Postulado 5

Exemplo

$$\hat{H}|n\rangle = E_n|n\rangle$$

$$\boxed{|1\rangle, |2\rangle, |3\rangle, |4\rangle}$$

$E_1 = E_2 = \mathcal{E}$

$E_3 = 2\mathcal{E}$

$E_4 = 3\mathcal{E}$

$$\boxed{|\psi\rangle = 3|1\rangle + |2\rangle - |3\rangle + 7|4\rangle}$$

a) Qual a prob. de medir $\mathcal{E}$?

b) Qual é o estado imediatamente após medir $\mathcal{E}$?

c) Qual o estado do sistema após $\Delta t = \tau$

a) $\langle\psi|\psi\rangle = 1 \rightarrow \left(3\langle 1| + 2\langle 2| \ldots\right)\left(3|1\rangle + |2\rangle - |3\rangle + 7|4\rangle\right)$

$$= 9\langle 1|1\rangle + 1\langle 2|2\rangle + 1\langle 3|3\rangle + 49\langle 4|4\rangle$$

$$= 60$$

$$|\psi'\rangle = \frac{|\psi\rangle}{\sqrt{60}} \rightarrow \langle\psi'|\psi'\rangle = 1$$

$$|\phi\rangle = \frac{|\psi\rangle}{\sqrt{\langle\psi|\psi\rangle}}$$

a) $$P_\mathcal{E} = \sum_{i=1}^{2} |c_n|^2 = \frac{9+1}{60} = \frac{1}{6}$$

$$|\psi_\mathcal{E}\rangle = \frac{3}{\sqrt{10}}|1\rangle + \frac{1}{\sqrt{10}}|2\rangle$$

b) $$|\psi_\mathcal{E}\rangle = \frac{\hat{P}_\mathcal{E}|\phi\rangle}{\sqrt{\langle\phi|\hat{P}_\mathcal{E}|\phi\rangle}} = \frac{1}{\sqrt{P_\mathcal{E}}}\hat{P}_\mathcal{E}|\phi\rangle$$

$$= \sqrt{6}\cdot\left(\frac{3}{\sqrt{60}}|1\rangle + \frac{1}{\sqrt{60}}|2\rangle\right)$$

$$\begin{cases} \hat{x} = x \cdot \\ \hat{P}_x = -i\hbar\partial_x \end{cases}$$

$$\hookrightarrow [\hat{x}, \hat{p}] = i\hbar$$

$$(\hat{x}\hat{p} - \hat{p}\hat{x})\psi$$

$$\hat{x}[\hat{p}[\cdot]] - \hat{p}[\hat{x}[\cdot]] \neq 0$$

$$x \cdot \partial_x(\cdot) - \partial_x(x \cdot) \neq 0$$

$$e^{i\frac{\hat{H}(t-t_0)}{\hbar}} \approx e^{ix} \rightsquigarrow e^{x}$$

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-ix} = 1 - ix - \frac{x^2}{2} + \frac{ix^3}{6} + \dots$$

$$e^{-i\hat{H}t/\hbar} = \hat{1} - \frac{i}{\hbar}\hat{H}t + \dots$$

$$\hat{A}[f(x)] \to g(x)$$

$$\hookrightarrow \hat{A} f(x) = x \cdot f(x)$$

$$\begin{cases} \psi(x) = \langle x | \psi \rangle \\ \psi(p) = \langle p | \psi \rangle \end{cases}$$

Exemplo 1: Suponha $|A\rangle = \sum_i A_i |e_i\rangle$ ortonormal

e que $|A\rangle = \sum_j A'_j |e'_j\rangle \Rightarrow$ Como determinar A'_j

Solução: usar a relação de completeza

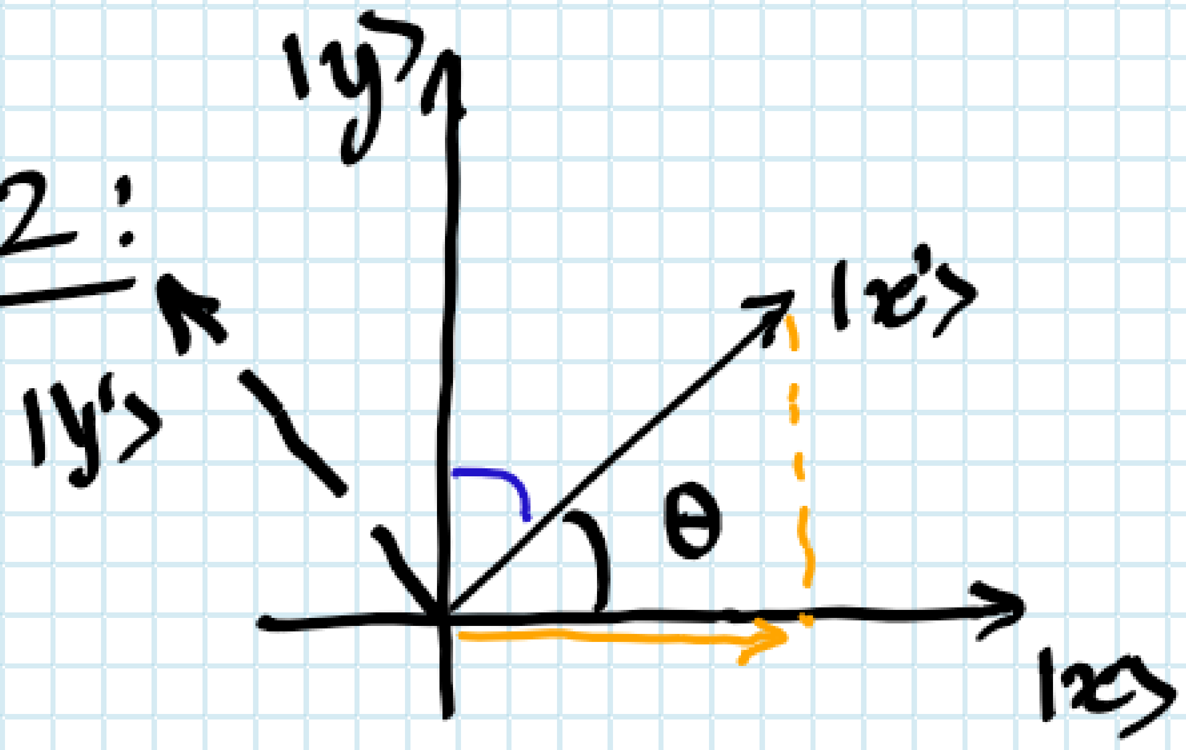
$$|A\rangle = \sum A_i \hat{1} |e_i\rangle = \sum_i A_i \sum_j |e'_j\rangle\langle e'_j|e_i\rangle$$

$$\hat{1} = \sum_j |e'_j\rangle\langle e'_j| \qquad |e_i\rangle$$

$$= \sum_j \underbrace{\sum_i A_i \langle e'_j|e_i\rangle}_{A'_j} |e'_j\rangle = \sum_j A'_j |e'_j\rangle$$

A'_j combinação linear

Exemplo 2:



$$\langle x'|x\rangle = \cos\theta$$
$$\rightarrow \langle x'|y\rangle = \cos(\pi/2 - \theta) = \operatorname{sen}\theta$$
$$\langle y'|y\rangle = \cos\theta$$
$$\langle y'|x\rangle = -\operatorname{sen}\theta$$

Alternativa 1

$$\langle x'|A\rangle = \langle x'|\hat{1}\,|A\rangle$$

$$= \langle x'| \underbrace{\left(|x\rangle\langle x| + |y\rangle\langle y|\right)}_{\hat{1} = \sum_i |e_i\rangle\langle e_i|} |A\rangle$$

$$= \langle x'|x\rangle \underbrace{\langle x|A\rangle}_{A_x} + \langle x'|y\rangle \underbrace{\langle y|A\rangle}_{A_y}$$

$$= \cos\theta \cdot A_x + \operatorname{sen}\theta \cdot A_y$$

$$\langle y'|A\rangle = (-\operatorname{sen}\theta) \cdot A_x + \cos\theta \cdot A_y$$

$$\Rightarrow \quad A'_x = \cos\theta\, A_x + \operatorname{sen}\theta\, A_y$$
$$A'_y = -\operatorname{sen}\theta\, A_x + \cos\theta\, A_y$$

$$\begin{pmatrix} A'_x \\ A'_y \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ -\operatorname{sen}\theta & \cos\theta \end{pmatrix}}_{R(\theta)} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

Alternativa 2

$$|A\rangle = \hat{1}\,|A\rangle = \left(|x'\rangle\langle x'| + |y'\rangle\langle y'|\right)|A\rangle$$

$$= \underbrace{\left(A_x\langle x'|x\rangle + A_y\langle x'|y\rangle\right)}_{A'_x}|x'\rangle + \underbrace{\left(A_x\langle y'|x\rangle + \langle y'|y\rangle A_y\right)}_{A'_y}|y'\rangle$$

Exemplo 3:

$$\langle E\rangle = \langle \hat{H}\rangle \Rightarrow \qquad \langle \hat{H}\rangle = \langle \psi|\hat{H}|\psi\rangle$$

$$\langle \hat{H}\rangle = \left(\sum_m c_m^* \langle e_m|\right) \hat{H} \left(\sum_n c_n |e_n\rangle\right)$$

$$= \sum_m \sum_n c_m^* c_n \langle e_m|\underbrace{\hat{H}|e_n\rangle}_{E_n|e_n\rangle} =$$

$$= \sum_m \sum_n c_m^* c_n E_n \underbrace{\langle e_m|e_n\rangle}_{\delta_{mn}}$$

$$\langle \hat{H}\rangle = \sum_n c_n^* c_n E_n = \sum_n |c_n|^2 E_n = \sum_n \underbrace{\rho_n E_n}$$

Regra de Born

coincide com média estatística

a

Sugestão: $\langle E^2\rangle$?? $\Rightarrow \langle \hat{H}^2\rangle$

$\hookrightarrow \hat{H}^2 = \hat{H}\cdot\hat{H}$

$$\langle \hat{H}^2\rangle = \langle \psi|H H|\psi\rangle \; ; \qquad H^\dagger = H$$

$$\underbrace{\langle \psi|H^\dagger}\, \underbrace{H|\psi\rangle}$$

$$= \left(\sum_m c_m^* \langle e_m|\right) H^\dagger H \left(\sum_n c_n |e_n\rangle\right) = \sum_{m,n} c_m^* \underbrace{\langle e_m|H^\dagger}\cdot H|e_n\rangle$$

$$E_m \underbrace{\langle e_m|e_n\rangle}_{\delta_{mn}} E_n$$

$$= \sum_{m,n} c_m^* c_n E_m E_n \delta_{m,n}$$

$$= \sum_n |c_n|^2 E_n^2 = \sum_n \rho_n E_n^2$$

$$\langle E^2\rangle = \sum_n \rho_n E^2$$

Sugestão: $\sigma_E^2, \sigma_E \rightarrow \sigma_E^2 = \langle E^2\rangle - \langle E\rangle^2$

$$\sigma_E = \sqrt{\langle E^2\rangle - \langle E\rangle^2}$$

Sugestão: Demonstre que relação de incerteza generalizada

Se $\hat{A}$, $\hat{B}$ são observáveis, então:

$$\sigma_A \cdot \sigma_B \geq \frac{\langle [A,B] \rangle}{2i}$$

$$(\Delta A)\cdot(\Delta B) \geq \frac{\langle [A,B] \rangle}{2i}$$

$$[\hat{x}, \hat{p}_x] = i\hbar$$

$$\Delta x \cdot \Delta p \geq \hbar/2$$

$$\sigma_x \cdot \sigma_p \geq \hbar/2$$

Exemplo: usando funções de onda.

$$[\hat{x}, \hat{p}_x] = \underbrace{\hat{x}\hat{p} - \hat{p}\hat{x}}_{(\hat{x}\hat{p} - \hat{p}\hat{x})\psi}$$

$\psi(x)$

$$\hat{x} = x \cdot$$

$$\hat{p}_x = -i\hbar\partial_x$$

$$\underbrace{(\hat{x}\hat{p} - \hat{p}\hat{x})}_{[\hat{x},\hat{p}]}\psi = \hat{x}(\hat{p}\psi) - \hat{p}(\hat{x}\psi)$$

$$= x\left(\cancel{-i\hbar\partial_x\psi}\right) + \underbrace{i\hbar\partial_x(x\psi)}_{i\hbar(\psi + \cancel{x\partial_x\psi})} = i\hbar\,\psi$$

$$\Downarrow$$

$$[\hat{x}, \hat{p}]\psi = i\hbar\psi \Rightarrow [\hat{x}, \hat{p}] = i\hbar$$

(c.q.d.)

Exemplo 4: Suponha que temos um operador $\hat{Q}$ com autovetores

$$\begin{cases} |q_1\rangle = \frac{1}{2}|e_1\rangle + \frac{\sqrt{3}}{2}|e_2\rangle \\ |q_2\rangle = \frac{\sqrt{3}}{2}|e_1\rangle - \frac{1}{2}|e_2\rangle \end{cases}$$

e Autovalores

$$q_1 = 2 \; ; \; q_2 = 3$$

$$\Rightarrow \hat{Q}|q_i\rangle = q_i|q_i\rangle$$

(a) Se o estado for $|\psi\rangle = |e_1\rangle$, qual é a prob. de medir q_1?

(b) Qual o valor $\langle\hat{Q}\rangle$?

(a) $\langle q_1|\psi\rangle = \langle q_1|e_1\rangle = \langle e_1|q_1\rangle^* = \langle e_1|q_1\rangle$

números complexos

$$\langle q_1|\psi\rangle = \left(\frac{1}{2}\langle e_1| + \frac{\sqrt{3}}{2}\langle e_2|\right)|e_1\rangle = \frac{1}{2}\langle e_1|e_1\rangle = \frac{1}{2}$$

$$P_1 = \left|\frac{1}{2}\right|^2 = 1/4$$

$$\sum_i P_i = 1$$

↳ verifique $\langle e_1|q_2\rangle \Rightarrow P_2 = \left|\frac{\sqrt{3}}{2}\right|^2 = 3/4$

$\langle q_2|\psi\rangle$

$$\langle\hat{Q}\rangle = P_1 \cdot q_1 + P_2 \cdot q_2 = \frac{1}{4}\cdot 2 + \frac{3}{4}\cdot 3 = \frac{11}{4}$$

Tratamento formal:

$$\langle \hat{Q} \rangle = \langle \psi | \hat{Q} | \psi \rangle = \langle \psi | \hat{Q} \underbrace{\hat{1} | \psi \rangle}_{|\psi\rangle}$$

$$= \langle \psi | \hat{Q} \left(\sum_i | q_i \rangle \langle q_i | \right) | \psi \rangle$$

Porque esta base??

$$\hookrightarrow \sum_i \langle \psi | \underbrace{\hat{Q} | q_i \rangle}_{q_i | q_i \rangle} \langle q_i | \psi \rangle = \sum_i q_i \underbrace{\overset{\langle q_i|\psi\rangle^*}{\langle \psi | q_i \rangle} \langle q_i | \psi \rangle}_{|\langle q_i|\psi\rangle|^2}$$

$$\langle \hat{Q} \rangle = \sum_i q_i |\langle q_i | \psi \rangle|^2 = \sum_i q_i P_i$$

generalizando p/ uma base contínua $|\alpha\rangle$

$$\boxed{\langle \hat{Q} \rangle = \int q_\alpha \cdot P(\alpha)\, d\alpha}$$

* **Sugestão**: Continuando o exemplo 4 ...

Suponha agora que $|\psi\rangle = \frac{1}{\sqrt{2}} \left(|e_1\rangle + |e_2\rangle \right)$

quanto é $\langle \hat{Q} \rangle$?!