# 45. Clebsch-Gordan Coefficients, Spherical Harmonics, and $d$ Functions

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

| $J$ | $J$ | ... |
|---|---|---|
| $M$ | $M$ | ... |
| $m_1$ $m_2$ | Coefficients | |
| $m_1$ $m_2$ | | |
| . . | | |
| . . | | |
| . . | | |

**1/2×1/2**

| $M$ | $J=1$ | $J=0$ |
|---|---|---|
| +1 | 1 | |
| +1/2 +1/2 | 1 | 0 |
| +1/2 −1/2 | 1/2 | 1/2 |
| −1/2 +1/2 | 1/2 | −1/2 |
| −1/2 −1/2 | 1 | |

**1×1/2**

| $M$ | $J=3/2$ | $J=1/2$ |
|---|---|---|
| +3/2 | 1 | |
| +1 +1/2 | 1 | |
| +1 −1/2 | 1/3 | 2/3 |
| 0 +1/2 | 2/3 | −1/3 |
| 0 −1/2 | 2/3 | 1/3 |
| −1 +1/2 | 1/3 | −2/3 |
| −1 −1/2 | 1 | |

**2×1**

| $M$ | $J=3$ | $J=2$ | $J=1$ |
|---|---|---|---|
| +3 | 1 | | |
| +2 +1 | 1 | | |
| +2 0 | 1/3 | 2/3 | |
| +1 +1 | 2/3 | −1/3 | |
| +2 −1 | 1/15 | 1/3 | 3/5 |
| +1 0 | 8/15 | 1/6 | −3/10 |
| 0 +1 | 2/5 | −1/2 | 1/10 |
| +1 −1 | 1/5 | 1/2 | 3/10 |
| 0 0 | 3/5 | 0 | −2/5 |
| −1 +1 | 1/5 | −1/2 | 3/10 |
| 0 −1 | 2/5 | 1/2 | 1/10 |
| −1 0 | 8/15 | −1/6 | −3/10 |
| −1 −1 | 2/3 | 1/3 | |
| −2 0 | 1/3 | −2/3 | |
| −2 −1 | 1 | | |

**1×1**

| $M$ | $J=2$ | $J=1$ | $J=0$ |
|---|---|---|---|
| +2 | 1 | | |
| +1 +1 | 1 | | |
| +1 0 | 1/2 | 1/2 | |
| 0 +1 | 1/2 | −1/2 | |
| +1 −1 | 1/6 | 1/2 | 1/3 |
| 0 0 | 2/3 | 0 | −1/3 |
| −1 +1 | 1/6 | −1/2 | 1/3 |
| 0 −1 | 1/2 | 1/2 | |
| −1 0 | 1/2 | −1/2 | |
| −1 −1 | 1 | | |

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

**2×1/2**

| $M$ | $J=5/2$ | $J=3/2$ |
|---|---|---|
| +5/2 | 1 | |
| +2 +1/2 | 1 | |
| +2 −1/2 | 1/5 | 4/5 |
| +1 +1/2 | 4/5 | −1/5 |
| +1 −1/2 | 2/5 | 3/5 |
| 0 +1/2 | 3/5 | −2/5 |
| 0 −1/2 | 3/5 | 2/5 |
| −1 +1/2 | 2/5 | −3/5 |
| −1 −1/2 | 4/5 | 1/5 |
| −2 +1/2 | 1/5 | −4/5 |
| −2 −1/2 | 1 | |

**3/2×1/2**

| $M$ | $J=2$ | $J=1$ |
|---|---|---|
| +2 | 1 | |
| +3/2 +1/2 | 1 | |
| +3/2 −1/2 | 1/4 | 3/4 |
| +1/2 +1/2 | 3/4 | −1/4 |
| +1/2 −1/2 | 1/2 | 1/2 |
| −1/2 +1/2 | 1/2 | −1/2 |
| −1/2 −1/2 | 3/4 | 1/4 |
| −3/2 +1/2 | 1/4 | −3/4 |
| −3/2 −1/2 | 1 | |

**3/2×1**

| $M$ | $J=5/2$ | $J=3/2$ | $J=1/2$ |
|---|---|---|---|
| +5/2 | 1 | | |
| +3/2 +1 | 1 | | |
| +3/2 0 | 2/5 | 3/5 | |
| +1/2 +1 | 3/5 | −2/5 | |
| +3/2 −1 | 1/10 | 2/5 | 1/2 |
| +1/2 0 | 3/5 | 1/15 | −1/3 |
| −1/2 +1 | 3/10 | −8/15 | 1/6 |
| +1/2 −1 | 3/10 | 8/15 | 1/6 |
| −1/2 0 | 3/5 | −1/15 | −1/3 |
| −3/2 +1 | 1/10 | −2/5 | 1/2 |
| −1/2 −1 | 3/5 | 2/5 | |
| −3/2 0 | 2/5 | −3/5 | |
| −3/2 −1 | 1 | | |

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

**3/2×3/2**

| $M$ | $J=3$ | $J=2$ | $J=1$ | $J=0$ |
|---|---|---|---|---|
| +3 | 1 | | | |
| +3/2 +3/2 | 1 | | | |
| +3/2 +1/2 | 1/2 | 1/2 | | |
| +1/2 +3/2 | 1/2 | −1/2 | | |
| +3/2 −1/2 | 1/5 | 1/2 | 3/10 | |
| +1/2 +1/2 | 3/5 | 0 | −2/5 | |
| −1/2 +3/2 | 1/5 | −1/2 | 3/10 | |
| +3/2 −3/2 | 1/20 | 1/4 | 9/20 | 1/4 |
| +1/2 −1/2 | 9/20 | 1/4 | −1/20 | −1/4 |
| −1/2 +1/2 | 9/20 | −1/4 | −1/20 | 1/4 |
| −3/2 +3/2 | 1/20 | −1/4 | 9/20 | −1/4 |
| +1/2 −3/2 | 1/5 | 1/2 | 3/10 | |
| −1/2 −1/2 | 3/5 | 0 | −2/5 | |
| −3/2 +1/2 | 1/5 | −1/2 | 3/10 | |
| −1/2 −3/2 | 1/2 | 1/2 | | |
| −3/2 −1/2 | 1/2 | −1/2 | | |
| −3/2 −3/2 | 1 | | | |

**2×3/2**

| $M$ | $J=7/2$ | $J=5/2$ | $J=3/2$ | $J=1/2$ |
|---|---|---|---|---|
| +7/2 | 1 | | | |
| +2 +3/2 | 1 | | | |
| +2 +1/2 | 3/7 | 4/7 | | |
| +1 +3/2 | 4/7 | −3/7 | | |
| +2 −1/2 | 1/7 | 16/35 | 2/5 | |
| +1 +1/2 | 4/7 | 1/35 | −2/5 | |
| 0 +3/2 | 2/7 | −18/35 | 1/5 | |
| +2 −3/2 | 1/35 | 6/35 | 2/5 | 2/5 |
| +1 −1/2 | 12/35 | 5/14 | 0 | −3/10 |
| 0 +1/2 | 18/35 | −3/35 | −1/5 | 1/5 |
| −1 +3/2 | 4/35 | −27/70 | 2/5 | −1/10 |
| +1 −3/2 | 4/35 | 27/70 | 2/5 | 1/10 |
| 0 −1/2 | 18/35 | 3/35 | −1/5 | −1/5 |
| −1 +1/2 | 12/35 | −5/14 | 0 | 3/10 |
| −2 +3/2 | 1/35 | −6/35 | 2/5 | −2/5 |
| 0 −3/2 | 2/7 | 18/35 | 1/5 | |
| −1 −1/2 | 4/7 | −1/35 | −2/5 | |
| −2 +1/2 | 1/7 | −16/35 | 2/5 | |
| −1 −3/2 | 4/7 | 3/7 | | |
| −2 −1/2 | 3/7 | −4/7 | | |
| −2 −3/2 | 1 | | | |

**2×2**

| $M$ | $J=4$ | $J=3$ | $J=2$ | $J=1$ | $J=0$ |
|---|---|---|---|---|---|
| +4 | 1 | | | | |
| +2 +2 | 1 | | | | |
| +2 +1 | 1/2 | 1/2 | | | |
| +1 +2 | 1/2 | −1/2 | | | |
| +2 0 | 3/14 | 1/2 | 2/7 | | |
| +1 +1 | 4/7 | 0 | −3/7 | | |
| 0 +2 | 3/14 | −1/2 | 2/7 | | |
| +2 −1 | 1/14 | 3/10 | 3/7 | 1/5 | |
| +1 0 | 3/7 | 1/5 | −1/14 | −3/10 | |
| 0 +1 | 3/7 | −1/5 | −1/14 | 3/10 | |
| −1 +2 | 1/14 | −3/10 | 3/7 | −1/5 | |
| +2 −2 | 1/70 | 1/10 | 2/7 | 2/5 | 1/5 |
| +1 −1 | 8/35 | 2/5 | 1/14 | −1/10 | −1/5 |
| 0 0 | 18/35 | 0 | −2/7 | 0 | 1/5 |
| −1 +1 | 8/35 | −2/5 | 1/14 | 1/10 | −1/5 |
| −2 +2 | 1/70 | −1/10 | 2/7 | −2/5 | 1/5 |
| +1 −2 | 1/14 | 3/10 | 3/7 | 1/5 | |
| 0 −1 | 3/7 | 1/5 | −1/14 | −3/10 | |
| −1 0 | 3/7 | −1/5 | −1/14 | 3/10 | |
| −2 +1 | 1/14 | −3/10 | 3/7 | −1/5 | |
| 0 −2 | 3/14 | 1/2 | 2/7 | | |
| −1 −1 | 4/7 | 0 | −3/7 | | |
| −2 0 | 3/14 | −1/2 | 2/7 | | |
| −1 −2 | 1/2 | 1/2 | | | |
| −2 −1 | 1/2 | −1/2 | | | |
| −2 −2 | 1 | | | | |

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

**Figure 45.1**: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).