining the points a and b exceeds 1 put an arrow on the lines directed towards the one whose corresponding simple root has a shorter lenght than the other.

When $K_{ab}K_{ba} = 2$ or 3 the corresponding simple roots, α_a and α_b , have different lenghts. In order to see this, remember that K_{ab} or K_{ba} is equal to -1. Taking $K_{ab} = -1$, we have $K_{ba} = -K_{ab}K_{ba} = -2$ or -3. But

$$\frac{\alpha_a^2}{\alpha_b^2} = \frac{K_{ab}}{K_{ba}} = \frac{1}{K_{ab}K_{ba}} \tag{2.185}$$

and consenquently $\alpha_b^2 \geq \alpha_a^2$. So the number of lines joining two points in a Dynkin diagram gives the ratio of the squared lenghts of the corresponding simple roots.

Example 2.16 *The Cartan matrix of the algebra of $SO(3)$ or $SU(2)$ is simply $K = 2$. It has only one simple root and therefore its Dynkin diagram is just a*

Figure 2.9: The Dynkin diagrams of rank 1 and 2 algebras.

point. The algebra of $SU(3)$ on the other hand has two simple roots. From its Cartan matrix given in example 2.13 and the rules above we see that its Dynkin diagram is formed by two points linked by just one line. Using the rules above one can easily construct the Dynkin diagrams for the algebras discussed in examples 2.11 - 2.15. They are given in figure 2.9.

The root system postulates, given in definition 2.16, impose severe restrictions on the possible Dynkin diagrams. In section 2.15 we will classifly the admissible diagrams, and we will see that there exists only nine types of simple Lie algebras.

We have said that for non simple algebras the Cartan matrix has a block diagonal form. This implies that the corresponding Dynkin diagram is not connect. Therefore a Lie algebra is simple only and if only its Dynkin diagram is connected.

We say a Lie algebra is simply laced if the points of its Dynkin diagram are joined by at most one link. This means all the roots of the algebra have the same length.

2.12 Root strings

We have shown in theorem 2.5 that if α and β are non proportional roots then $\alpha+\beta$ is a root whenever $\alpha.\beta<0$, and $\alpha-\beta$ is a root whenever $\alpha.\beta>0$. We can use this result further to see if $\alpha+m\beta$ or $\alpha-n\beta$ (for m,n integers) are roots. In this way we can obtain a set of roots forming a string. We then come to the concept of the *α-root string through β*. Let p be the largest positive integer for which $\beta+p\alpha$ is a root, and let q be largest positive integer for which $\beta-q\alpha$ is a root. We will show that the set of vectors

$$\beta+p\alpha\ ;\ \beta+(p-1)\alpha\ ;\ ...\ \beta+\alpha\ ;\ \beta\ ;\ \beta-\alpha\ ;\ ...\ \beta-q\alpha \tag{2.186}$$

are all roots. They constitute the α-root string through β.

Suppose that $\beta+p\alpha$ and $\beta-q\alpha$ are roots and that the string is broken, let us say, on the positive side. That is, there exist positive integers r and s with $p>r>s$ such that

1. $\beta+(r+1)\alpha$ is a root but $\beta+r\alpha$ is not a root

2. $\beta+(s+1)\alpha$ is not a root but $\beta+s\alpha$ is a root

According to theorem 2.5, since $\beta+r\alpha$ is not a root then we must have

$$\alpha.\left(\beta+(r+1)\alpha\right)\leq 0 \tag{2.187}$$

For the same reason, since $\beta+(s+1)\alpha$ is not a root we have

$$\alpha.\left(\beta+s\alpha\right)\geq 0 \tag{2.188}$$

Therefore we get that

$$\left((r+1)-s\right)\alpha^2\leq 0 \tag{2.189}$$

and since $\alpha^2>0$

$$s-r\geq 1 \tag{2.190}$$

But this is a contradiction with our assumption that $r>s>0$. So this proves that the string can not be broken on the positive side. The proof that the string is not broken on the negative side is similar.

Notice that the action of the Weyl reflection σ_α on a given root is to add or subtract a multiple of the root α. Since all roots of the form $\beta+n\alpha$ are contained in the α-root string through β, we conclude that this root string is invariant under σ_α. In fact σ_α reverses the α-root string. Clearly the image of $\beta + p\alpha$ under σ_α has to be $\beta - q\alpha$, and vice versa, since they are the roots that are most distant from the hyperplane perpendicular to α. We then have

$$\sigma_\alpha(\beta - q\alpha) = \beta - q\alpha - \frac{2\alpha.(\beta - q\alpha)}{\alpha^2}\alpha = \beta + p\alpha \tag{2.191}$$

and since the only possible values of $\frac{2\alpha.\beta}{\alpha^2}$ are 0, ±1, ±2 and ±3 we get that

$$q - p = \frac{2\alpha.\beta}{\alpha^2} = 0,\ \pm1,\ \pm2,\ \pm3 \tag{2.192}$$

Denoting $\beta - q\alpha$ by γ we see that for the α-root string through γ we have $q = 0$ and therefore the possible values of p are 0, 1, 2 and 3. Consequently the number of roots in any string can not exceed 4.

For a simply laced Lie algebra the only possible values of $\frac{2\alpha.\beta}{\alpha^2}$ are 0 and ±1. Therefore the root strings, in this case, can have at most two roots.

Notice that if α and β are distinct simple roots, we necessarily have $q = 0$, since $\beta - \alpha$ is never a root in this case. So

$$[E_{-\alpha}, E_\beta] = [E_\alpha, E_{-\beta}] = 0 \tag{2.193}$$

If, in addition, $\alpha.\beta = 0$ we get from (2.192) that $p = 0$ and consequently $\alpha + \beta$ is not a root either. For a semisimple Lie algebra, since if α is a root then $-\alpha$ is also a root, it follows that

$$[E_\alpha, E_\beta] = [E_{-\alpha}, E_{-\beta}] = 0 \tag{2.194}$$

for α and β simple roots and $\alpha.\beta = 0$. We can read this result from the Dynkin diagram since, if two points are not linked then the corresponding simple roots are orthogonal.

Example 2.17 *For the algebra of $SU(3)$ we see from the diagram shown in figure 2.6 that the α_1-root string through α_2 contains only two roots namely 2 and 3= 2+1.*

Example 2.18 *From the root diagram shown in figure 2.7 we see that, for the algebra of $SO(5)$, the α_1-root string through α_2 contains thre roots α_2, $\alpha_3 = \alpha_1 + \alpha_2$, and $\alpha_4 = \alpha_2 + 2\alpha_1$.*

Example 2.19 *The algebra G_2 is the only simple Lie algebra which can have root strings with four roots. From the diagram shown in figure 2.8 we see that the α_1-root string through α_2 contains the roots α_2, $\alpha_3 = \alpha_2 + \alpha_1$, $\alpha_5 = \alpha_2 + 2\alpha_1$ and $\alpha_6 = 2\alpha_2 + 3\alpha_1$.*

2.13 Commutation relations from Dynkin diagrams

We now explain how one can obtain from the Dynkin diagram of a Lie algebra, the corresponding root system and then the commutation relations. The fact that this is possible to be done is a demonstration of how powerful the information encoded in the Dynkin diagram is.

We start by introducing the concept of *height of a root* . In theorem 2.7 we have shown that any root can be written as a linear combination of the simple roots with integer coefficients all of the same sign (see eq. (2.172)). The height of a root is the sum of these integer coefficients, i.e.

$$h(\alpha) \equiv \sum_{a=1}^{rank\mathcal{G}} n_a \tag{2.195}$$

where n_a are given by (2.172). The only roots of height one are the simple roots. This definition classifies the roots according to a hierarchy. We can reconstruct the root system of a Lie algebra from its Dynkin diagram starting from the roots of lowest height as we now explain.

Given the Dynkin diagram we can easily construct the Cartan matrix. We know that the diagonal elements are always 2. The off diagonal elements are zero whenever the corresponding points (simple roots) of the diagram are not linked. When they are linked we have K_{ab} (or K_{ba}) equals to -1 and K_{ba} (or K_{ab}) equal to minus the number of links between those points.

Example 2.20 *The Dynkin diagram of $SO(7)$ is given in figure 2.10*

We see that the simple root 3 (according to the rules of section 2.11) has a length smaller than that of the other two. So we have $K_{23} = -2$ and $K_{32} = -1$. Since the roots 1 and 2 have the same length we have $K_{12} = K_{21} = -1$. K_{13} and K_{31} are zero because there are no links between the roots 1 and 3. Therefore

$$K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \tag{2.196}$$

Once the Cartan matrix has been determined from the Dynkin diagram, one obtain all the roots of the algebra from the Cartan matrix. We are interested in semisimple Lie algebras. Therefore, since in such case the roots come in pairs α and $-\alpha$, we have to find just the positive roots. We now give an algorithm for determining the roots of a given height n from those of height $n-1$. The steps are

Figure 2.10: The Dynkin diagram of $so(7)$.

1. The roots of height 1 are just the simple roots.

2. We have seen in (2.194) that if two simple roots are orthogonal then their sum is not a root. On the other hand if they are not orthogonal then their sum is necessarily a root. From theorem 2.6 one has $\alpha.\beta \leq 0$ for α and β simple, and therefore from theorem 2.5 one gets their sum is a root (if they are not orthogonal). Consequently to obtain the roots of height 2 one just look at the Dynkin diagram. The sum of pairs of simple roots which corresponding points are linked, by one or more lines, are roots. These are the only roots of height 2.

3. The procedure to obtain the roots of height 3 or greater is the following: suppose $\alpha^{(l)} = \sum_{a=1}^{rank \mathcal{G}} n_a \alpha_a$ is a root o height l, i.e. $\sum_{a=1}^{rank \mathcal{G}} n_a = l$. Using the Cartan matrix one evaluates
$$\frac{2\alpha^{(l)}.\alpha_b}{\alpha_b^2} = \sum_{a=1}^{rank \mathcal{G}} n_a K_{ab} \tag{2.197}$$
where α_b is a simple root. If this quantity is negative one gets from theorem 2.5 that $\alpha^{(l)} + \alpha_b$ is a root of height $l+1$. If it is zero or positive on uses (2.192) to write
$$p = q - \sum_{a=1}^{rank \mathcal{G}} n_a K_{ab} \tag{2.198}$$
where p and q are the highest positive integers such that $\alpha^{(l)} + p\alpha_b$ and $\alpha^{(l)} - q\alpha_b$ are roots. The integer q can be determined by looking at the set of roots of height smaller than l (which have already been determined) and checking what is the root of smallest height of the form $\alpha^{(l)} - m\alpha_b$. One then finds p from (2.198). If p does not vanish, $\alpha^{(l)} + \alpha_b$ is a root. Notice that if $p \geq 2$ one also determines roots of height greater than $l+1$. By applying this procedure using all simple roots and all roots of height l one determines all roots of height $l+1$.

4. The process finishes when no roots of a given height $l+1$ is found. That is because there can not exists roots of height $l+2$ if there are no roots of height $l+1$.

Therefore we have shown that the root system of a Lie algebra can be determined from its Dynkin diagram. In some cases it is more practical to determine the root system using the Weyl reflections through hyperplanes perpendicular to the simple roots.

The root which has the highest height is said the *highest root* of the algebra and it is generally denoted ψ. For simple Lie algebras the highest root is unique. The integer $h(\psi) + 1 = \sum_{a=1}^{rank\mathcal{G}} m_a + 1$, where $\psi = \sum_{a=1}^{rank\mathcal{G}} m_a \alpha_a$, is said the *Coxeter number* of the algebra.

Example 2.21 *In example 2.20 we have determined the Cartan matrix of* $SO(7)$ *from its Dynkin diagram. We now determine its root system following the procedure described above. The dimension of* $SO(7)$ *is* 21 *and its rank is* 3*. So, the number of positive roots is* 9*. The first three are the simple roots* α_1 *,* α_2 *and* α_3 *. Looking at the Dynkin diagram in figure 2.10 we see that* $\alpha_1 + \alpha_2$ *and* $\alpha_2 + \alpha_3$ *are the only roots of height* 2*, since* α_1 *and* α_3 *are orthogonal. We have* $\frac{2(\alpha_1+\alpha_2).\alpha_a}{\alpha_a^2} = K_{1a} + K_{2a}$ *which, from (2.196), is equal to* 1 *for* $a = 1, 2$ *and* -2 *for* $a = 3$*. Therefore, from (2.198), we get that* $2\alpha_1 + \alpha_2$ *and* $\alpha_1 + 2\alpha_2$ *are not roots but* $\alpha_1 + \alpha_2 + \alpha_3$ *and* $\alpha_1 + \alpha_2 + 2\alpha_3$ *are roots. Analogously we have* $\frac{2(\alpha_2+\alpha_3).\alpha_a}{\alpha_a^2} = K_{2a} + K_{3a}$ *which is equal to* -1 *for* $a = 1$*,* 1 *for* $a = 2$ *and* 0 *for* $a = 3$*. Therefore the only new root we obtain is* $\alpha_2 + 2\alpha_3$*. This exhausts the roots of height* 3*. One can check that the only root of height* 4 *is* $\alpha_1 + \alpha_2 + 2\alpha_3$ *which we have obtained before. Now* $\frac{2(\alpha_1+\alpha_2+2\alpha_3).\alpha_a}{\alpha_a^2} = K_{1a} + K_{2a} + 2K_{3a}$ *which is equal to* 1*,* -1 *and* 2 *for* $a = 1, 2, 3$ *respectively. Since it is negative for* $a = 2$ *we get that* $\alpha_1 + 2\alpha_2 + 2\alpha_3$ *is a root. This is the only root of height* 5*, and it is in fact the highest root of* $SO(7)$*. So the Coxeter number of* $SO(7)$ *is* 6*. Summarizing we have that the positive roots of* $SO(7)$ *are*

roots of height 1	α_1; α_2; α_3
roots of height 2	$(\alpha_1 + \alpha_2)$; $(\alpha_2 + \alpha_3)$
roots of height 3	$(\alpha_1 + \alpha_2 + \alpha_3)$; $(\alpha_2 + 2\alpha_3)$
roots of height 4	$(\alpha_1 + \alpha_2 + 2\alpha_3)$
roots of height 5	$(\alpha_1 + 2\alpha_2 + 2\alpha_3)$

These could also be determined starting from the simple roots and using Weyl reflections.

We now show how to determine the commutation relations from the root system of the algebra. We have been using the Cartan-Weyl basis introduced in (2.139). However the commutation relations take a simpler form in the so called *Chevalley basis* . In this basis the Cartan subalgebra generators are given by

$$H_a \equiv \frac{2\alpha_a.H}{\alpha_a^2} \tag{2.199}$$

where α_a ($a = 1, 2, ...$ rank $\mathcal{G}$) are the simple roots and $\alpha_a.H = \alpha_a^i H^i$, where H_i are the Cartan subalgebra generators in the Cartan-Weyl basis and α_a^i are the components of the simple root α_a in that basis, i.e. $[H_i, E_{\alpha_a}] = \alpha_a^i E_{\alpha_a}$. The generators H_a are not orthonormal like the H_i . From (2.139) and (2.175) we have that

$$Tr(H_a H_b) = \frac{4\alpha_a.\alpha_b}{\alpha_a^2 \alpha_b^2} = \frac{2}{\alpha_a^2} K_{ab} \tag{2.200}$$

The generators H_a obviously commute among themselves

$$[H_a, H_b] = 0 \tag{2.201}$$

The commutation relations between H_a and step operators are given by (see (2.129))

$$[H_a, E_\alpha] = \frac{2\alpha.\alpha_a}{\alpha_a^2} E_\alpha = K_{\alpha a} E_\alpha \tag{2.202}$$

where we have defined $K_{\alpha a} \equiv \frac{2\alpha.\alpha_a}{\alpha_a^2}$. Since α can be written as in (2.172) we see that $K_{\alpha a}$ is a linear combination with integer coefficients, all of the same sign, of the a-columm of the Cartan matrix

$$K_{\alpha a} = \frac{2\alpha.\alpha_a}{\alpha_a^2} = \sum_{b=1}^{rank\mathcal{G}} n_b K_{ba} \tag{2.203}$$

where $\alpha = \sum_{b=1}^{rank\mathcal{G}} n_b \alpha_b$. Notice that the factor multiplying E_α on the r.h.s of (2.202) is an integer. In fact this is a property of the Chevalley basis. All the structure constants of the algebra in this basis are integer numbers. The commutation relations (2.202) are determined once one knows the root system of the algebra.

We now consider the commutation relations between step operators. From (2.130)

$$[E_\alpha, E_\beta] \;=\; \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ H_\alpha = m_a H_a & \text{if } \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.204}$$

where m_a are integers in the expansion $\frac{\alpha}{\alpha^2} = \sum_{a=1}^{rank\mathcal{G}} m_a \frac{\alpha_a}{\alpha_a^2}$. The structure constants $N_{\alpha\beta}$, in the Chevalley basis, are integers and can be determined from the root system of the algebra and also from the Jacobi identity . Let us explain now how to do that.

Notice that from the antisymmetry of the Lie bracket

$$N_{\alpha\beta} = -N_{\beta\alpha} \tag{2.205}$$

for any pair of roots α and β. The structure constants $N_{\alpha\beta}$ are defined up to rescaling of the step operators. If we make the transformation

$$E_\alpha \rightarrow \rho_\alpha E_\alpha \tag{2.206}$$

keeping the Cartan subalgebra generators unchanged, then from (2.204) the structure constants $N_{\alpha\beta}$ must transform as

$$N_{\alpha\beta} \rightarrow \frac{\rho_\alpha \rho_\beta}{\rho_{\alpha+\beta}} N_{\alpha\beta} \tag{2.207}$$

and

$$\rho_\alpha \rho{-}\alpha = 1 \tag{2.208}$$

As we have said in section 2.9, any symmetry of the root diagram can be elevated to an automorphism of the corresponding Lie algebra. In any semisimple Lie algebra the transformation $\alpha \rightarrow -\alpha$ is a symmetry of the root diagram since if α is a root so is $-\alpha$. We then define the transformation $\sigma : \mathcal{G} \rightarrow \mathcal{G}$ as

$$\sigma(H_a) = -H_a \; ; \quad \sigma(E_\alpha) = \eta_\alpha E_{-\alpha} \tag{2.209}$$

and $\sigma^2 = 1$. From the commutation relations (2.201), (2.202) and (2.204) one sees that such transformation is an automorphism if

$$\begin{aligned} \eta_\alpha \eta_{-\alpha} &= 1 \\ N_{\alpha\beta} &= \frac{\eta_\alpha \eta_\beta}{\eta_{\alpha+\beta}} N_{-\alpha,-\beta} \end{aligned} \tag{2.210}$$

Using the freedom to rescale the step operators as in (2.207) one sees that it is possible to satisfy (2.210) and make (2.209) an automorphism. In particular it is possible to choose all η_α equals to -1 and therefore

$$N_{\alpha\beta} = -N_{-\alpha,-\beta} \tag{2.211}$$

Consider the α-root string through β given by (2.186). Using the Jacobi identity for the step operators E_α, $E_{-\alpha}$ and $E_{\beta+n\alpha}$, where $p > n > 1$ and p is the highest integer such that $\beta + p\alpha$ is a root, we obtain from (2.204) that

$$N_{\beta+n\alpha,-\alpha} N_{\beta+(n-1)\alpha,\alpha} - N_{\beta+n\alpha,\alpha} N_{\beta+(n+1)\alpha,-\alpha} = \frac{2\alpha.(\beta+n\alpha)}{\alpha^2} \tag{2.212}$$

Notice that the second term on the l.h.s of this equation vanishes when $n = p$, since $\beta + (p+1)\alpha$ is not a root. Adding up the equations (2.212) for n taking the values 1, 2, ... p , we obtain that

$$\begin{aligned} N_{\beta+\alpha,-\alpha} N_{\beta\alpha} &= \frac{2\alpha.\beta}{\alpha^2} p + 2\left(p + (p-1) + (p-2) + \ldots + 1\right) \\ &= p(q+1) \end{aligned} \tag{2.213}$$

where we have used (2.192).

From the fact that the Killing form is invariant under the adjoint representation (see (2.48) it follows that it is invariant under inner automorphisms, i.e. $Tr(\sigma(T)\sigma(T')) = Tr(TT')$ with $\sigma(T) = gTg^{-1}$. However one can show that the Killing form is invariant any automorphism (inner or outer). Using this fact for the automorphism (2.209) (with $\eta_\alpha = -1$), the invariance property (2.46) and the normalization (2.139) one gets

$$\begin{aligned} Tr([E_\alpha, E_\beta]E_{-\alpha-\beta}) &= N_{\alpha\beta}\frac{2}{(\alpha+\beta)^2} \\ &= -Tr([E_{-\alpha}, E_{-\beta}]E_{\alpha+\beta}) \\ &= -Tr([E_{\alpha+\beta}, E_{-\alpha}]E_{-\beta}) \\ &= -N_{\alpha+\beta,-\alpha}\frac{2}{\beta^2} \end{aligned} \tag{2.214}$$

Consequently

$$N_{\alpha+\beta,-\alpha} = -\frac{\beta^2}{(\alpha+\beta)^2} N_{\alpha\beta} \tag{2.215}$$

Substituting this into (2.213) we get

$$N_{\alpha\beta}^2 = \frac{(\alpha+\beta)^2}{\beta^2} p(q+1) \tag{2.216}$$

Therefore, up to a sign, the structure constants $N_{\alpha\beta}$ defined in (2.204) can be determined from the root system of the algebra.

Using the Jacobi identity for the step operators E_α, E_α and $E_{\beta-n\alpha}$, with n varying from 1 to q where q is the highest integer such that $\beta - q\alpha$ is a root, and doing similar calculations we obtain that

$$N_{\beta,-\alpha}^2 = \frac{(\beta-\alpha)^2}{\beta^2} q(p+1) \tag{2.217}$$

The relation (2.216) can be put in a simpler form. From (2.192) we have that (see section 25.1 of [HUM 72])

$$\begin{aligned}(q+1) - p\frac{(\alpha+\beta)^2}{\beta^2} &= p + \frac{2\alpha.\beta}{\alpha^2} + 1 - p\frac{(\alpha+\beta)^2}{\beta^2} \\ &= \frac{2\alpha.\beta}{\alpha^2} + 1 - p\frac{\alpha^2}{\beta^2} - p\frac{2\alpha.\beta}{\beta^2} \\ &= \left(\frac{2\alpha.\beta}{\alpha^2} + 1\right)\left(1 - p\frac{\alpha^2}{\beta^2}\right) \end{aligned} \tag{2.218}$$

We want to show the r.h.s of this relation is zero. We distinguish two cases:

1. In the case where $\alpha^2 \geq \beta^2$ we have $\mid \frac{2\alpha.\beta}{\alpha^2} \mid \leq \mid \frac{2\alpha.\beta}{\beta^2} \mid$. From table 2.2 we see that the possible values of $\frac{2\alpha.\beta}{\alpha^2}$ are -1, 0 or 1. In the first case we get that the first factor on the r.h.s of (2.218) vanishes. On the other two cases we have that $\alpha.\beta \geq 0$ and then $(\alpha+\beta)^2$ is strictly larger than both, α^2 and β^2 . Since we are assuming $\alpha+\beta$ is a root and since, as we have said at the end of section 2.8, there can be no more than two different root lengths in each component of a root system, we conclude that $\alpha^2 = \beta^2$. For the same reason $\beta + 2\alpha$ can not be a root since $(\beta+2\alpha)^2 > (\beta+\alpha)^2$ and therefore $p = 1$. But this implies that the second factor on the r.h.s of (2.218) vanishes.

2. For the case of $\alpha^2 < \beta^2$ we have that $(\alpha+\beta)^2 = \alpha^2$ or β^2, since otherwise we would have three different root lengths. This forces $\alpha.\beta$ to be strictly negative. Therefore we have $(\beta-\alpha)^2 > \beta^2 > \alpha^2$ and consequently $\beta-\alpha$ is not a root and so $q = 0$. But $\mid \frac{2\alpha.\beta}{\beta^2} \mid < \mid \frac{2\alpha.\beta}{\alpha^2} \mid$ and therefore $\frac{2\alpha.\beta}{\beta^2} = -1, 0$ or 1. Since $\alpha.\beta < 0$ we have $\frac{2\alpha.\beta}{\beta^2} = -1$. Then from (2.192) we have $p = -\frac{2\alpha.\beta}{\beta^2}\frac{\beta^2}{2\alpha.\beta} = \frac{\beta^2}{\alpha^2}$. Therefore the second factor on the r.h.s of (2.218) vanishes.

Then, we have shown that

$$q + 1 = p\frac{(\alpha+\beta)^2}{\beta^2} \tag{2.219}$$

and from (2.216)

$$N_{\alpha\beta}^2 = (q+1)^2 \tag{2.220}$$

This shows that the structure constants $N_{\alpha\beta}$ are integer numbers. From (2.201), (2.202) and (2.204) we see that all structure contants in the Chevalley basis are integers. Summarizing we have

$$[H_a, H_b] = 0 \tag{2.221}$$

$$[H_a, E_\alpha] = \frac{2\alpha.\alpha_a}{\alpha_a^2} E_\alpha = K_{\alpha a} E_\alpha \tag{2.222}$$

$$[E_\alpha, E_\beta] = \begin{cases} (q+1)\varepsilon(\alpha,\beta)E_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ H_\alpha = \frac{2\alpha.H}{\alpha^2} = m_a H_a & \text{if } \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.223}$$

where we have denoted $\varepsilon(\alpha, \beta)$ the sign of the structure constant $N_{\alpha\beta}$, i.e. $N_{\alpha\beta} = (q+1)\varepsilon(\alpha, \beta)$. These signs, also called cocycles, are determined through the Jacobi identity as explained in section 2.14. As we have said before q is the highest positive integer such that $\beta - q\alpha$ is a root. However when $\alpha + \beta$ is a root, which is the case we are interested in (2.223), it is true that q is also the highest positive integer such that $\alpha - q\beta$ is a root. The reason is the following: in a semisimple Lie algebra the roots always appear in pairs (α and $-\alpha$). Therefore if $\beta - \alpha$ is a root so is $\alpha - \beta$. In addition we have seen in section 2.12 that the root strings are unbroken and they can have at most four roots. Therefore, since we are assuming that $\alpha + \beta$ is a root, the only possible way of not satisfying what we said before is to have, let us say, the α-root string through β as $\beta - 2\alpha$, $\beta - \alpha$, β, $\beta + \alpha$; and the β-root string through α as $\alpha - \beta$, α, $\alpha + \beta$ or $\alpha - \beta$, α, $\alpha + \beta$, $\alpha + 2\beta$. But from (2.192) we have

$$\frac{2\alpha.\beta}{\alpha^2} = 1 \tag{2.224}$$

and

$$\frac{2\alpha.\beta}{\beta^2} = 0 \;\; or \;\; -1 \tag{2.225}$$

which are clearly incompatible.

We have said in section 2.12 that for a simply laced Lie algebra there can be at most two roots in a root string. Therefore if $\alpha + \beta$ is a root $\alpha - \beta$ is not, and therefore $q = 0$. Consequently the structure constants $N_{\alpha\beta}$ are always ± 1 for a simply laced algebra.

2.14 Finding the cocycles $\varepsilon(\alpha, \beta)$

As we have seen the Dynkin diagram of an algebra contains all the necessary information to construct the commutation relations (2.221)-(2.223). However that information is not enough to determine the cocycles $\varepsilon(\alpha, \beta)$ defined in (2.223). For that we need the Jacobi identity. We now explain how to use such identities to determine the cocycles. We will show that the consistency conditions imposed on the cocycles are such that they can be split into a number of sets equal to the number of positive non simple roots. The sign of a cocycle in a given set completly determines the signs of all other cocycles of that set, but has no influence in the determination of the cocycles in the other sets. Therefore the cocycles $\varepsilon(\alpha, \beta)$ are determined by the Jacobi identities up to such "gauge freedom" in fixing independently the signs of the cocycles of different sets.

From the antisymmetry of the Lie bracket the cocycles have to satisfy

$$\varepsilon(\alpha, \beta) = -\varepsilon(\beta, \alpha) \tag{2.226}$$

In addition, from the choice made in (2.211) one has

$$\varepsilon(\alpha, \beta) = -\varepsilon(-\alpha, -\beta) \tag{2.227}$$

Consider three roots α, β and γ such that their sum vanishes. The Jacobi identity for their corresponding step operators yields, using (2.221) - (2.223)

$$\begin{aligned} 0 &= [[E_\alpha, E_\beta], E_\gamma] + [[E_\gamma, E_\alpha], E_\beta] + [[E_\beta, E_\gamma], E_\alpha] \\ &= -((q_{\alpha\beta}+1)\varepsilon(\alpha, \beta)\frac{2\gamma.H}{\gamma^2} + (q_{\gamma\alpha}+1)\varepsilon(\gamma, \alpha)\frac{2\beta.H}{\beta^2} \\ &\quad +(q_{\beta\gamma}+1)\varepsilon(\beta, \gamma)\frac{2\alpha.H}{\alpha^2}) \\ &= -(((q_{\beta\gamma}+1)\varepsilon(\beta, \gamma) - \frac{\alpha^2}{\gamma^2}(q_{\alpha\beta}+1)\varepsilon(\alpha, \beta))\frac{2\alpha.H}{\alpha^2} \\ &\quad +((q_{\gamma\alpha}+1)\varepsilon(\gamma, \alpha) - \frac{\beta^2}{\gamma^2}(q_{\alpha\beta}+1)\varepsilon(\alpha, \beta))\frac{2\beta.H}{\beta^2}) \end{aligned} \tag{2.228}$$

Since the integers $q's$ are non negative we get

$$\varepsilon(\alpha, \beta) = \varepsilon(\beta, \gamma) = \varepsilon(\gamma, \alpha) \tag{2.229}$$

and also

$$\frac{1}{\gamma^2}(q_{\alpha\beta}+1) = \frac{1}{\alpha^2}(q_{\beta\gamma}+1) = \frac{1}{\beta^2}(q_{\gamma\alpha}+1) \tag{2.230}$$

Further relations are found by considering Jacobi identities for three step operators corresponding to roots adding up to a fourth root. Now such identities yield relations involving products of two cocycles. However, in many situations there are only two non vanishing terms in the Jacobi identity. Consider three roots α, β and γ such that $\alpha+\beta$, $\beta+\gamma$ and $\alpha+\beta+\gamma$ are roots but $\alpha+\gamma$ is not a root. Then the Jacobi identity for the corresponding step operators yields

$$\begin{aligned} 0 &= [[E_\alpha, E_\beta], E_\gamma] + [[E_\gamma, E_\alpha], E_\beta] + [[E_\beta, E_\gamma], E_\alpha] \\ &= (q_{\alpha\beta}+1)(q_{\alpha+\beta,\gamma}+1)\varepsilon(\alpha,\beta)\varepsilon(\alpha+\beta,\gamma) \\ &\quad +(q_{\beta\gamma}+1)(q_{\beta+\gamma,\alpha}+1)\varepsilon(\beta,\gamma)\varepsilon(\beta+\gamma,\alpha) \end{aligned} \tag{2.231}$$

Therefore one gets

$$\varepsilon(\alpha,\beta)\varepsilon(\alpha+\beta,\gamma) = \varepsilon(\beta,\gamma)\varepsilon(\alpha,\beta+\gamma) \tag{2.232}$$

and

$$(q_{\alpha\beta}+1)(q_{\alpha+\beta,\gamma}+1) = (q_{\beta\gamma}+1)(q_{\beta+\gamma,\alpha}+1) \tag{2.233}$$

There remains to consider the cases where the three terms in the Jacobi identity for three step operators do not vanish. Such thing happens when we have three roots α, β and γ such that $\alpha+\beta$, $\alpha+\gamma$, $\beta+\gamma$ and $\alpha+\beta+\gamma$ are roots as well. We now classify all cases where that happens. We shall denote long roots by μ, ν, ρ, ... and short roots by e, f, g, ... From the properties of roots discussed in section 2.8 one gets that $\frac{2\mu.\nu}{\mu^2}$, $\frac{2\mu.e}{\mu^2}$, $\frac{2e.f}{e^2} = 0, \pm 1$. Let us consider the possible cases:

1. *All three roots are long.* If $\mu+\nu$ is a root then $\frac{(\mu+\nu)^2}{\mu^2} = 2 + \frac{2\mu.\nu}{\mu^2}$. Since $\mu+\nu$ can not be a longer than μ one gets $\frac{2\mu.\nu}{\mu^2} = -1$. So $\mu+\nu$ is a long root and if $\mu+\nu+\rho$ is also a root one gets by the same argument that $\frac{2(\mu+\nu).\rho}{\mu^2} = -1$. Therefore $\mu+\rho$ and $\nu+\rho$ can not be roots simultaneously since that would imply, by the same arguments, $\frac{2\mu.\rho}{\mu^2} = \frac{2\nu.\rho}{\mu^2} = -1$ which is a contradiction with the result above.

2. *Two roots are long and one short.* If $\mu+e$ is a root then $\frac{(\mu+e)^2}{\mu^2} = 1 + \frac{e^2}{\mu^2} + \frac{2\mu.e}{\mu^2}$. Since $\mu+e$ can not be longer than μ it follows that $\frac{2\mu.e}{\mu^2} = -1$. Therefore $\mu+e$ is a short root since $(\mu+e)^2 = e^2$. So, if $\mu+e+\nu$ is a root then $\frac{(\mu+e+\nu)^2}{\nu^2} = 1 + \frac{(\mu+e)^2}{\mu^2} + \frac{2(\mu+e).\nu}{\nu^2}$ and therefore $\frac{2(\mu+e).\nu}{\nu^2} = -1$. Consequently $\mu+\nu$ and $\nu+e$ can not be roots simultaneously since that would imply, by the same arguments, $\frac{2\mu.\nu}{\nu^2} = \frac{2\nu.e}{\nu^2} = -1$.

3. *Two roots are short and one long.* Analogously if $e+f$ and $\mu+e+f$ are roots one gets $\frac{2(e+f).\mu}{\mu^2}=-1$ independently of $e+f$ being shost or long. So, it is impossible for $\mu+e$ and $\mu+f$ to be both roots since one would get $\frac{2\mu.e}{\mu^2}=\frac{2\mu.f}{\mu^2}=-1$.

4. *All three roots are short.* If $e+f$ is a root then $\frac{(e+f)^2}{e^2}=2+\frac{2e.f}{e^2}$ and there exists three possibilities:

   (a) $\frac{2e.f}{e^2}=-1$ and $e+f$ is a short root.

   (b) $\frac{2e.f}{e^2}=1$ and $\frac{(e+f)^2}{e^2}=3$ (can only happen in G_2).

   (c) $\frac{2e.f}{e^2}=0$ and $\frac{(e+f)^2}{e^2}=2$ (can only happen in B_n, C_n and F_4).

In section 2.8 we have seen that the possible ratios of squared length of the roots are 1, 2 and 3. Therefore there can not exists roots with three different lengths in the same irreducible root system since if $\frac{\alpha^2}{\beta^2}=2$ and $\frac{\gamma^2}{\beta^2}=3$ then $\frac{\gamma^2}{\alpha^2}=\frac{3}{2}$.

Consider the case 4.b and let g be the third short root. Then if $e+g$ is a root we have $\frac{(e+g)^2}{(e+f)^2}=\frac{2}{3}+\frac{2e.g}{(e+f)^2}=1$ or $\frac{1}{3}$. But this is impossible since $\frac{2e.g}{(e+f)^2}$ would not be an integer. So the second case is ruled out since we would not have $e+f$, $e+g$, $f+g$ and $e+f+g$ all roots.

Consider the case 4.c. If $e+g$ is a root then $\frac{(e+g)^2}{(e+f)^2}=1+\frac{1}{2}\frac{2e.g}{g^2}=1$ or $\frac{1}{2}$. Therefore $\frac{2e.g}{g^2}=0$ or -1. Similarly if $f+g$ is a root we get $\frac{2f.g}{g^2}=0$ or -1. But if $e+f+g$ is also a root then it has to be a short root since $\frac{(e+f+g)^2}{(e+f)^2}=\frac{3}{2}+\frac{2(e+f).g}{(e+f)^2}$. Consequently $\frac{2(e+f).g}{(e+f)^2}=-1$ and $\frac{(e+f+g)^2}{(e+f)^2}=\frac{1}{2}$. It then follows that $\frac{2e.g}{g^2}+\frac{2f.g}{g^2}=\frac{2(e+f).g}{(e+f)^2}\frac{(e+f)^2}{g^2}=-2$. Therefore in the case 4.c we can have $e+f$, $e+g$, $f+g$ and $e+f+g$ all roots if $e.f=0$, $\frac{2e.g}{g^2}=\frac{2f.g}{g^2}=-1$.

Consider the case 4.a. Again if $e+g$ is a root then $\frac{(e+g)^2}{g^2}=2+\frac{2e.g}{g^2}=1$ or 2. So, $\frac{2e.g}{g^2}=0$ or -1. Similarly if $f+g$ is a root $\frac{2f.g}{g^2}=0$ or -1. If $e+f+g$ is also a root then $\frac{(e+f+g)^2}{g^2}=2+\frac{2(e+f).g}{g^2}=1$ or 2. Therefore $\frac{2(e+f).g}{g^2}=0$ or -1. Consequently $\frac{2e.g}{g^2}$ and $\frac{2f.g}{g^2}$ can not be both -1. Suppose then $\frac{2e.g}{g^2}=0$ and consequently $e+g$ is a long root, i.e. $\frac{(e+g)^2}{g^2}=2$. According to the arguments used in case 4.c we get $e+f+g$ is a short root and then $\frac{2f.g}{g^2}=-1$.

We then conclude that the only possibility for the ocurrence of three short roots e, f and g such that the sum of any two of them and $e+f+g$ are all roots is that two of them are ortoghonal, let us say $e.f=0$ and $\frac{2e.g}{g^2}=\frac{2f.g}{g^2}=-1$. This can only happen in the algebras C_n or F_4. Therefore none of the three terms in the Jacobi identity for the corresponding step operators will vanish. We have

$$\begin{aligned} 0 &= [[E_e, E_f], E_g] + [[E_g, E_e], E_f] + [[E_f, E_g], E_e] \\ &= (q_{ef} + 1)(q_{e+f,g} + 1)\varepsilon(e, f)\varepsilon(e + f, g) \\ &\quad +(q_{ge} + 1)(q_{g+e,f} + 1)\varepsilon(g, e)\varepsilon(g + e, f) \\ &\quad +(q_{fg} + 1)(q_{f+g,e} + 1)\varepsilon(f, g)\varepsilon(f + g, e) \end{aligned} \tag{2.234}$$

According to the discussion in section 2.12 any root string in an algebra where the ratio of the squared lengths of roots is 1 or 2 can have at most 3 roots. From (2.192) we see that $q_{ef} = 1$ and $q_{ge} = q_{fg} = q_{e+f,g} = q_{g+e,f} = q_{f+g,e} = 0$. Therefore

$$\varepsilon(e, f)\varepsilon(e + f, g) = \varepsilon(g, e)\varepsilon(f, g + e) = \varepsilon(f, g)\varepsilon(e, f + g) \tag{2.235}$$

We can then determine the cocycles using the following algorithm:

1. The cocycles involving two negative roots, $\varepsilon(-\alpha, -\beta)$ with α and β both positive, is determined from those involving two positive roots through the relation (2.227).

2. The cocycles involving one positive and one negative root, $\varepsilon(-\alpha, \beta)$ with both α and β both positive, are also determined from those involving two positive roots through the relations (2.229) and (2.227). Indeed, if $-\alpha + \beta$ is a positive root we write $-\alpha + \beta = \gamma$ and if it is negative we write $-\alpha+\beta = -\gamma$ with γ positive in both cases. Therefore from (2.229) and (2.227) it follows $\varepsilon(-\alpha, \beta) = \varepsilon(-\gamma, -\alpha) = -\varepsilon(\gamma, \alpha)$ in the first case, and $\varepsilon(-\alpha, \beta) = \varepsilon(\beta, \gamma)$ in the second case.

3. Let ρ be a positive non simple root which can be written as $\rho = \alpha + \beta = \gamma + \delta$ with α, β, γ and δ all positive roots. Then the cocycles $\varepsilon(\alpha, \beta)$ and $\varepsilon(\gamma, \delta)$ can be related to each other by using combinations of the relations (2.232)

Using such algorithm one can then verify that there will be one cocycle to be chosen freely, for each positive non-simple root of the algebra. Once those cocycles are chosen, all the other are uniquely determined.

2.15 The classification of simple Lie algebras

The simple Lie algebras are, as we have seen, the building blocks for constructing all Lie algebras and therefore the classification of those is very important. We have also seen that there exists, up to isomorphism, only one Lie algebra associoated to a given Dynkin diagram. Since the Dynkin diagram for a simple Lie algebra is necessarily connected, we see that the classification of the simple algebras is equivalent to the classification of possible connected Dynkin diagrams. We now give such classification.

We will firstly look for the possible Dynkin diagrams ignoring the arrows on them. We then define unit vectors in the direction of the simple roots as

$$\epsilon_a = \frac{\alpha_a}{\sqrt{\alpha_a^2}} \tag{2.236}$$

Therefore each point of the diagram will be associated to a unit vector ϵ_a, and these are all linearly independent. They satisfy

$$2\epsilon_a \cdot \epsilon_b = \frac{2\alpha_a \cdot \alpha_b}{\sqrt{\alpha_a^2 \alpha_b^2}} = -\sqrt{K_{ab}K_{ba}} \tag{2.237}$$

Now, from theorem 2.6 we have that $\epsilon_a \cdot \epsilon_b \leq 0$, and therefore from (2.179)

$$2\epsilon_a \cdot \epsilon_b = 0, -1, -\sqrt{2}, -\sqrt{3} \tag{2.238}$$

which correspond to minus the square root of the number of lines joining the points a and b. We shall call a set of unit vectors satisfying (2.238) an admissible set.

One notices that by ommiting some ϵ_a's, the remaining ones form an admissible set, which diagram is obtained from the original one by ommiting the corresponding points and all lines attached to them. So we have the obvious lemma.

Lemma 2.2 *Any subdiagram of an admissible diagram is an admissible diagram.*

Lemma 2.3 *The number of pairs of vertices in a Dynkin diagram linked by at least one line is strictly less than r, the rank of the algebra (or number of vertices).*

Proof: Consider the vector

$$\epsilon = \sum_{a=1}^{r} \epsilon_a \tag{2.239}$$

Since the vectors ϵ_a's are linearly independent we have $\epsilon \neq 0$ and then

$$0 < \epsilon^2 = r + 2 \sum_{\text{pairs}} \epsilon_a \cdot \epsilon_b \tag{2.240}$$

And from (2.238) we see that if a and b are linked, then $2\epsilon_a \cdot \epsilon_b \leq -1$. In order to keep the inequality we see that the number of linked pairs of points must be smaller or equal to $r - 1$. □

Corollary 2.1 *There are no loops in a Dynkin diagram.*

Proof: If a diagram has a loop we see from lemma 2.2 that the loop itself would be an admissible diagram. But that would violate lemma 2.3 since the number o linked pairs of vertices is equal to the number of vertices. □

Lemma 2.4 *The number of lines attached to a given vertice can not exceed three.*

Proof: Let η be a unit vector corresponding to a vertex and let ϵ_1, ϵ_2, $\ldots \epsilon_k$ be the set of unit vectors which correspond to the vertices linked to it. Since the diagram has no loops we must have

$$\epsilon_a \cdot \epsilon_b = 0 \qquad\qquad a, b = 1, 2, 3, \ldots k \tag{2.241}$$

So we can write

$$\eta = \sum_{a=1}^{k} (\eta \cdot \epsilon_a)\, \epsilon_a + (\eta \cdot \epsilon_0)\, \epsilon_0 \tag{2.242}$$

where ϵ_0 is a unit vector in a subspace perpendicular to the set ϵ_1, ϵ_2, $\ldots \epsilon_k$. Then

$$\eta^2 = 1 = \sum_{a=1}^{k} (\eta \cdot \epsilon_a)^2 + (\eta \cdot \epsilon_0)^2 \tag{2.243}$$

But the number of lines linked to η is (see (2.237) and (2.238))

$$4 \sum_{a=1}^{k} (\eta \cdot \epsilon_a)^2 = 4 - 4\, (\eta \cdot \epsilon_0)^2 \leq 4 \tag{2.244}$$

Figure 2.11: Possible links a vertex can have.

Figure 2.12: The only connected diagram with triple link.

The equality is only possible if $\eta \cdot \epsilon_0 = 0$. But that is impossible since it means η is a linear combination of ϵ_1, ϵ_2, $\ldots \epsilon_k$. Therefore, the number of lines linked to η is strictly less than 4 and the lemma is proved. □

Consequently we see that the possible links a vertex can have are shown in figure 2.11 and then it follows the corollary 2.2.

Corollary 2.2 *The only connected diagram which has a triple link is the one shown in figure 2.12 and it corresponds to the exceptional Lie algebra G_2.*

Corollary 2.3 *If an admissible diagram D has a subdiagram Γ given in figure 2.13, then the diagram D' obtained from D by the contraction of the Γ is also an admissible diagram. By constraction we mean the reduction of Γ to the point*

$$\epsilon = \sum_{a=l}^{a+k} \epsilon_a \tag{2.245}$$

which corresponds to a new simple root $\alpha = \sum_{a=l}^{a+k} \alpha_a$. Therefore, the simple roots of D' are α together with the simple roots of D which do not correspond to ϵ_a, ϵ_{a+1}, $\ldots \epsilon_{a+k}$.

Proof: We have to show that D' is an admissible diagram. The vector ϵ, defined in (2.245), together with the remaining ϵ_a's in D are linearly indepen-

Figure 2.13: Diagram Γ.

dent. ϵ has unit length since

$$\epsilon^2 = k + 2 \sum_{\text{pairs}} \epsilon_a \cdot \epsilon_b \tag{2.246}$$

But since $2\epsilon_a \cdot \epsilon_b = -1$, for a amd b being nearest neighbours, we have

$$\epsilon^2 = k + (k-1)(-1) = 1 \tag{2.247}$$

Any η belonging to $D - \Gamma$ can be linked at most to one of the points of Γ. Otherwise we would have a loop. Therefore, either

$$\eta \cdot \epsilon = \eta \cdot \epsilon_a \qquad \text{for a given } \epsilon_a \text{ in } \Gamma \tag{2.248}$$

or

$$\eta \cdot \epsilon = 0 \tag{2.249}$$

But since η and ϵ_a belong to an admissible diagram we have that they satisfy (2.238). Therefore, η and ϵ also satisfy (2.238) and consequently D' is an admissible diagram.

Corollary 2.4 *Any admissible diagram can not have subdiagrams of the form shown in figure 2.14.*

The reason is that by lemma 2.3 we would obtain that the diagrams shown in figure 2.15 are subdiagrams of admissible diagrams. From lemmas 2.2 and 2.4 we see that this is impossible.

So, from the results obtained so far we see that an admissible diagram has to have one of the forms shown in figure 2.16.

Consider the diagram B) of figure 2.16, and define the vectors

$$\epsilon = \sum_{a=1}^{p} a\epsilon_a \qquad ; \qquad \eta = \sum_{a=1}^{q} a\epsilon_a \tag{2.250}$$

Figure 2.14: Non-admissible subdiagrams.

Figure 2.15:

Figure 2.16:

Therefore

$$\begin{aligned}\epsilon^2 &= \sum_{a=1}^{p} a^2 + 2\sum_{\text{pairs}} ab\,\epsilon_a\cdot\epsilon_b \\ &= \sum_{a=1}^{p} a^2 - \sum_{a=1}^{p-1} a\,(a+1) \\ &= p^2 - \sum_{a=1}^{p-1} a = p^2 - p\,(p-1)\,/2 \\ &= p\,(p+1)\,/2 \end{aligned} \tag{2.251}$$

where we have used the fact that $2\epsilon_a\cdot\epsilon_b = -1$ for a and b being nearest neighbours and $2\epsilon_a\cdot\epsilon_b = 0$ otherwise. In a similar way we obtain that

$$\eta^2 = q\,(q+1)\,/2 \tag{2.252}$$

Since the points p and$q$ are linked by a double line we have

$$2\epsilon_p\cdot\eta_q = -\sqrt{2} \tag{2.253}$$

and so

$$\epsilon\cdot\eta = pq\;\epsilon_p\cdot\eta_q = -pq/\sqrt{2} \tag{2.254}$$

Using Schwartz inequality

$$(\epsilon\cdot\eta)^2 \leq \epsilon^2\eta^2 \tag{2.255}$$

we have from (2.251), (2.252) and (2.254) that

$$p^2q^2 < p\,(p+1)\;\;q\,(q+1)\,/2 \tag{2.256}$$

Since the equality can not hold because ϵ and η are linearly independent, eq. (2.256) can be written as

$$(p-1)\,(q-1) < 2 \tag{2.257}$$

There are three possibilities for $p, q \geq 1$, namely

1. $p = q = 2$
2. $p = 1$ and q any positive integer
3. $q = 1$ and p any positive integer

Figure 2.17:

Figure 2.18:

In the first case we have the diagram 2.17 which corresponds to the exceptional Lie algebra of rank 4 denoted F_4. In the other two cases we obtain the diagram of figure 2.18 which corresponds to the classical Lie algebras $so(2r+1)$ or $Sp(r)$ depending on the direction of the arrow.

Consider now the diagram D) of figure 2.16 and define the vectors

$$\epsilon = \sum_{a=1}^{p-1} a\epsilon_a \qquad \eta = \sum_{a=1}^{q-1} a\eta_a \qquad = \sum_{a=1}^{s-1} a\rho_a \tag{2.258}$$

Doing similar calculations to those leading to (2.251) we obtain

$$\epsilon^2 = p(p-1)/2 \qquad \eta^2 = q(q-1) \qquad \rho^2 = s(s-1) \tag{2.259}$$

The vectors ϵ, η, ρ and ψ (see diagram D) in figure 2.16) are linearly independent. Since $\psi^2 = 1$ we have from (2.259)

$$\begin{aligned} \cos^2(\epsilon, \psi) &= \frac{(\epsilon \cdot \psi)^2}{\epsilon^2 \psi^2} = \frac{(p-1)\,(\epsilon_{p-1} \cdot \psi)^2}{\epsilon^2} \\ &= \frac{(1 - 1/p)}{2} \end{aligned} \tag{2.260}$$

where we have used that $2\epsilon_{p-1} \cdot \psi = -1$.

Analogously we have

$$\cos^2(\eta, \psi) = \frac{(1 - 1/q)}{2} \tag{2.261}$$

and

$$\cos^2(\rho, \psi) = \frac{(1 - 1/s)}{2} \tag{2.262}$$

We can write ψ as

$$\psi = (\psi \cdot \epsilon)\, \frac{\epsilon}{\mid \epsilon \mid^2} + (\psi \cdot \eta)\, \frac{\eta}{\mid \eta \mid^2} + (\psi \cdot \rho)\, \frac{\rho}{\mid \rho \mid^2} + (\psi \cdot \epsilon_0)\, \epsilon_0 \tag{2.263}$$

Figure 2.19:

where ϵ_0 is a unit vector in the subspace perpendicular to ϵ, η and ρ. Then

$$\psi^2 = 1 = \frac{(\psi \cdot \epsilon)^2}{\epsilon^2} + \frac{(\psi \cdot \eta)^2}{\eta^2} + \frac{(\psi \cdot \rho)^2}{\rho^2} + (\psi \cdot \epsilon_0)^2 \tag{2.264}$$

Notice that $(\psi \cdot \rho)$ has to be different from zero, since ϵ, η, ρ and ψ are linarly independent, we get the inequality

$$\cos^2(\epsilon, \psi) + \cos^2(\eta, \psi) + \cos^2(\rho, \psi) < 1 \tag{2.265}$$

and so from (2.260-2.260)

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{s} > 1 \tag{2.266}$$

Whithou any loss of generality we can assume $p \geq q \geq s$. Then the possibilities are

1. $(p, q, s) = (p, 2, 2)$ with p any positive integer. The diagram we obtain is given in figure 2.19 which corresponds to the classical Lie algebra $so(2r)$.

2. $(p, q, s) = (p, 3, 2)$ with p taking the values 3, 4 or 5. The diagrams we obtain correspond to the exceptional Lie algebras E_6, E_7 and E_8 respectively, given in figure 2.20.

This ends the search for connected admissible diagrams. We have only to consider the arrows on the diagrams with double and triple links. When that is done we obtain all possible connected Dynkin diagrams corresponding to the simple Lie algebras. We list in figure 2.21 the diagrams we have obtained giving the name of the corresponding algebra in both, the physicist's and mathematician's notations.

Figure 2.20:

Figure 2.21: The Dynkin diagrams of the simple Lie algebras.

Chapter 3

Representation theory of Lie algebras

3.1 Introduction

In this chapter we shall develop further the concepts introduced in section 1.5 for group representations. The concept of a representation of a Lie algebra is analogous to that of a group. A set of operators D_1, D_2, ... acting on a vector space V is a *representation of a Lie algebra* in the *representation space* V if we can define an operation between any two of these operators such that it reproduces the commutation relations of the Lie algebra. We will be interested mainly on matrix representations and the operation will be the usual commutator of matrices. In addition we shall consider the representations of compact Lie algebras and Lie groups only, since the representation theory of non compact Lie groups is beyond the scope of these lecture notes.

Some results on the representation theory of finite groups can be extended to the case of compact Lie groups. In some sense this this is true because the volume of the group space is finite for the case of compact Lie groups, and therefore the integration over the group elements converge. We state without proof two important results on the representation theory of compact Lie groups which are also true for finite groups:

Theorem 3.1 *A finite dimensional representation of a compact Lie group is equivalent to a unitary one.*

Theorem 3.2 *A unitary representation can be decomposed into unitary irreducible representations.*

We then see that the irreducible representations (irreps.) constitute the building blocks for constructing finite dimensional representations of compact Lie groups. The aim of this chapter is to show how to classify and construct the irreducible representations of compact Lie groups and Lie algebras.

3.2 The notion of weights

We have defined in section 2.6 (see definition 2.12) the Cartan subalgebra of a semisimple Lie algebra as the maximal abelian subalgebra wich can be diagonalized simultaneously. Therefore we can take the basis of the representation space V as the eigenstates of the Cartan subalgebra generators. Then we have

$$H_i \mid \mu\rangle = \mu_i \mid \mu\rangle \qquad i = 1, 2, 3...r(\text{rank}) \tag{3.1}$$

The eigenvalues of the Cartan subalgebra generators constitute r-component vectors and they are called *weights*. Like the roots, the weights live in a r-dimensional Euclidean space. There can be more than one base state associated to a single weight. So the base states can be degenerated.

In section 2.8 we have seen that the operator $H_\alpha = 2\alpha \cdot H/\alpha^2$, has integer eigenvalues. Therefore from (3.1) we have

$$H_\alpha \mid \mu\rangle = \frac{2\alpha \cdot \mu}{\alpha^2} \mid \mu\rangle \tag{3.2}$$

and consenquently we have that

$$\frac{2\alpha \cdot \mu}{\alpha^2} \qquad \text{is an integer for any root } \alpha \tag{3.3}$$

Any vector μ satisfying this condition is a weight, and in fact this is the only condition a weight has to satisfy. From (2.153) we see that any root is a weight but the converse is not true. Notice that $\frac{2\alpha \cdot \mu}{\mu^2}$ does not have to be an integer and therefore the table 2.2 does not apply to the weights.

A weight is called *dominant* if it lies in the Fundamental Weyl Chamber or on its borders. Obviously a dominant weight has a non negative scalar product with any positive root. It is possible to find among the dominant weights, r weights λ_a, $a = 1, 2...r$, satisfying

$$\frac{2\lambda_a \cdot \alpha_b}{\alpha_b^2} = \delta_{ab} \qquad \text{for any simple root } \alpha_b \tag{3.4}$$

In orther words we can find r dominant weights which are orthogonal to all simple roots except one. These weights are called *fundamental weights*. They play an important role in representation theory as we will see below.

Consider now a simple root α_a and any weight μ. From (3.3) we have that

$$\frac{2\mu \cdot \alpha_a}{\alpha_a^2} = m_a = \text{integer} \tag{3.5}$$

Using (3.4) we have

$$\frac{2\alpha_a}{\alpha_a^2} \cdot \left(\mu - \sum_{a=1}^{r} m_a \lambda_a\right) = 0 \tag{3.6}$$

Since the simple roots constitute a basis of an r-dimensional Euclidean space we conclude that

$$\mu = \sum_{a=1}^{r} m_a \lambda_a \tag{3.7}$$

Therefore any weight can be written as a linear combination of the fundamental weights with integer coefficients. We now want to show that any vector formed by an integer linear combination of the fundamental weights is also a weight, i.e., it satisfies the condition (3.3). In order to do that we introduce the concept of *co-root* , which is a root devided by its squared lenght

$$\alpha^v \equiv \frac{\alpha}{\alpha^2} \tag{3.8}$$

Since

$$(\alpha^v)^2 = \frac{1}{\alpha^2} \tag{3.9}$$

and

$$\frac{2\alpha^v \cdot \beta^v}{(\alpha^v)^2} = \frac{2\alpha \cdot \beta}{\beta^2} \tag{3.10}$$

one sees that the co-roots satisfy all the properties of roots and consequently are also roots. However the co-roots of a given algebra $\mathcal{G}$ are the roots of another algebra $\mathcal{G}^v$, called the dual algebra to $\mathcal{G}$. The simply laced algebras, $su(N)$ (A_{N_1}), $so(2N)$ (D_N), E_6 , E_7 and E_8, together with the exceptional algebras G_2 and F_4 are self-dual algebras, in the sense that $\mathcal{G} = \mathcal{G}^v$. However $so(2N+1)$ (B_N) is the dual algebra to $sp(N)$ (C_N) and vice versa. The Cartan matrix of the dual algebra $\mathcal{G}^v$ is the transpose of the Cartan matrix of $\mathcal{G}$ since

$$(K_{ab})^v = \frac{2\alpha_a^v \cdot \alpha_b^v}{(\alpha_b^v)^2} = \frac{2\alpha_a \cdot \alpha_b}{\alpha_a^2} = K_{ba} \tag{3.11}$$

where we have used the fact that the simple co-roots are given by

$$\alpha_a^v = \frac{\alpha_a}{\alpha_a^2} \tag{3.12}$$

Any co-root can be written as a linear combination of the simple co-roots with integer coefficients all of the same sign. To show that we observe from theorem 2.7 that

$$\alpha^v = \frac{\alpha}{\alpha^2} = \sum_{a=1}^{r} n_a \frac{\alpha_a^2}{\alpha^2} \alpha_a^v \tag{3.13}$$

and from (3.4) we get

$$n_a = \frac{2\lambda_a \cdot \alpha}{\alpha_a^2} \tag{3.14}$$

Therefore

$$\alpha^v = \sum_{a=1}^{r} \frac{2\lambda_a \cdot \alpha}{\alpha^2} \alpha_a^v \equiv \sum_{a=1}^{r} m_a \alpha_a^v \tag{3.15}$$

since from (3.3) we have that $\frac{2\lambda_a \cdot \alpha}{\alpha^2}$ is an integer. In additon these integers are all of the same sign since all λ_a's lie on the Fundamental Weyl Chamber or on its border.

Let ν be a vector defined by

$$\nu = \sum_{a=1}^{r} k_a \lambda_a \tag{3.16}$$

where λ_a are the fundamental weights and k_a are arbitrary integers. Using (3.15) and (3.4) we get

$$\frac{2\alpha \cdot \nu}{\alpha^2} = 2\alpha^v \cdot \nu = \sum_{a,b} m_a k_b \frac{2\lambda_b \cdot \alpha_a}{\alpha_a^2} = \sum_{a} m_a k_a \tag{3.17}$$

Therefore ν is a weight. So we have shown that any integer linear combination of the fundamental weights is a weigtht and that all weights are of this form. Consequently the weights constitute a lattice Λ called the *weight lattice*. This quantized spectra of weights is a consequence of the fact that H_α has integer eigenvalues and is an important feature of representation theory of compact Lie algebras.

As we have said any root is a weight and consequently belong to Λ. We can also form a lattice by taking all vectors which are integer linear combinations of the simple roots. This lattice is called the *root lattice* and is denoted by Λ_r . All points in Λ_r are weights and therefore Λ_r is a sublattice of Λ. The weight lattice forms an abelian group under the addition of vectors. The root lattice is an invariant subgroup and consequently the coset space Λ/Λ_r has the structure of a group (see section 1.4). One can show that Λ/Λ_r corresponds to the center of the covering group corresponding to the algebra which weight lattice is Λ. We will show that all the weights of a given irreducible representation of a compact Lie algebra lie in the same coset.

Before giving some examples we would like to discuss the relation between the simple roots and the fundamental weights, which constitute two basis for the root (or weight) space. Since any root is a weight we have that the simple roots can be written as integer linear combination of the fundamental weights. Using (3.4) one gets that the integer coefficients are the entries of the Cartan matrix, i.e.

$$\alpha_a = \sum_b K_{ab} \lambda_b \tag{3.18}$$

and then

$$\lambda_a = \sum_b K^{-1}_{ab} \alpha_b \tag{3.19}$$

So the fundamental weights are not, in general, written as integer linear combination of the simple roots.

Example 3.1 *$SU(2)$ has only one simple root and consequently only one fundamental weight. Choosing a normalization such that $\alpha = 1$, we have that*

$$\frac{2\lambda \cdot \alpha}{\alpha^2} = 1 \qquad \text{and so} \qquad \lambda = \frac{1}{2} \tag{3.20}$$

Therefore the weight lattice of $SU(2)$ is formed by the integers and half integer numbers and the root lattice only by the integers. Then

$$\Lambda/\Lambda_r = \mathbb{Z}_2 \tag{3.21}$$

which is the center of $SU(2)$.

Example 3.2 *$SU(3)$ has two fundamental weights since it has rank two. They can be constructed solving (3.4) or equivalently (3.19). The Cartan matrix of $SU(3)$ and its inverse are given by (see example 2.13)*

$$K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad K^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{3.22}$$

So, from (3.19), we get that fundamental weights are

$$\lambda_1 = \frac{1}{3}(2\alpha_1 + \alpha_2) \qquad \lambda_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2) \tag{3.23}$$

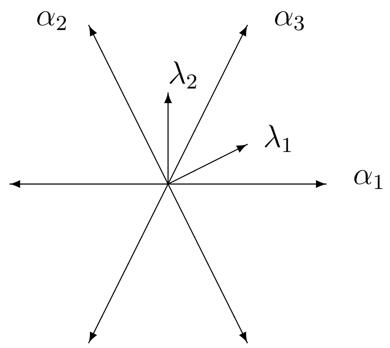

Figure 3.1: The fundamental weights of A_2 ($SU(3)$ or $SL(3)$)

In example 2.10 we have seen that the simple roots of $SU(3)$ *are given by* $\alpha_1 = (1, 0)$ *and* $\alpha_2 = \left(-1/2, \sqrt{3}/2\right)$. *Therefore*

$$\lambda_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right) \qquad \lambda_2 = \left(0, \frac{\sqrt{3}}{3}\right) \tag{3.24}$$

The vectors representing the fundamental weights are given in figure 3.1.

The root lattice, Λ_r , generated by the simple roots α_1 and α_2, corresponds to the points on the intersection of lines shown in the figure 3.2. The weight lattice, generated by the fundamental weights λ_1 and λ_2 , are all points of Λ_r plus the centroid of the triangles, shown by circles and plus signs on the figure 3.2.

The points of the weight lattice can be obtained from the origin, λ_1 and λ_2 by adding to them all points of the root lattice. Therefore the coset space Λ/Λ_r has three points which can be represented by 0, λ_1 and λ_2. Since $\lambda_1 + \lambda_2 = \alpha_1 + \alpha_2$ and $3\lambda_1 = 2\alpha_1 + \alpha_2$ lie in the same coset as 0, we see that Λ/Λ_r has the structure of the cyclic group $\mathbb{Z}_3$ which is the center of $SU(3)$.

3.3 The highest weight state

In a irreducible representation one can obtain all states of the representation by starting with a given state and applying sequences of step operators on it. If that was not possible the representation would have an invariant subspace and therefore would not be irreducible.

Figure 3.2: The weight lattice of $SU(3)$.

Consider a state with weight μ satisfying (3.1). The state defined by

$$\mid \mu' \rangle \equiv E_\alpha \mid \mu \rangle \tag{3.25}$$

satisfies

$$\begin{aligned} H_i \mid \mu' \rangle &= H_i E_\alpha \mid \mu \rangle \\ &= \left(E_\alpha H_i + [\, H_i \, , \, E_\alpha \,] \right) \mid \mu \rangle \\ &= \left(\mu_i + \alpha_i \right) E_\alpha \mid \mu \rangle \end{aligned} \tag{3.26}$$

and therefore it has weight $\mu + \alpha$. Therefore the state

$$E_{\alpha_1} E_{\alpha_2} \ldots E_{\alpha_n} \mid \mu \rangle \tag{3.27}$$

has weight $\mu + \alpha_1 + \ldots + \alpha_n$.

For this reason the weights in an irreducible representation differ by a sum of roots, and consequently they all lie in the same coset in Λ / Λ_r. Since that is the center of the covering group we see that the weights of an irreducible representation is associated to only one element of the center.

In a finite dimensional representation, the number of weights is finite, since this is at most the number of base states (remember the weights can be degenerated). Therefore, by applying sequences of step operators corresponding to positive roots on a given state we will eventually get zero. So, an irreducible finite dimensional representation possesses a state such that

$$E_\alpha \mid \lambda \rangle = 0 \qquad \text{for any } \alpha > 0 \tag{3.28}$$

This state is called the *highest weight state* of the representation, and λ is the *highest weight*. It is possible to show that there is only one highest weight in an irrep. and only one highest weight state associated to it. That is, the highest weight is *unique and non degenerate*.

All other states of the representation are obtained from the highest weight state by the application of a sequence of step operators corresponding to negative roots. The state defined by

$$\mid \mu \rangle \equiv E_{-\alpha_1} E_{-\alpha_2} \ldots E_{-\alpha_n} \mid \lambda \rangle \tag{3.29}$$

according to (3.26) has weight $\lambda - \alpha_1 - \alpha_2 \ldots - \alpha_n$. All the basis states are of the form (3.29). If one applies a positive step operator on the state (3.29) the resulting state of the representation can be written as a linear combination of states of the form (3.29). To see this, let β be a a positive root and α any of the negative roots appearing in (3.29). Then we have

$$E_\beta \mid \mu\rangle = \left(E_{-\alpha_1} E_\beta + \left[\, E_\beta \, , \, E_{-\alpha_1} \,\right]\right) E_{-\alpha_2} \ldots E_{-\alpha_n} \mid \lambda\rangle \tag{3.30}$$

In the cases where $\beta - \alpha_1$ is a negative root or it is not a root or even $\beta - \alpha_1 = 0$, we obtain that the second term on the r.h.s. of (3.30) is a state of the form of (3.29). In the case $\beta - \alpha_1$ is a positive root we contiunue the process until all positive step operators act directly on the highest state $\mid \lambda\rangle$, and consequently annihilate it. Therefore the state (3.30) is a linear combination of the states (3.29).

The weight lattice Λ is invariant by the Weyl group. If μ is a weight, and therefore satisfies (3.3), it follows that $\sigma_\beta(\mu)$ also satisfies (3.3) for any root β, and so is a weight. To show this we use the fact that $\sigma_\beta(x) \cdot \sigma_\beta(y) = x \cdot y$ and $\sigma_\beta^2 = 1$. Then (denoting $\gamma = \sigma_\beta(\alpha)$)

$$\frac{2\alpha \cdot \sigma_\beta(\mu)}{\alpha^2} = \frac{2\mu \cdot \sigma_\beta(\alpha)}{\sigma_\beta(\alpha)^2} = \frac{2\gamma \cdot \mu}{\gamma^2} = \text{integer} \tag{3.31}$$

However we can show that the set of weights of a given representation, which is a finite subset of Λ, is invariant by the Weyl group. The state defined by

$$\mid \bar{\mu}\rangle \equiv S_\alpha \mid \mu\rangle \tag{3.32}$$

where $\mid \mu\rangle$ is a state of the representation and S_α is defined in (2.159), is also a state of the representation since it is obtained from $\mid \mu\rangle$ by the action of an operator of the representation. Using (2.160) we get

$$\begin{aligned} x \cdot H \mid \bar{\mu}\rangle &= S_\alpha S_\alpha^{-1} x \cdot H S_\alpha \mid \mu\rangle \\ &= S_\alpha \sigma_\alpha(x) \cdot H \mid \mu\rangle \\ &= \sigma_\alpha(x) \cdot \mu \mid \bar{\mu}\rangle \\ &= \sigma_\alpha(\mu) \cdot x \mid \bar{\mu}\rangle \end{aligned} \tag{3.33}$$

Since the vector x is arbitrary we obtain that the state $\mid \bar{\mu}\rangle$ has, weight $\sigma_\alpha(\mu)$

$$H_i \mid \bar{\mu}\rangle = H_i S_\alpha \mid \mu\rangle = \sigma_\alpha(\mu)_i \, S_\alpha \mid \mu\rangle = \sigma_\alpha(\mu)_i \mid \bar{\mu}\rangle \tag{3.34}$$

Therefore if μ is a weight of the representation so is $\sigma_\alpha(\mu)$ for any root α. One can easily check that the root lattice Λ_r is also invariant by the Weyl reflections.

A consequence of the above result is that the highest weight λ of an irrep. is a dominant weight. By taking its Weyl reflection

$$\sigma_\alpha(\lambda) = \lambda - \frac{2\lambda \cdot \alpha}{\alpha^2}\alpha \tag{3.35}$$

one obtains that $2\lambda \cdot \alpha$ has to be non negative if α is a positive root, since $\sigma_\alpha(\lambda)$ is also a weight of the representation and consequenlty can not exceed λ by a multiple of a positive root. Therefore

$$\lambda \cdot \alpha \geq 0 \qquad \text{for any positive root } \alpha \tag{3.36}$$

and the highest weight λ is a dominant weight.

The highest weight λ can be used to label the representation. This is one of the consequences of the following theorem which we state without proof.

Theorem 3.3 *There exists a unique irreducible representation of a compact Lie algebra (up to equivalence) with highest weight state $\mid \lambda\rangle$ for each λ of the weight lattice in the Fundamental Weyl Chamber or on its border.*

The importance of this theorem is that it provides some sort of classification of all irreps. of a compact Lie algebra. All other reducible representations are constructed from these ones. The irreps. can be labelled by their highest weight λ as D^λ or $D^{(n_1,n_2,\ldots n_r)}$ where the n_a's are non-negative integers appearing in the expansion of λ in terms of the fundamental weights λ_a, i.e. $\lambda = \sum_{a=1}^r n_a \lambda_a$, and $n_a = \frac{2\lambda\cdot\alpha_a}{\alpha_a^2}$.

An irrep. is called a *fundamental representation* when its highest weight is a fundamental weight. Therefore the number of fundamental representations of a semisimple compact Lie algebra is equal to its rank.

The highest weight of the adjoint representation is the highest positive root (see section 2.13). It follows that the weights of the adjoint representation are all roots of the algebra together with zero which is a weight r-fold degenerated (r= rank).

We say a weight μ is a *minimal weight* if it satisfies

$$\frac{2\mu \cdot \alpha}{\alpha^2} = 0 \text{ or } \pm 1 \text{ for any root } \alpha \tag{3.37}$$

The representation for which the highest weight is minimal is said to be a *minimal representation*. These representations play an important role in grand unified theories (GUT) in the sense that the constituent fermions prefer, in general, to form multiplets in such minimal representations.

Example 3.3 *In the example 3.1 we have seen that the only fundamental weight of* $SU(2)$ *is* $\lambda = \frac{1}{2}$ *. Therefore the dominant weights of* $SU(2)$ *are the positive integers and half integers. Each one of these dominant weights corresponds to an irreducible representation of* $SU(2)$*. Then we have that* $\lambda = 0$ *corresponds to the scalar representation,* $\lambda = \frac{1}{2}$ *the spinorial rep. which is the fundamental rep. of* $SU(2)$ *(dim* $= 2$*),* $\lambda = 1$ *is the vectorial rep. which is the adjoint of* $SU(2)$ *(dim* $= 3$*) and so on.*

Example 3.4 *In the case of* $SU(3)$ *we have two fundamental representations with highest weights* λ_1*, and* λ_2 *(see example 3.2. They are respectively the triplet and antitriplet representations of* $SU(3)$*. The rep. with highest weight* $\lambda_1 + \lambda_2 = \alpha_3$ *is the adjoint. All representations with highest weight of the form with* $\lambda = n_1\lambda_1 + n_2\lambda_2$*, with* n_1 *and* n_2 *non negative integers are irreducible representations of SU(3).*

3.4 Weight strings and multiplicities

If we apply the step operator E_α or $E_{-\alpha}$, for a fixed root α, successively on a state of weight μ of a finite dimensional representation, we will eventually get zero. That means that there exist positive integer numbers p and q such that

$$E_\alpha \mid \mu + p\alpha\rangle \qquad \text{and} \qquad E_{-\alpha} \mid \mu - q\alpha\rangle \tag{3.38}$$

p and q are the greatest positive integers for which $\mu+p\alpha$ and $\mu-q\alpha$ are weights of the representation. One can show that all vectors of the form $\mu + n\alpha$ with n integer and $-q < n < p$, are weights of the representation. Therefore the weights form unbroken strings, called *weight strings* , of the form

$$\mu + p\alpha \,;\, \mu + (p-1)\,\alpha \,;\, \ldots \,\mu + \alpha \,;\, \mu \,;\, \mu - \alpha \,;\, \ldots \,\mu - q\alpha \tag{3.39}$$

We have shown in the last section that the set of weights of a representation is invariant under the Weyl group. The effect of the action of the Weyl reflection σ_α on a weight is to add or subtract a multiple of the root α, since $\sigma_\alpha(\mu) = \mu - \frac{2\mu\cdot\alpha}{\alpha^2}\alpha$, and from (3.3) we have that $\frac{2\mu\cdot\alpha}{\alpha^2}$ is an integer. Therefore the weight string (3.39) is invariant by the Weyl reflection σ_α. In fact, σ_α reverses the string (3.39) and consenquently we have that

$$\sigma_\alpha(\mu + p\alpha) = \mu - q\alpha = \mu - \frac{2\mu\cdot\alpha}{\alpha^2}\alpha - p\alpha \tag{3.40}$$

and so

$$\frac{2\mu\cdot\alpha}{\alpha^2} = q - p \tag{3.41}$$

This result is similar to (2.192) which was obtained for root strings. However, notice that the possible values of $q - p$, in this case, are not restrict to the values given in (2.192) ($q-p$ can, in principle, have any integer value). In the case where μ is the highest weight of the representation we have that p is zero if α is a positive root, and q is zero if α is negative. The relation (3.41) provides a practical way of finding the weights of the representation. In some cases it is easier to find some weights of a given representation by taking successive Weyl reflections of the highest weight. However, this method does not provide, in general, all the weights of the representation.

Once the weights are known one has to calculate their multiplicities. There exists a formula, due to Kostant, which expresses the multiplicities directly as a sum over the elements of the Weyl group. However, it is not easy to use this formula in practice. There exists a recursive formula, called *Freudenthal's* *formula* , which is much easier to use. According to it the multiplicity $m(\mu)$ of a weight μ in an irreducible representation of highest weight λ is given recursively as (see sections 22.3 and 24.2 of [HUM 72])

$$\left((\lambda+\delta)^2-(\mu+\delta)^2\right) m(\mu)=2\sum_{\alpha>0}\sum_{n=1}^{p(\alpha)}\alpha\cdot(\mu+n\alpha)\, m(\mu+n\alpha) \tag{3.42}$$

where

$$\delta\equiv\frac{1}{2}\sum_{\alpha>0}\alpha \tag{3.43}$$

The first summation on the l.h.s. is over the positive roots and the second one over all positive integers n such that $\mu+n\alpha$ is a weight of the representation, and we have denoted by $p(\alpha)$ the highest value of n. By starting with $m(\lambda)=1$ one can use (3.43) to calculate the multiplicities of the weights from the higher ones to the lower ones.

If the states $\mid\mu\rangle_1$ and $\mid\mu\rangle_2$ have the same weight, i.e., μ is degenerated, then the weight $\sigma_\alpha(\mu)$ is also degenerate and has the same multiplicity as μ. Using (3.32) we obtain that the states

$$\mid\sigma_\alpha(\mu)\rangle_1=S_\alpha\mid\mu\rangle_1 \qquad \text{and} \qquad \mid\sigma_\alpha(\mu)\rangle_2=S_\alpha\mid\mu\rangle_2 \tag{3.44}$$

have weight $\sigma_\alpha(\mu)$ and their linear independence follows from the linear independence of $\mid\mu\rangle_1$ and $\mid\mu\rangle_2$. Indeed,

$$0=x_1\mid\sigma_\alpha(\mu)\rangle_1+x_2\mid\sigma_\alpha(\mu)\rangle_2=S_\alpha(x_1\mid\mu\rangle_1+x_2\mid\mu\rangle_2) \tag{3.45}$$

So, if $\mid\mu\rangle_1$ and $\mid\mu\rangle_2$ are linearly independent one gets that one must have $x_1=x_2=0$ and so, $\mid\sigma_\alpha(\mu)\rangle_1$ and $\mid\sigma_\alpha(\mu)\rangle_2$ are also linearly independent.

Therefore all the weights of a representation which are conjugate under the Weyl group have the same multiplicity. This fact can be used to make the Freudenthal's formula more efficient in the calculation of the multiplicities.

Example 3.5 *Using the results of example 2.14 we have that the Cartan matrix of so*(5) *ond its inverse are*

$$K=\begin{pmatrix}2&-1\\-2&2\end{pmatrix} \qquad K^{-1}=\frac{1}{2}\begin{pmatrix}2&1\\2&2\end{pmatrix} \tag{3.46}$$

Then, using (3.19), we get that the fundamental weights of so(5) *are*

$$\lambda_1=\frac{1}{2}(2\alpha_1+\alpha_2) \qquad \lambda_2=\alpha_1+\alpha_2 \tag{3.47}$$

Figure 3.3: The weights of the spinor representation of $so(5)$.

where α_1 and α_2 are the simple roots of $so(5)$. Let us consider the fundamental representation with highest weight λ_1. The scalar products of λ_1 with the positive roots of $so(5)$ are

$$
\begin{aligned}
\frac{2\lambda_1 \cdot \alpha_1}{\alpha_1^2} &= 1 \qquad \frac{2\lambda_1 \cdot \alpha_2}{\alpha_2^2} = 0 \\
\frac{2\lambda_1 \cdot (\alpha_1 + \alpha_2)}{(\alpha_1 + \alpha_2)^2} &= 1 \qquad \frac{2\lambda_1 \cdot (2\alpha_1 + \alpha_2)}{(2\alpha_1 + \alpha_2)^2} = 1
\end{aligned}
\tag{3.48}
$$

Therefore using (3.41) (with $p = 0$ since λ_1 is the highest weight) we get that

$$
\lambda_1 \; ; \quad (\lambda_1 - \alpha_1) \; ; \quad (\lambda_1 - \alpha_1 - \alpha_2) \; ; \quad (\lambda_1 - 2\alpha_1 - \alpha_2) \tag{3.49}
$$

are weights of the representation. By taking Weyl reflections of these weights or using (3.41) further one can check that these are the only weights of the fundamental rep. with highest weight λ_1.

Since all weights are conjugate under the Weyl group they all have the same multiplicity as λ_1 , which is one. Therefore they are not degenerate and the representation has dimension 4. This is the spinor representation of $so(5)$. One can check that the weights of the fundamental representation of $so(5)$ with highest weight λ_2 are

$$
\begin{aligned}
&\lambda_2 \; ; \quad \lambda_2 - \alpha_2 = \alpha_1 \; ; \quad \lambda_2 - \alpha_1 - \alpha_2 = 0 \; ; \\
&\lambda_2 - 2\alpha_1 - \alpha_2 = -\alpha_1 \; ; \quad \lambda_2 - 2\alpha_1 - 2\alpha_2 = -\left(\alpha_1 + \alpha_2\right)
\end{aligned}
\tag{3.50}
$$

Again these weights are not degenerate and the representation has dimension 5*. This is the vector representation of* $so(5)$*.*

Example 3.6 *Consider the irrep. of* $su(3)$ *with highest weight* $\lambda = \alpha_3 = \alpha_1 + \alpha_2$ *, i.e., the highest positive root. Using (3.41) and performing Weyl reflections one can check that the weights of such rep. are all roots plus the zero weight. Since the roots are conjugated to* $\alpha_3 = \lambda$ *under the Weyl group we conclude that they are non degenerated weights. The multiplicity of the zero weight can be calculated from the Freundenthal's formula. From (3.43) we have that, in this case,* $\delta = \alpha_3$ *and so from (3.42) we get*

$$\left(4\alpha_3^2 - \alpha_3^2\right) m\left(0\right) = 2\left(m\left(\alpha_1\right)\alpha_1^2 + m\left(\alpha_2\right)\alpha_2^2 + m\left(\alpha_3\right)\alpha_3^2\right) \tag{3.51}$$

Since $m\left(\alpha_1\right) = m\left(\alpha_2\right) = m\left(\alpha_3\right) = 1$ *and* $\alpha_1^2 = \alpha_2^2 = \alpha_3^2$ *we obtain that* $m\left(0\right) = 2$*. So there are two states with zero weight and consequently the representation has dimension* 8*. This is the adjoint of* $su(3)$*.*

3.5 The weight δ

A vector which plays an important role in the representation theory of Lie algebras is the vector δ defined in (3.43). It is half of the sum of all positive roots. In same cases δ is a root, but in general that is not so. However δ is always a dominant weight of the algebra. In other to show that we need some results which we now prove.

Let α_a be a simple root and let β be a positive root non proportional to α_a. If we write $\beta = \sum_{b=1}^{r} n_b \alpha_b$ we have that $n_b \neq 0$ for some $b \neq a$. Now, the coefficient of α_b in $\sigma_{\alpha_a}\left(\beta\right)$ is still n_b, and consequently $\sigma_{\alpha_a}\left(\beta\right)$ has at least one positive coefficient. So, $\sigma_{\alpha_a}\left(\beta\right)$ is a positive root, and it is different from α_a, since α_a is the image of $-\alpha_a$ under σ_{α_a}. Therefore we have proved the following lemma.

Lemma 3.1 *If* α_a *is a simple root, then* σ_{α_a} *permutes the positive roots other than* α_a*.*

From this lemma it follows that

$$\sigma_{\alpha_a}\left(\delta\right) = \delta - \alpha_a \tag{3.52}$$

and consequently

$$\frac{2\delta \cdot \alpha_a}{\alpha_a^2} = 1 \qquad \text{for any simple root } \alpha_a \tag{3.53}$$

From the definition (3.43) it follows that δ is a vector on the root (or weight) space and therefore can be written in terms of the simple roots or the fundamental weights. Writing

$$\delta = \sum_{b=1}^{r} x_b \lambda_b \tag{3.54}$$

we get from (3.4) and (3.53) that

$$\frac{2\delta \cdot \alpha_a}{\alpha_a^2} = 1 = \sum_{b=1}^{r} x_b \frac{2\lambda_b \cdot \alpha_a}{\alpha_a^2} = x_a \tag{3.55}$$

So we have shown that

$$\delta = \sum_{b=1}^{r} \lambda_b \tag{3.56}$$

and consequently δ is a dominant weight.

3.6 Casimir operators

Let $\Gamma^{s_1 s_2 \ldots s_n}$ be a tensor invariant under the adjoint representation of a Lie group G. By that we mean

$$\Gamma^{s_1 s_2 \ldots s_n} = d^{s_1}_{s'_1}(g)\, d^{s_2}_{s'_2}(g) \ldots d^{s_n}_{s'_n}(g)\, \Gamma^{s'_1 s'_2 \ldots s'_n} \tag{3.57}$$

for any $g \in G$, and where $d^{s_j}_{s'_j}(g)$ is the matrix representing g in the adjoint representation, i.e. $g T_s g^{-1} = T_{s'} d^{s'}_{s}(g)$ (see (2.31)).

Consider now a representation D of G and construct the operator

$$C^{(D)}_n \equiv \Gamma^{s_1 s_2 \ldots s_n}\, D(T_{s_1})\, D(T_{s_2}) \ldots D(T_{s_n}) \tag{3.58}$$

Notice that such operator can only be defined on a given representation since it involves the product of operators and not Lie brackets of the generators.

We then have

$$\begin{aligned} D(g)\, C^{(D)}_n &= \Gamma^{s_1 s_2 \ldots s_n}\, D\left(g T_{s_1} g^{-1}\right) D\left(g T_{s_2} g^{-1}\right) \ldots D\left(g T_{s_n} g^{-1}\right) D(g) \\ &= d^{s'_1}_{s_1}(g) \ldots d^{s'_n}_{s_n}(g)\, \Gamma^{s_1 \ldots s_n} D\left(T_{s'_1}\right) \ldots D\left(T_{s'_n}\right) D(g) \\ &= \Gamma^{s'_1 \ldots s'_n} D\left(T_{s'_1}\right) \ldots D\left(T_{s'_n}\right) D(g) \\ &= C^{(D)}_n D(g) \end{aligned} \tag{3.59}$$

So, we have shown that $C^{(D)}_n$ commutes with any matrix of the representation

$$\left[\, C^{(D)}_n \,,\, D(g) \right] = 0 \tag{3.60}$$

We are interested in operators that can not be reduced to lower orders. That implies that the tensor $\Gamma^{s_1 s_2 \ldots s_n}$ has to be totally symmetric. Indeed, suppose that $\Gamma^{s_1 s_2 \ldots s_n}$ has an antisymmetric part in the indices s_j and s_{j+1}. Then we write

$$\begin{aligned} D\left(T_{s_j}\right) D\left(T_{s_{j+1}}\right) &= \frac{1}{2}\{D\left(T_{s_j}\right) \,,\, D\left(T_{s_{j+1}}\right)\} + \frac{1}{2}\left[\, D\left(T_{s_j}\right) , D\left(T_{s_{j+1}}\right) \right] \\ &= \frac{1}{2}\{D\left(T_{s_j}\right) \,,\, D\left(T_{s_{j+1}}\right)\} + f^{t}_{s_j s_{j+1}} D(T_t) \end{aligned} \tag{3.61}$$

and so, $C^{(D)}_n$ will have terms involving the product of $(n-1)$ operators. Therefore, by totally symmetrizing the tensor $\Gamma^{s_1 s_2 \ldots s_n}$ we get operators $C^{(D)}_n$ which are monomials of order n in $D(T_s)$'s. Such operators are called *Casimir operators*, and n is called their *order*. They play an important role in representation

A_r	$SU(r+1)$	2, 3, 4, $\ldots$ $r+1$
B_r	$SO(2r+1)$	2, 4, 6, $\ldots$ $2r$
C_r	$Sp(r)$	2, 4, 6 $\ldots$ $2r$
D_r	$SO(2r)$	2, 4, 6 $\ldots$ $2r-2$, r
E_6		2, 5, 6, 8, 9, 12
E_7		2, 6, 8, 10, 12, 14, 18
E_8		2, 8, 12, 14, 18, 20, 24, 30
F_4		2, 6, 8, 12
G_2		2, 6

Table 3.1: The orders of the Casimir operators for the simple Lie Groups

theory. From Schur's lemma 1.1 it follows that in an irreducible representation the Casimir operators have to be proportional to the identity.

One way of constructing tensors which are invariant under the adjoint representation, is by considering traces of products of generators in a given representation D', since

$$\text{Tr}\left(D'\left(T_{s_1}T_{s_2}\ldots T_{s_n}\right)\right) = \text{Tr}\left(D'\left(gT_{s_1}g^{-1}gT_{s_2}g^{-1}\ldots gT_{s_n}g^{-1}\right)\right) \tag{3.62}$$

Then taking

$$\Gamma_{s_1 s_2 \ldots s_n} \equiv \frac{1}{n!}\sum_{\text{permutations}} \text{Tr}\left(D'\left(T_{s_1}T_{s_2}\ldots T_{s_n}\right)\right) \tag{3.63}$$

we get Casimir operators. However, one finds that after the symetrization procedure very few tensors of the form above survive. It follows that a semisimple Lie algebra of rank r possesses r invariant Casimir operators functionally independent. Their orders, for the simple Lie algebras, are given in table 3.1.

3.6.1 The Quadratic Casimir operator

Notice from table 3.1 that all simple Lie groups have a quadratic Casimir operator. That is because all such groups have an invariant symmetric tensor of order two which is the Killing form (see section 2.4)

$$\eta_{st} = \text{Tr}\left(d\left(T_s\right)d\left(T_t\right)\right) \tag{3.64}$$

and

$$C_2^{(D)} \equiv \eta^{st} D\left(T_s\right) D\left(T_t\right) \tag{3.65}$$

where η^{st} is the inverse of η_{st}.

Using the normalization (2.139) of the Killing form, we have that the Casimir operator in the Cartan-Weyl basis is given by

$$C_2^{(D)} = \sum_{i=1}^{r} D\left(H_i\right) D\left(H_i\right) + \sum_{\alpha>0} \frac{\alpha^2}{2} \left(D\left(E_\alpha\right) D\left(E_{-\alpha}\right) + D\left(E_{-\alpha}\right) D\left(E_\alpha\right)\right) \quad (3.66)$$

Since the Casimir operator commutes with all generators, we have from the Schur's lemma 1.1 that in an irreducible representation it must be proportional to the unit matrix. Denoting by λ the highest weight of the irreducible representation D we have

$$\begin{aligned} C_2^{(D)} \mid \lambda\rangle &= \left(\sum_{i=1}^{r} \lambda_i^2 + \sum_{\alpha>0} \frac{\alpha^2}{2} \left[\, D\left(E_\alpha\right),\, D\left(E_{-\alpha}\right)\right]\right) \mid \lambda\rangle \\ &= \left(\lambda^2 + \sum_{\alpha>0} \frac{\alpha^2}{2} H_\alpha^2\right) \mid \lambda\rangle \\ &= \left(\lambda^2 + \sum_{\alpha>0} \alpha \cdot \lambda\right) \mid \lambda\rangle \end{aligned} \quad (3.67)$$

where we have used (3.28) and (2.130). So, if D, with highest weight λ, is irreducible, we can write using (3.43) that

$$C_2^{(D)} = \lambda \cdot (\lambda + 2\delta)\, \mathbb{1} = \left((\lambda + \delta)^2 - \delta^2\right) \mathbb{1} \quad (3.68)$$

where $\mathbb{1}$ is the unit matrix in the representation D under consideration.

Example 3.7 *In the case of $SU(2)$ the quadratic operator is J^2 , i.e., the square of the angular momentum. Indeed, from example 3.1 we have that $\alpha = 1$, and then $\delta = 1/2$ and therefore $C_2^{(D)} = \lambda\,(\lambda + 1)$. Since λ is a positive integer or half integer we see that these are really the eigenvalues of J^2.*

3.7 Characters

In definition 1.13 we defined the *character* of an element g of a group G in a given finite dimensional representation of G, with highest weight λ, as being the trace of the matrix that represents that element, i.e.

$$\chi^{\lambda}(g) \equiv \mathrm{Tr}\,(D\,(g)) \quad (3.69)$$

Obviously equivalent representations (see section 1.5) have the same characters. Analogously, two conjugate elements, $g_1 = g_3 g_2 g_3^{-1}$, have the same character in all representations. Therefore the conjugacy classes can be labelled by the characters.

Example 3.8 *Using (2.27) and the commutation relations (2.63) for the algebra of* $so(3)$ *(or* $su(2)$*) one gets that*

$$e^{i\frac{\pi}{2}T_2}\,T_3\,e^{-i\frac{\pi}{2}T_2} = T_1 \tag{3.70}$$

and consequently

$$e^{i\frac{\pi}{2}T_2}\,e^{i\theta T_3}\,e^{-i\frac{\pi}{2}T_2} = e^{i\theta T_1} \tag{3.71}$$

An analogous result is obtained if we interchange the roles of the generators T_1 *,* T_2 *and* T_3*. Therefore the rotations by a given angle* θ*, no matter the axis, are conjugate. The conjugacy classes of* $SO(3)$ *are defined by the angle of rotation, and the characters in a representation of spin* j *are given by*

$$\chi^j(\theta) = \chi^j\left(e^{i\theta T_3}\right) = \sum_{m=-j}^{j} e^{im\theta} \tag{3.72}$$

where m *are the eigenvalues of* T_3 *(see section 2.5). We have a geometric progression and therefore*

$$\chi^j(\theta) = \frac{e^{i\left(j+\frac{1}{2}\right)\theta} - e^{-i\left(j+\frac{1}{2}\right)\theta}}{e^{i\theta/2} - e^{-i\theta/2}} \tag{3.73}$$

Notice that rotations by θ *and* $-\theta$ *have the same character.*

The relation (3.71) can be generalized for any compact Lie group. Any element of a compact group is conjugate to an element of the abelian subgroup which is the exponentiation of the Cartan subalgebra, i.e.

$$g = g' e^{i\theta\cdot H} g'^{-1} \tag{3.74}$$

Therefore the conjugacy classes, and consequently the characters, can be labelled by r parameters or "angles" (r = rank).

However, the elements of the abelian group parametrized by θ and $\sigma_\alpha(\theta)$ have the same character, since from (2.160) we have

$$S_\alpha e^{i\theta\cdot H} S_\alpha^{-1} = e^{i\sigma_\alpha(\theta)\cdot H} \tag{3.75}$$

Thus the parameter θ and its Weyl reflections parametrize the same conjugacy class.

The generalization of (3.73) to any compact group was done by H. Weyl in 1935. In a representation with highest weight the elements of the conjugacy class labelled by have a character given by

$$\chi^\lambda(\theta) = \frac{\sum_{\sigma\in W}(\mathrm{sign}\sigma)\,e^{i\sigma(\lambda+\delta)\cdot\theta}}{e^{i\delta\cdot\theta}\prod_{\alpha>0}\left(1-e^{-i\alpha\cdot\theta}\right)} \tag{3.76}$$

where the summation is over the elements σ of the Weyl group W, and where sign is 1 (-1) if the element σ of the Weyl group is formed by an even (odd) number of reflections. δ is the same as the one defined in (3.43). This relation is called the *Weyl character formula.*

The character can also be calculated once one knows the multiplicities of the weights of the representation. From (3.69) and (3.74) we have that

$$\chi^{\lambda}(\theta) = \mathrm{Tr}\, D^{\lambda}\left(e^{i\theta\cdot H}\right) = \sum_{\mu} m(\mu)\, e^{i\theta\cdot\mu} \tag{3.77}$$

where the summation is over the weights of the representation and $m(\mu)$ are their multiplicities. These can be obtained from Freudenthal's formula (3.42).

In the scalar representation the elements of the group are represented by the unity and the highest weight is zero. So setting $\lambda = 0$ in (3.76) we obtain what is called the *Weyl denominator formula*

$$\sum_{\sigma\in W} (\mathrm{sign}\sigma)\, e^{i\sigma(\delta)\cdot\theta} = e^{i\delta\cdot\theta} \prod_{\alpha>0}\left(1 - e^{-i\alpha\cdot\theta}\right) \tag{3.78}$$

In general, such formula provides a nontrivial relation between a product and a sum. Substituting (3.78)in (3.76) we can write the Weyl character formula as the ratio of two sums:

$$\chi^{\lambda}(\theta) = \frac{\sum_{\sigma\in W} (\mathrm{sign}\sigma)\, e^{i\sigma(\lambda+\delta)\cdot\theta}}{\sum_{\sigma\in W} (\mathrm{sign}\sigma)\, e^{i\sigma(\delta)\cdot\theta}} \tag{3.79}$$

The dimension of the representation can be obtained from the Weyl character formula (3.76) noticing that

$$\mathrm{dim}D^{\lambda} = \mathrm{Tr}\,(\mathbb{1}) = \chi^{\lambda}(0) \tag{3.80}$$

we then obtain the so called *Weyl dimensionality formula*

$$\mathrm{dim}D^{\lambda} = \frac{\prod_{\alpha>0} (\lambda+\delta)\cdot\alpha}{\prod_{\alpha>0} \delta\cdot\alpha} \tag{3.81}$$

Example 3.9 *In the case of $SO(3)$ (or $SU(2)$) we have that $\alpha = 1$, $\delta = 1/2$ and consequently we have from (3.81) that*

$$\mathrm{dim}\; D^{j} = 2j+1 \tag{3.82}$$

This result can also be obtained from (3.73) by taking the limit $\theta \to 0$ and using L'Hospital's rule

(m_1, m_2)	dimension
$(1,0)$	(triplet) 3
$(0,1)$	(anti-triplet) 3
$(2,0)$	6
$(0,2)$	6
$(1,1)$	(adjoint) 8
$(3,0)$	10
$(0,3)$	10
$(2,1)$	15
$(1,2)$	15

Table 3.2: The dimensions of the smallest irreps. of $SU(3)$

Example 3.10 *Consider an irrep. of* $SU(3)$ *with highest weight* λ. *We can write* $\lambda = m_1\lambda_1 + m_2\lambda_2$ *where* λ_1 *and* λ_2 *are the fundamental weights and* m_1 *and* m_2 *are non-negative integers. From (3.56) we have that* $(\delta + \lambda) = (m_1 + 1)\,\lambda_1 + (m_2 + 1)\,\lambda_2$. *Normalizing the roots of* $SU(3)$ *as* $\alpha^2 = 2$ *we have (from (3.4)) that* $\lambda_a \cdot \alpha_b = \delta_{ab}$ $(a, b = 1, 2)$, *where* α_1 *and* α_2 *are the simple roots and therefore (* $\alpha_3 = \alpha_1 + \alpha_2$ *)*

$$\begin{aligned} (\delta+\lambda)\cdot\alpha_1 &= m_1+1\;; \quad (\delta+\lambda)\cdot\alpha_2 = m_2+1\;; \quad (\delta+\lambda)\cdot\alpha_3 = m_1+m_2+2 \\ \delta\cdot\alpha_1 &= \delta\cdot\alpha_2 = 1\;; \qquad \delta\cdot\alpha_3 = 2 \end{aligned} \tag{3.83}$$

So, from (3.81) the dimension of the irrep. of $SU(3)$ *with highest weight* λ *is*

$$\dim D^{\lambda} = \dim D^{\lambda} = \frac{1}{2}\,(m_1+1)\,(m_2+1)\,(m_1+m_2+2) \tag{3.84}$$

In table 3.2 we give the dimensions of the smallest irreps. of $SU(3)$.

Example 3.11 *Similarly let us consider the irreps. of* $SO(5)$ *(or* $Sp(2)$*) with highest weight* $\lambda = m_1\lambda_1 + m_2\lambda_2$. *From example 2.14 we have that the positive roots of* $SO(5)$ *are* α_1, α_2, $\alpha_3 \equiv \alpha_1 + \alpha_2$, *and* $\alpha_4 \equiv 2\alpha_1 + \alpha_2$, *and so using (3.4) and (3.56) we get (setting* $\alpha_1^2 = 1$, $\alpha_2^2 = 2$*)*

$$\begin{aligned} \frac{2\delta\cdot\alpha_1}{\alpha_1^2} = \frac{2\delta\cdot\alpha_2}{\alpha_2^2} &= 1\;; \quad \frac{2\delta\cdot\alpha_3}{\alpha_3^2} = \frac{3}{2}1\;; \quad \frac{2\delta\cdot\alpha_4}{\alpha_4^2} = 2 \\ \frac{2\,(\delta+\lambda)\cdot\alpha_1}{\alpha_1^2} &= m_1+1\;; \quad \frac{2\,(\delta+\lambda)\cdot\alpha_2}{\alpha_2^2} = m_2+1 \\ \frac{2\,(\delta+\lambda)\cdot\alpha_3}{\alpha_3^2} &= \frac{1}{2}\,(m_1+2m_2+3)\;; \quad \frac{2\,(\delta+\lambda)\cdot\alpha_4}{\alpha_4^2} = \frac{1}{2}\,(m_1+m_2+2) \end{aligned} \tag{3.85}$$

(m_1, m_2)	dimension
$(1,0)$	(spinor) 4
$(0,1)$	(vector) 5
$(2,0)$	(adjoint) 10
$(0,2)$	14
$(1,1)$	16
$(3,0)$	20
$(0,3)$	30
$(2,1)$	35
$(1,2)$	40

Table 3.3: The dimensions of the smallest irreps. of $SO(5)$ (or $Sp(2)$)

Therefore from (3.81)

$$\dim D^{(m_1,m_2)} = \frac{1}{6}\,(m_1+1)\,(m_2+1)\,(m_1+m_2+2)\,(m_1+2m_2+3) \qquad (3.86)$$

The smallest irreps. of $SO(5)$ (or $Sp(2)$) are shown in table 3.3.

We give in figures 3.4 and 3.5 the dimensions of the fundamental representations of the simple Lie algebras (extracted from [DYN 57]).

Figure 3.4: The dimensions of the fundamental representations of the classical Lie groups.

Figure 3.5: The dimensions of of the fundamental representations of the exceptional Lie groups.

3.8 Construction of matrix representations

We have seen that finite dimensional representations of compact Lie groups are equivalent to unitary ones (see theorem 3.1). In such representations the Cartan subalgebra generators and step operators can be chosen to satisfy[1]

$$H_i^\dagger = H_i \; ; \qquad E_\alpha^\dagger = E_{-\alpha} \tag{3.87}$$

We have chosen the basis of the representation to be formed by the eigenstates of the Cartan subalgebra generators. Using (3.1) and (3.87) we have

$$\langle \mu' \mid H_i \mid \mu \rangle = \mu_i \langle \mu' \mid \mu \rangle = \mu'_i \langle \mu' \mid \mu \rangle \tag{3.88}$$

and so

$$(\mu' - \mu) \langle \mu' \mid \mu \rangle = 0 \tag{3.89}$$

and consequently states with different weights are orthogonal. In the case a weight is degenerate, it is possible to find an orthogonal basis for the subspace generated by the states corresponding to that degenerate weight. We then shall denote the base states of the representation by $\mid \mu, k\rangle$ where μ is the corresponding weight and k is an integer number that runs from 1 to $m(\mu)$, the multiplicity of μ. We can always normalize these states such that

$$\langle \mu', k' \mid \mu, k \rangle = \delta_{\mu,\mu'}\, \delta_{kk'} \tag{3.90}$$

If T denotes an operator of the representation of the algebra then the matrix

$$D\,(T)_{(\mu',k')\,(\mu,k)} \equiv \langle \mu', k' \mid T \mid \mu, k \rangle \tag{3.91}$$

form a matrix representation since they reproduce the commutation relations of the algebra. Indeed

$$\begin{aligned}
[\, D\,(T)\,,\, D\,(T')\,]_{(\mu',k')\,(\mu,k)} &= \sum_{\mu'',k''} \langle \mu', k' \mid T \mid \mu'', k'' \rangle \langle \mu'', k'' \mid T' \mid \mu', k' \rangle \\
&- \sum_{\mu'',k''} \langle \mu', k' \mid T' \mid \mu'', k'' \rangle \langle \mu'', k'' \mid T \mid \mu', k' \rangle \\
&= \langle \mu', k' \mid [\, T\,,\, T'\,] \mid \mu', k' \rangle \\
&= D\,([\, T\,,\, T'\,])_{(\mu',k')\,(\mu,k)}
\end{aligned} \tag{3.92}$$

[1] In order to simplify the notation we will denote the operators $D\,(H_i)$ and $D\,(E_\alpha)$ by H_i and E_α respectively.

where we have used the fact that

$$\mathbb{1} = \sum_{\mu,k} \mid \mu, k\rangle\langle\mu, k \mid \tag{3.93}$$

is the identity operator.

When a step operator E_α acts on a state of weight μ, it either annihilates it or produces a state of weight $\mu+\alpha$. Therefore, using (3.93) and (3.90) one gets

$$\begin{aligned} E_\alpha \mid \mu, k\rangle &= \sum_{\mu',k'} \mid \mu', k'\rangle\langle\mu', k' \mid E_\alpha \mid \mu, k\rangle \\ &= \sum_{l=1}^{m(\mu+\alpha)} \mid \mu+\alpha, l\rangle\langle\mu+\alpha, l \mid E_\alpha \mid \mu, k\rangle \end{aligned} \tag{3.94}$$

where the sum is over the states of weight $\mu+\alpha$. Therefore, from (3.91) one has

$$D\left(E_\alpha\right)_{(\mu',k')\,(\mu,k)} = \langle\mu+\alpha, k' \mid E_\alpha \mid \mu, k\rangle\delta_{\mu',\mu+\alpha} \tag{3.95}$$

The matrix elements of H_i are known once we have the weights of the representation, since from (3.1) and (3.90)

$$D\left(H_i\right)_{(\mu',k')\,(\mu,k)} = \langle\mu', k' \mid H_i \mid \mu, k\rangle = \mu_i\, \delta_{\mu',\mu}\, \delta_{k',k} \tag{3.96}$$

Therefore, in order to construct the matrix representation of the algebra we have to calculate the "transition amplitudes" $\langle\mu+\alpha, l \mid E_\alpha \mid \mu, k\rangle$. Notice that from (3.87)

$$\langle\mu+\alpha, l \mid E_\alpha \mid \mu, k\rangle^\dagger = \langle\mu, k \mid E_{-\alpha} \mid \mu+\alpha, l\rangle \tag{3.97}$$

Now, using the commutation relation (see (2.223))

$$[\, E_\alpha\,,\, E_{-\alpha}\,] = \frac{2\alpha\cdot H}{\alpha^2} \tag{3.98}$$

one gets

$$\begin{aligned} \langle\mu, k \mid [\, E_\alpha\,,\, E_{-\alpha}\,] \mid \mu, k\rangle &= \langle\mu, k \mid \frac{2\alpha\cdot H}{\alpha^2} \mid \mu, k\rangle \\ &= \frac{2\alpha\cdot\mu}{\alpha^2} \\ &= \langle\mu, k \mid E_\alpha E_{-\alpha} \mid \mu, k\rangle - \langle\mu, k \mid E_{-\alpha} E_\alpha \mid \mu, k\rangle \\ &= \sum_{l=1}^{m(\mu-\alpha)} \langle\mu, k \mid E_\alpha \mid \mu-\alpha, l\rangle\langle\mu-\alpha, l \mid E_{-\alpha} \mid \mu, k\rangle \\ &- \sum_{l=1}^{m(\mu+\alpha)} \langle\mu, k \mid E_{-\alpha} \mid \mu+\alpha, l\rangle\langle\mu+\alpha, l \mid E_\alpha \mid \mu, k\rangle \end{aligned} \tag{3.99}$$

and so, using (3.97)

$$\sum_{l=1}^{m(\mu-\alpha)} \mid \langle \mu, k \mid E_\alpha \mid \mu - \alpha, l \rangle \mid^2 - \sum_{l=1}^{m(\mu+\alpha)} \mid \langle \mu + \alpha, l \mid E_\alpha \mid \mu, k \rangle \mid^2 = \frac{2\alpha \cdot \mu}{\alpha^2} \tag{3.100}$$

where $m(\mu + \alpha)$ and $m(\mu - \alpha)$ are the multiplicities of the weights $\mu + \alpha$ and $\mu - \alpha$ respectively.

The relation (3.100) can be used to calculate the modules of the transition amplitudes recursively. By taking α to be a positive root and μ the highest weight λ of the representation we have that the second term on the l.h.s. of (3.100) vanishes. Since, in a irrep., λ is not degenerate we can neglect the index k and write

$$\sum_{l=1}^{m(\mu-\alpha)} \mid \langle \lambda \mid E_\alpha \mid \mu - \alpha, l \rangle \mid^2 = \frac{2\alpha \cdot \lambda}{\alpha^2} = q \tag{3.101}$$

where, according to (3.41), q is the highest positive integer such that $\lambda - q\alpha$ is a weight of the representation. Taking now the second highest weight we repeat the process and so on.

The other relations that the transition amplitudes have to satisfy come from the commutation relations between step operators. If $\alpha + \beta$ is a root we have from (2.223)

$$\langle \mu + \alpha + \beta, l \mid [\, E_\alpha \,,\, E_\beta \,] \mid \mu, k \rangle = (q + 1)\, \varepsilon(\alpha, \beta) \langle \mu + \alpha + \beta, l \mid E_{\alpha+\beta} \mid \mu, k \rangle \tag{3.102}$$

Then using (3.90) and (3.94) one gets

$$\begin{aligned} & \sum_{k'=1}^{m(\mu+\beta)} \langle \mu + \alpha + \beta, l \mid E_\alpha \mid \mu + \beta, k' \rangle \langle \mu + \beta, k' \mid E_\beta \mid \mu, k \rangle \\ - & \sum_{k'=1}^{m(\mu+\alpha)} \langle \mu + \alpha + \beta, l \mid E_\beta \mid \mu + \alpha, k' \rangle \langle \mu + \alpha, k' \mid E_\alpha \mid \mu, k \rangle \\ = & \; (q + 1)\, \varepsilon(\alpha, \beta) \langle \mu + \alpha + \beta, l \mid E_{\alpha+\beta} \mid \mu, k \rangle \end{aligned} \tag{3.103}$$

where q is the highest positive integer such that $\beta - q\alpha$ (or equivalently $\alpha - q\beta$, since we are assuming $\alpha + \beta$ is a root) is a root, and $\varepsilon(\alpha, \beta)$ are signs determined from the Jacobi identities (see section 2.14)

We now give some examples to ilustrate how to use (3.100) and (3.103) to construct matrix representations. This method is very general and consequently difficult to use when the representation (or the algebra) is big. There are other methods which work better in specific cases.

3.8.1 The irreducible representations of $SU(2)$

In section 2.5 we have studied the representations of $SU(2)$. We have seen that the weights of $SU(2)$, denoted by m, are integers or half integers, and on a given irreducible representation with highest weight j they run from $-j$ to j in integer steps. The weights are non-degenerated and so the representations have dimensions $2j+1$. As we did in section 2.5 we shall denote the basis of the representation space as

$$\mid j, m\rangle \qquad\qquad m = -j, -j+1, \ldots, j-1, j \tag{3.104}$$

and they are orthonormal

$$\langle j, m' \mid j, m\rangle = \delta_{m,m'} \tag{3.105}$$

The Chevalley basis for $SU(2)$ satisfy the commutation relations

$$[\, H \,,\, E_{\pm} \,] = \pm E_{\pm} \qquad [\, E_{+} \,,\, E_{-} \,] = H \tag{3.106}$$

where $H = 2\alpha \cdot H/\alpha^2$, with α being the only positive root of $SU(2)$. In section 2.5 we have used the basis

$$[\, T_3 \,,\, T_{\pm} \,] = \pm T_{\pm} \qquad [\, T_{+} \,,\, T_{-} \,] = 2T_3 \tag{3.107}$$

and so we have $E_{\pm} \equiv T_{\pm}$ and $H \equiv 2T_3$. Since m are eigenvalues of T_3

$$T_3 \mid j, m\rangle = m \mid j, m\rangle \tag{3.108}$$

we get from (3.91) the matrix representing T_3 as

$$D^{(j)}_{m',m}\,(T_3) = \langle j, m' \mid T_3 \mid j, m\rangle = m\,\delta_{m,m'} \tag{3.109}$$

Using the relation (3.100), which is the same as taking the expectation value on the state $\mid j, m\rangle$ of both sides of the second relation in (3.107), we get

$$\mid \langle j, m \mid T_{+} \mid j, m-1\rangle \mid^2 - \mid \langle j, m+1 \mid T_{+} \mid j, m\rangle \mid^2 = 2m \tag{3.110}$$

where we have used the fact that $T_{+}^{\dagger} = T_{-}$ (see (3.87)). Notice that $T_{+} \mid j, j\rangle = 0$, since j is the highest weight and so

$$\mid \langle j, j \mid T_{+} \mid j, j-1\rangle \mid^2 = 2j \tag{3.111}$$

Clearly, such result could also be obtained directly from (3.101). The other matrix elements of T_{+} can then be obtained recursively from (3.110). Indeed, denoting $c_m \equiv\mid \langle j, m+1 \mid T_+ \mid j, m\rangle \mid^2$, we get $c_{j-1} = 2j$, $c_{j-2} = 2j + 2(j-1)$, $c_{j-3} = 2j + 2(j-1) + 2(j-2)$, and so

$$c_m = \sum_{l=0}^{j-m-1} 2(j-l) = (j-m)(j+m+1) = j(j+1) - m(m+1)$$

Therefore

$$\mid \langle j, m+1 \mid T_+ \mid j, m\rangle \mid^2 = j(j+1) - m(m+1) \qquad (3.112)$$

and since

$$\langle j, m+1 \mid T_+ \mid j, m\rangle^\dagger = \langle j, m \mid T_- \mid j, m+1\rangle \qquad (3.113)$$

we get

$$\mid \langle j, m-1 \mid T_- \mid j, m\rangle \mid^2 = j(j+1) - m(m-1) \qquad (3.114)$$

The phases of such matrix elements can be chosen to vanish, since in $SU(2)$ we do not have a relation like (3.103) to relate them. Therefore, we get

$$T_\pm \mid j, m\rangle = \sqrt{j(j+1) - m(m\pm 1)} \mid j, m\pm 1\rangle \qquad (3.115)$$

and so,

$$\begin{aligned} D^{(j)}_{m',m}(T_+) &= \langle j, m' \mid T_+ \mid j, m\rangle \\ &= \sqrt{j(j+1) - m(m+1)}\; \delta_{m',m+1} \\ D^{(j)}_{m',m}(T_-) &= \langle j, m' \mid T_- \mid j, m\rangle \\ &= \sqrt{j(j+1) - m(m-1)}\; \delta_{m',m-1} \end{aligned} \qquad (3.116)$$

3.8.2 The triplet representation of $SU(3)$

Consider the fundamental representation of $SU(3)$ with highest weight λ_1. In example 3.10 we have seen it has dimension 3, and in fact it is the so called triplet representation of $SU(3)$. From (3.4) we have

$$\frac{2\lambda_1 \cdot \alpha_1}{\alpha_1^2} = \frac{2\lambda_1 \cdot \alpha_3}{\alpha_3^2} = 1 \qquad (3.117)$$

where $\alpha_3 = \alpha_1 + \alpha_2$, α_1 and α_2 are the the simple roots of $SU(3)$. So,from (3.41) we get that λ_1, $(\lambda_1 - \alpha_1)$ and $(\lambda_1 - \alpha_3)$ are weights of the representation. Since the representation has dimension 3 it follows that they are the only weights and they are non-degenerate. Those weights are shown in figure 3.6.

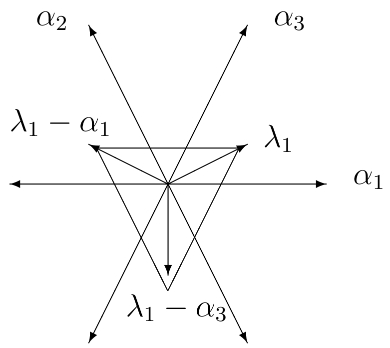

Figure 3.6: The weights of the triplet representation of $SU(3)$

Taking the Cartan subalgebra generators in the Chevalley basis we have

$$\langle \mu' \mid H_a \mid \mu \rangle = \frac{2\alpha_a \cdot \mu}{\alpha_a^2} \delta_{\mu',\mu} \qquad\qquad a = 1, 2 \tag{3.118}$$

where we have used (3.90), and where we have neglected the degeneracy index. From (3.4) and the Cartan matrix of $SU(3)$ (see example 2.13) we have

$$\begin{aligned} \frac{2\alpha_1 \cdot (\lambda_1 - \alpha_1)}{\alpha_1^2} &= -1 \qquad \frac{2\alpha_2 \cdot (\lambda_1 - \alpha_3)}{\alpha_2^2} = 1 \\ \frac{2\alpha_1 \cdot (\lambda_1 - \alpha_3)}{\alpha_1^2} &= 0 \qquad \frac{2\alpha_2 \cdot (\lambda_1 - \alpha_1)}{\alpha_2^2} = 1 \end{aligned} \tag{3.119}$$

Denoting the states as (as a matter of ordering the rows and columus of the matrices)

$$\mid 1 \rangle \equiv \mid \lambda_1 \rangle \; ; \qquad \mid 2 \rangle \equiv \mid \lambda_1 - \alpha_1 \rangle \; ; \qquad \mid 3 \rangle \equiv \mid \lambda_1 - \alpha_3 \rangle \tag{3.120}$$

we obtain from (3.117), (3.118), (3.119) and that the matrices representing the Cartan subalgebra generators are

$$D^{\lambda_1}(H_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad D^{\lambda_1}(H_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3.121}$$

Using (3.101) and (3.117) we have that

$$\mid \langle \lambda_1 \mid E_{\alpha_1} \mid \lambda_1 - \alpha_1 \rangle \mid^2 = \mid \langle \lambda_1 \mid E_{\alpha_3} \mid \lambda_1 - \alpha_3 \rangle \mid^2 = 1 \tag{3.122}$$

Making $\mu = \lambda_1 - \alpha_1$ and $\alpha = \alpha_2$ in (3.100) and using the fact that

$$\langle \lambda_1 - \alpha_1 + \alpha_2 \mid E_{\alpha_2} \mid \lambda_1 - \alpha_1 \rangle = 0 \tag{3.123}$$

since $\lambda_1 - \alpha_1 + \alpha_2$ is not weight, we get

$$\mid \langle \lambda_1 - \alpha_1 \mid E_{\alpha_2} \mid \lambda_1 - \alpha_1 - \alpha_2 \rangle \mid^2 = 1 \tag{3.124}$$

These are the only non vanishing "transition amplitudes". From (3.95) and (3.120) we see that the only non vanishing elements of the matrices representing the step operators are

$$\begin{aligned} D^{\lambda_1}(E_{\alpha_1}) &= \langle \lambda_1 \mid E_{\alpha_1} \mid \lambda_1 - \alpha_1 \rangle \equiv e^{i\theta} \\ D^{\lambda_1}(E_{\alpha_2}) &= \langle \lambda_1 - \alpha_1 \mid E_{\alpha_2} \mid \lambda_1 - \alpha_3 \rangle \equiv e^{i\varphi} \\ D^{\lambda_1}(E_{\alpha_3}) &= \langle \lambda_1 \mid E_{\alpha_3} \mid \lambda_1 - \alpha_3 \rangle \equiv e^{i\phi} \end{aligned} \tag{3.125}$$

where, according to (3.122) and (3.124), we have introduced the angles θ, ϕ and φ. The negative step operators are obtained from these ones using (3.87).

Choosing the cocycle $\varepsilon(\alpha_1, \alpha_2) = 1$ and since $\alpha_2 - \alpha_1$ is not a root, we have from (3.103) that the fases have to satisfy (set $\mu = \lambda_1 - \alpha_3$, $\alpha = \alpha_1$ and $\beta = \alpha_2$ in (3.103))

$$\theta + \varphi = \phi \tag{3.126}$$

There are no futher restrictions on these fases.

Therefore we get that the matrices which represent the step operators in the triplet representation are

$$\begin{aligned} D^{\lambda_1}(E_{\alpha_1}) &= \begin{pmatrix} 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & D^{\lambda_1}(E_{-\alpha_1}) &= \begin{pmatrix} 0 & 0 & 0 \\ e^{-i\theta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ D^{\lambda_1}(E_{\alpha_2}) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i\varphi} \\ 0 & 0 & 0 \end{pmatrix} & D^{\lambda_1}(E_{-\alpha_2}) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e^{-i\varphi} & 0 \end{pmatrix} \\ D^{\lambda_1}(E_{\alpha_3}) &= \begin{pmatrix} 0 & 0 & e^{i(\theta+\varphi)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & D^{\lambda_1}(E_{-\alpha_3}) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e^{-i(\theta+\varphi)} & 0 & 0 \end{pmatrix} \end{aligned} \tag{3.127}$$

In general, the fases θ and φ are chosen to vanish. The algebra of $SU(3)$ is generated by taking real linear combination of the matrices H_a $(a = 1, 2)$, $(E_\alpha + E_{-\alpha})$ and $(E_\alpha - E_{-\alpha})$. On the other hand the algebra of $SL(3)$ is generated by the same matrices but the third one does not have the factor i. Notice that in this way the triplet representation of the group $SU(3)$ is unitary whilst the triplet of $SL(3)$ is not.

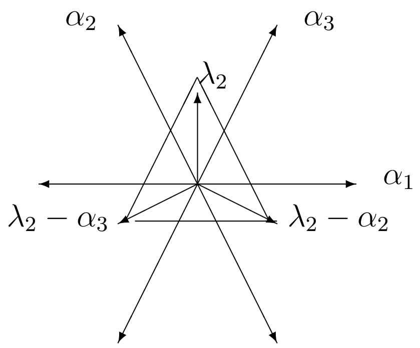

Figure 3.7: The weights of the anti-triplet representation of $SU(3)$

3.8.3 The anti-triplet representation of $SU(3)$

We now consider the other fundamental representation of $SU(3)$ which has highest weight λ_2 . In example 3.10 we saw it also has diemnsion 3 and it is the anti-triplet of $SU(3)$. Using (3.4) we get that the weight are λ_2, $\lambda_2 - \alpha_2$ and $\lambda_2 - \alpha_3$ and consequently they are not degenerate. They are shown in figure 3.7.

We shall denote the states as

$$\mid 1\rangle \equiv\mid \lambda_2\rangle \; ; \qquad \mid 2\rangle \equiv\mid \lambda_2 - \alpha_2\rangle \; ; \qquad \mid 3\rangle \equiv\mid \lambda_2 - \alpha_3\rangle \tag{3.128}$$

Using the Cartan matrix of $SU(3)$ (see example 2.13), (3.4) and (3.118) we get that the matrices which represent the Cartan subalgebra generators in the Chevalley basis are

$$D^{\lambda_2}(H_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad D^{\lambda_2}(H_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.129}$$

Using (3.101) we have that

$$\mid \langle \lambda_2 \mid E_{\alpha_2} \mid \lambda_2 - \alpha_2\rangle \mid^2 = \mid \langle \lambda_2 \mid E_{\alpha_3} \mid \lambda_2 - \alpha_3\rangle \mid^2 = 1 \tag{3.130}$$

and from (3.100)

$$\mid \langle \lambda_2 - \alpha_2 \mid E_{\alpha_1} \mid \lambda_2 - \alpha_1 - \alpha_2\rangle \mid^2 = 1 \tag{3.131}$$

Using (3.95) we get that the only non vanishing matrix elements of the step operators are

$$\begin{aligned} D^{\lambda_2}(E_{\alpha_1}) &= \langle \lambda_2 - \alpha_2 \mid E_{\alpha_1} \mid \lambda_2 - \alpha_3 \rangle \equiv e^{i\theta} \\ D^{\lambda_2}(E_{\alpha_2}) &= \langle \lambda_2 \mid E_{\alpha_2} \mid \lambda_2 - \alpha_2 \rangle \equiv e^{i\varphi} \\ D^{\lambda_2}(E_{\alpha_3}) &= \langle \lambda_2 \mid E_{\alpha_3} \mid \lambda_2 - \alpha_3 \rangle \equiv e^{i\phi} \end{aligned} \tag{3.132}$$

where, according to (3.130) and (3.131), we have introduced the fases θ, φ and ϕ. From (3.87) we obtain the matrices for the negative step operators. Using the fact that $(q+1)\,\varepsilon(\alpha_1, \alpha_2) = 1$ we get from (3.103) that these fases have to satisfy

$$\theta + \varphi = \phi + \pi \tag{3.133}$$

Therefore the matrices which represent the step operators in the anti-triplet representation are

$$\begin{aligned} D^{\lambda_2}(E_{\alpha_1}) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i\theta} \\ 0 & 0 & 0 \end{pmatrix} \qquad D^{\lambda_2}(E_{-\alpha_1}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e^{-i\theta} & 0 \end{pmatrix} \\ D^{\lambda_2}(E_{\alpha_2}) &= \begin{pmatrix} 0 & e^{i\varphi} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad D^{\lambda_2}(E_{-\alpha_2}) = \begin{pmatrix} 0 & 0 & 0 \\ e^{-i\varphi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ D^{\lambda_2}(E_{\alpha_3}) &= -\begin{pmatrix} 0 & 0 & e^{i(\theta+\varphi)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad D^{\lambda_2}(E_{-\alpha_3}) = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e^{-i(\theta+\varphi)} & 0 & 0 \end{pmatrix} \end{aligned} \tag{3.134}$$

So, these matrices are obtained from those of the triplet by making the change $E_{\pm\alpha_1} \leftrightarrow E_{\pm\alpha_2}$ and $E_{\pm\alpha_3} \leftrightarrow -E_{\pm\alpha_3}$. From (3.121) and (3.129) we see the Cartan subalgebra generators are also interchanged.

3.9 Tensor product of representations

We have seen in definition 1.12 of section 1.5 the concept of tensor product of representations. The idea is quite simple. Consider two irreducible representations D^{λ} and $D^{\lambda'}$ of a Lie group G, with highest weights λ and λ' and representation spaces V^{λ} and $V^{\lambda'}$ respectively. We can construct a third representation by considering the tensor product space $V^{\lambda\otimes\lambda'} \equiv V^{\lambda} \otimes V^{\lambda'}$. The operators representing the group elements in the tensor product representation are

$$D^{\lambda\otimes\lambda'}(g) \equiv D^{\lambda}(g) \otimes D^{\lambda'}(g) \tag{3.135}$$

and they act as

$$D^{\lambda\otimes\lambda'}(g)\, V^{\lambda\otimes\lambda'} = D^{\lambda}(g)\, V^{\lambda} \otimes D^{\lambda'}(g)\, V^{\lambda'} \tag{3.136}$$

They form a representation since

$$\begin{aligned} D^{\lambda\otimes\lambda'}(g_1)\, D^{\lambda\otimes\lambda'}(g_2) &= D^{\lambda}(g_1)\, D^{\lambda}(g_2) \otimes D^{\lambda'}(g_1)\, D^{\lambda'}(g_2) \\ &= D^{\lambda}(g_1 g_2) \otimes D^{\lambda'}(g_1 g_2) \\ &= D^{\lambda\otimes\lambda'}(g_1 g_2) \end{aligned} \tag{3.137}$$

The operators representing the elements T of the Lie algebra $\mathcal{G}$ of G are given by

$$D^{\lambda\otimes\lambda'}(T) \equiv D^{\lambda}(T) \otimes \mathbb{1} + \mathbb{1} \otimes D^{\lambda'}(T) \tag{3.138}$$

Indeed

$$\begin{aligned} \left[\, D^{\lambda\otimes\lambda'}(T_1)\, ,\, D^{\lambda\otimes\lambda'}(T_2) \right] &= \left[\, D^{\lambda}(T_1)\, ,\, D^{\lambda}(T_1) \right] \otimes \mathbb{1} \\ &+ \mathbb{1} \otimes \left[\, D^{\lambda'}(T_1)\, ,\, D^{\lambda'}(T_1) \right] \\ &= D^{\lambda}([\,T_1\, ,\, T_2\,]) \otimes \mathbb{1} + \mathbb{1} \otimes D^{\lambda'}([\,T_1\, ,\, T_2\,]) \\ &= D^{\lambda\otimes\lambda'}([\,T_1\, ,\, T_2\,]) \end{aligned} \tag{3.139}$$

Notice that if $\mid \mu, l\rangle$ and $\mid \mu', l'\rangle$ are states of the representations V^{λ} and $V^{\lambda'}$ with weights μ and μ' respectively, one gets

$$\begin{aligned} D^{\lambda\otimes\lambda'}(H_i) \mid \mu, l\rangle \otimes \mid \mu', l'\rangle &= D^{\lambda}(H_i) \mid \mu, l\rangle \otimes \mid \mu', l'\rangle \\ &+ \mid \mu, l\rangle \otimes D^{\lambda'}(H_i) \mid \mu', l'\rangle \\ &= (\mu_i + \mu'_i) \mid \mu, l\rangle \otimes \mid \mu', l'\rangle \end{aligned} \tag{3.140}$$

It then follows that the weigths of the representation $V^{\lambda\otimes\lambda'}$ are the sums of all weights of V^{λ} with all weights of $V^{\lambda'}$. If λ and λ' are the highest weights of V^{λ} and $V^{\lambda'}$ respectively, then the highest weight of $V^{\lambda\otimes\lambda'}$ is $\lambda+\lambda'$, and the corresponding state is

$$\mid \lambda + \lambda'\rangle = \mid \lambda\rangle \otimes \mid \lambda'\rangle \tag{3.141}$$

which is clearly non-degenerate.

In general, the representation $V^{\lambda\otimes\lambda'}$ is reducible and one can split it as the sum of irreducible representations of G

$$V^{\lambda\otimes\lambda'} = \oplus_{\lambda''} V^{\lambda''} \tag{3.142}$$

where $V^{\lambda''}$ are irreducible representations with highest weight λ''. The decomposition (3.142) is called the *branching* of the representation $V^{\lambda\otimes\lambda'}$.

Taking orthonormal basis $\mid \mu, l\rangle$ and $\mid \mu', l'\rangle$ for V^{λ} and $V^{\lambda'}$ respectively, we can construct an orthonormal basis for $V^{\lambda\otimes\lambda'}$ as

$$\mid \mu + \mu', k\rangle = \sum_{l=1}^{m(\mu)} \sum_{l'=1}^{m(\mu')} C^{k}_{l,l'} \mid \mu, l\rangle \otimes \mid \mu', l'\rangle \tag{3.143}$$

where $m(\mu)$ and $m(\mu')$ are the multiplicities of μ and μ' in V^{λ} and $V^{\lambda'}$ respectively, and $k = 1, 2, \ldots m(\mu + \mu')$, with $m(\mu + \mu')$ being the multiplicity of $\mu + \mu'$ in $V^{\lambda\otimes\lambda'}$. Clearly, $m(\mu + \mu') = m(\mu)\, m(\mu')$. The constants $C^{k}_{l,l'}$ are the so-called *Clebsch-Gordan coefficients*.

Example 3.12 *Let us consider the tensor product of two spinorial representations of $SU(2)$. As discussed in section 3.8.1 it is a two dimensional representation with states $\mid \frac{1}{2}, \frac{1}{2}\rangle$ and $\mid \frac{1}{2}, -\frac{1}{2}\rangle$, and satisfying*

$$T_3 \mid \tfrac{1}{2}, \pm\tfrac{1}{2}\rangle = \pm\frac{1}{2} \mid \tfrac{1}{2}, \pm\tfrac{1}{2}\rangle \tag{3.144}$$

and (see (3.115))

$$\begin{aligned} T_{+} \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle &= 0\,; & T_{+} \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle &= \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \\ T_{-} \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle &= \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle\,; & T_{-} \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle &= 0 \end{aligned} \tag{3.145}$$

One can easily construct the irreducible components by taking the highest weight state $\mid \frac{1}{2}, \frac{1}{2}\rangle \otimes \mid \frac{1}{2}, \frac{1}{2}\rangle$ and act with the lowering operator. One gets

$$\begin{aligned} D^{\frac{1}{2}\otimes\frac{1}{2}}(T_{-}) \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle &= (T_{-} \otimes \mathbb{1} + \mathbb{1} \otimes T_{-}) \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \\ &= \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle + \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \end{aligned}$$

and

$$\left(D^{\frac{1}{2}\otimes\frac{1}{2}}(T_{-})\right)^2 \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle = 2 \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \tag{3.146}$$

and

$$\left(D^{\frac{1}{2}\otimes\frac{1}{2}}(T_{-})\right)^3 \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle = 0 \tag{3.147}$$

On the other hand notice that

$$D^{\frac{1}{2}\otimes\frac{1}{2}}(T_{\pm}) \left(\mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle - \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle\right) = 0 \tag{3.148}$$

Therefore, one gets that the states

$$\begin{aligned} \mid 1, 1\rangle &\equiv \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \\ \mid 1, 0\rangle &\equiv \left(\mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle + \mid \tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle\right)/\sqrt{2} \\ \mid 1, -1\rangle &\equiv \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes \mid \tfrac{1}{2}, -\tfrac{1}{2}\rangle \end{aligned} \tag{3.149}$$

constitute a triplet representation (spin 1*) of* $SU(2)$*.*

The state

$$\mid 0,0\rangle \equiv \left(\mid \tfrac{1}{2},-\tfrac{1}{2}\rangle\otimes \mid \tfrac{1}{2},\tfrac{1}{2}\rangle - \mid \tfrac{1}{2},\tfrac{1}{2}\rangle\otimes \mid \tfrac{1}{2},-\tfrac{1}{2}\rangle\right)/\sqrt{2} \qquad (3.150)$$

constitute a scalar representation (spin 0*) of* $SU(2)$*.*

The branching of the tensor product representation is usually denoted in terms of the dimensions of the irreducible representations, and in such case we have

$$\mathbf{2}\otimes\mathbf{2}=\mathbf{3}+\mathbf{1} \qquad (3.151)$$

Given an irreducible representation D of a group G one observes that it is also a representation of any subgroup H of G. However, it will in general be a reducible representation of the subgroup. The decomposition of D in terms of irreducible representations of H is called the branching of D. In order to illustrate it let us discuss some examples.

Example 3.13 *The operator* T_3 *generates a subgroup* $U(1)$ *of* $SU(2)$ *(see (3.107)). From the considerations in 3.8.1 one observes that each state* $\mid j,m\rangle$ *constitutes a scalar representation of such* $U(1)$ *subgroup. Therefore, each spin* j *representation of* $SU(2)$ *decomposes into* $2j+1$ *scalars representation of* $U(1)$*.*

Example 3.14 *In example 3.6 we have seen that weights of the adjoint representation of* $SU(3)$ *are its roots plus the null weight which is two-fold degenerate. So, let us denote the states as*

$$\mid \pm\alpha_1\rangle\;;\quad \mid \pm\alpha_2\rangle\;;\quad \mid \pm\alpha_3\rangle\;;\quad \mid 0\rangle\;;\quad \mid 0'\rangle \qquad (3.152)$$

Consider the $SU(2)\otimes U(1)$ *subgroup of* $SU(3)$ *generated by*

$$\begin{aligned} SU(2) &\equiv \{E_{\pm\alpha_1},\frac{2\alpha_1\cdot H}{\alpha_1^2}\} \\ U(1) &\equiv \{\frac{2\lambda_2\cdot H}{\alpha_2^2}\} \end{aligned} \qquad (3.153)$$

One can define the state $\mid 0\rangle$ *as*

$$\mid 0\rangle \equiv E_{-\alpha_1}\mid \alpha_1\rangle \qquad (3.154)$$

and consequently the states

$$\mid \alpha_1\rangle\;;\qquad \mid 0\rangle\;;\qquad \mid -\alpha_1\rangle \qquad (3.155)$$

constitute a triplet representation of the $SU(2)$ *defined above. In addition, the states*

$$\mid \alpha_2\rangle \; ; \qquad \mid \alpha_3\rangle \tag{3.156}$$

and

$$\mid -\alpha_3\rangle \; ; \qquad \mid -\alpha_2\rangle \tag{3.157}$$

constitute two dublet representations of the same $SU(2)$*.*

By taking $\mid 0'\rangle$ *to be orthogonal to* $\mid 0\rangle$ *one gets that it is a singlet representation of* $SU(2)$*.*

Clearly, each state $\mid \mu\rangle$ *in (3.152) constitute a scalar representation of the* $U(1)$ *subgroup with eigenvalue* $2\lambda_2 \cdot \mu/\alpha_2^2$*. Since,* $U(1)$ *commutes with the* $SU(2)$ *it follows the states of a given irreducible representation of* $SU(2)$ *have to have the same eigenvalue fo the* $U(1)$*. Therefore, we have got the following branching of the adjoint of* $SU(3)$ *in terms of irreps. of* $SU(2) \otimes U(1)$

$$\mathbf{8} = \mathbf{3}(0) + \mathbf{2}(1) + \mathbf{2}(-1) + \mathbf{1}(0) \tag{3.158}$$

where the numbers inside the parenteses are the $U(1)$ *eigenvalues.*

Bibliography

[ALD 86] R. Aldrovandi and J.G. Pereira, *An introduction to geometrical Physics*, World Scientific (1995).

[AUM 77] L. Auslander and R.E. Mackenzie, *Introduction to Differential Manifolds*, Dover Publ., Inc., New York (1977).

[BAR 77] A. O. Barut and R. Raczka, *Theory of group representations and applications*, Polish Scientific Publishers (1977).

[BUD 72] F. J. Budden; "The Fascination of Groups"; Cambridge University Press (1972).

[CBW 82] Y. Choquet-Bruhat, C. De Witt-Morette and M. Dillard-Bleick, *Analysis, Manifolds and Physics*, North-Holland Publ. Co. (1982).

[COR 84] J.F. Cornwell; *Group theory in Physics*; vols. I, II and III; Techniques in Physics 7; Academic Press (1984).

[DYN 57] E.B. Dynkin, *Transl. Amer. Math. Soc.* **(2) 6** (1957) 111, and **(1) 9** (1962) 328.

[FLA 63] H. Flanders, *Differential Forms with Applications to the Physical Sciences*, Academic Press (1957).

[HAM 62] M. Hamermesh; "Group Theory and its Applications to Physical Problems" ; Addison-Wesley Publ. Comp. (1962).

[HEL 78] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press (1978).

[HUM 72] J.E. Humphreys, *Introduction to Lie Algebra and Representation Theory*, Graduate Texts in Mathematics, Vol. 9, Springer-Verlag (1972).

[JAC 79] Nathan Jacobson; *Lie Algebras*, Dover Publ., Inc. (1979).

[LEZ 92] A. N. Leznov and M. V. Saveliev, *Group–Theoretical Methods for Integration of Nonlinear Dynamical Systems*, Progress in Physics Series, v. 15, Birkhaüser–Verlag, Basel, 1992.

[OLI 82] D. I. Olive, *Lectures on gauge theories and Lie algebras: with some applications to spontaneous symmetry breaking and integrable dynamical systems*, University of Virginia preprint (1982).

Index