Equações de Maxwell

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \widetilde{F}_{\mu\nu} = \frac{1}{2!}\,\varepsilon_{\mu\nu\rho\sigma}\,F^{\rho\sigma}$$

$$S = \int d^4x \left[-\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} - A_\mu\,j^\mu \right]$$

$$\partial_\mu F^{\mu\nu} = j^\nu$$

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0$$

$$\partial_\rho \tilde{F}_{\mu\nu} + \partial_\nu \tilde{F}_{\rho\mu} + \partial_\mu \tilde{F}_{\nu\rho} = \tilde{j}_{\rho\mu\nu}$$

Euler-Lagrange equations

$$\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0$$

Identidade de Bianchi

$$j^\lambda = \frac{1}{3!}\varepsilon^{\lambda\rho\mu\nu}\,\tilde{j}_{\rho\mu\nu}$$

Simetria de gauge: $A_\mu \to A_\mu + \partial_\mu \alpha \qquad F_{\mu\nu} \to F_{\mu\nu}$

Força de Lorentz: $$\frac{d\,p^\mu}{d\,\tau} = q\,F^{\mu\nu}\,v_\nu$$

$$p^\mu = (\gamma\, m\, c\,,\, p_x\,,\, p_y\,,\, p_z) \qquad v_\mu = \gamma\,(c\,,\, -v_x\,,\, v_y\,,\, v_z) \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

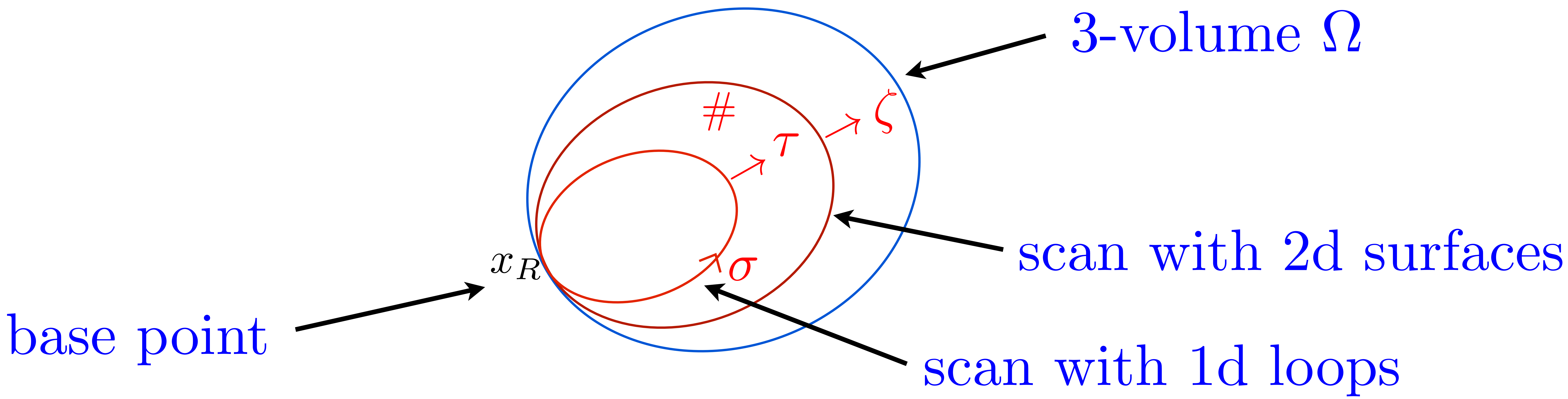
$$\eta_{\mu\nu} = \mathrm{diag.}\,(1\,,\,-1\,,\,-1\,,\,-1)$$

Abelian Stokes Theorem

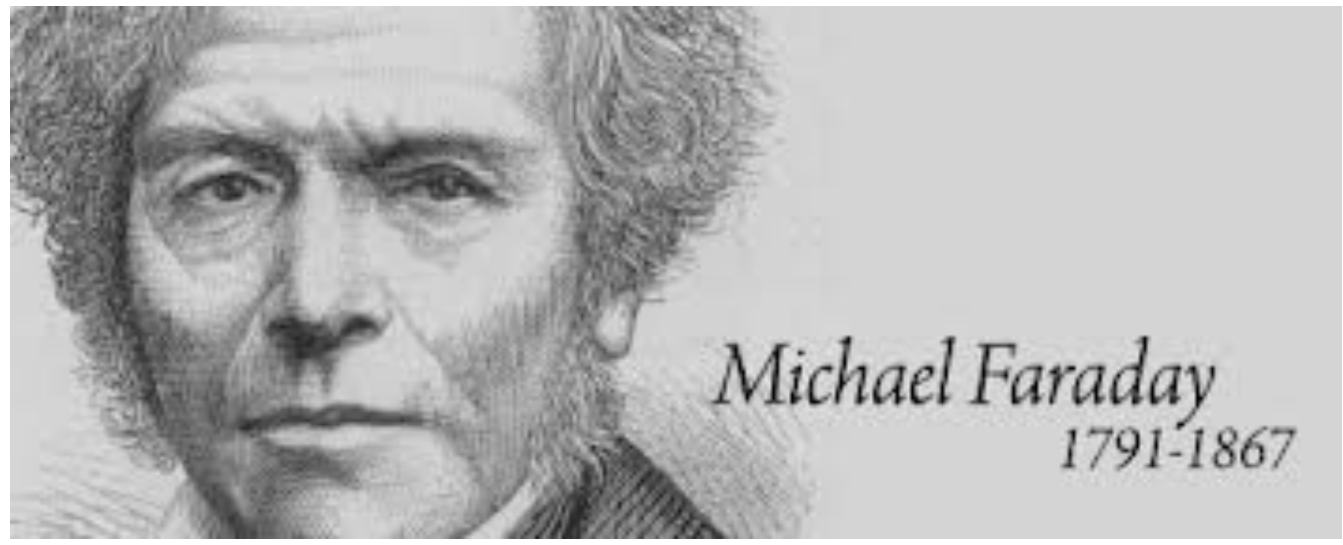
$$\int_{\partial\Omega} B = \int_{\Omega} d \wedge B$$

For an abelian two-form $B_{\mu\nu}$ and a 3-volume Ω

$$\boxed{\int_{\partial\Omega} B_{\mu\nu} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau} = \int_{\Omega} \left[\partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} + \partial_\mu B_{\nu\rho}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta}}$$



Back to Faraday: Integral Equations



$$\partial_\mu F^{\mu\nu} = j^\nu \qquad \longrightarrow \qquad \partial_\rho \tilde{F}_{\mu\nu} + \partial_\nu \tilde{F}_{\rho\mu} + \partial_\mu \tilde{F}_{\nu\rho} = \tilde{\jmath}_{\rho\mu\nu}$$

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0 \qquad\qquad \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0$$

$$j^\lambda = \frac{1}{3!} \varepsilon^{\lambda\rho\mu\nu} \, \tilde{\jmath}_{\rho\mu\nu}$$

In Stokes theorem, take $B_{\mu\nu} \equiv \alpha \, F_{\mu\nu} + \beta \, \tilde{F}_{\mu\nu}$ to get

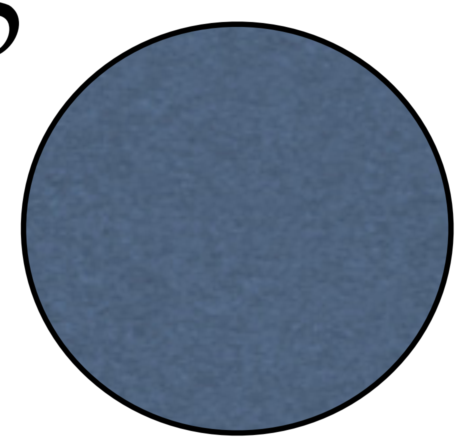
$$\boxed{\int_{\partial\Omega} \left[\alpha \, F_{\mu\nu} + \beta \, \tilde{F}_{\mu\nu} \right] \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} = \beta \int_\Omega \tilde{\jmath}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\zeta}}$$

For $\alpha = 0$, $\beta = 1$, and Ω purely spatial

$$\int_{\partial\Omega} \widetilde{F}_{ij} \frac{\partial \, x^i}{\partial \, \sigma} \frac{\partial \, x^j}{\partial \, \tau} \, d\sigma \, d\tau = - \int_{\partial\Omega} \vec{E} \cdot d\vec{\Sigma} = - \int_\Omega \frac{\rho}{\varepsilon_0} = - \frac{Q}{\varepsilon_0} \qquad \text{Gauss law}$$

Integral form of Maxwell's equations

S



Closed two dimensional surface in space-time

$$\Phi(S) = \oint_S F_{\mu\nu} \frac{\partial\, x^\mu}{\partial\, \sigma} \frac{\partial\, x^\nu}{\partial\, \tau}\, d\sigma\, d\tau$$

$$\widetilde{\Phi}(S) = \oint_S \widetilde{F}_{\mu\nu} \frac{\partial\, x^\mu}{\partial\, \sigma} \frac{\partial\, x^\nu}{\partial\, \tau}\, d\sigma\, d\tau$$

Maxwell's equations

$$\Phi(S) = 0 \qquad \beta = 0$$

$$\widetilde{\Phi}(S) = -\frac{Q}{\varepsilon_0} \qquad \alpha = 0$$

charge inside S

$$Q = -\varepsilon_0 \int_V \widetilde{J}_{\mu\nu\rho} \frac{\partial\, x^\mu}{\partial\, \sigma} \frac{\partial\, x^\nu}{\partial\, \tau} \frac{\partial\, x^\rho}{\partial\, \zeta}\, d\sigma\, d\tau d\zeta \qquad \widetilde{J}_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma} j^\sigma$$

$S \equiv$ closed spatial surface

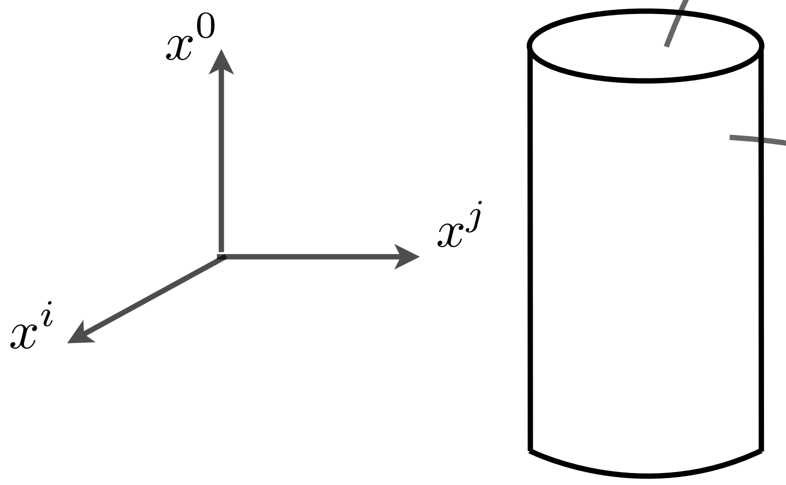
$$\Phi(S) = \oint_S F_{ij} \frac{\partial x^i}{\partial \sigma} \frac{\partial x^j}{\partial \tau} d\sigma \, d\tau = -c \oint_S \vec{B} \cdot d\vec{\Sigma} = 0$$

$$F_{ij} = -c \, \varepsilon_{ijk} B_k \qquad d\vec{\Sigma} = \varepsilon_{ijk} \frac{\partial x^i}{\partial \sigma} \frac{\partial x^j}{\partial \tau} d\sigma \, d\tau \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\widetilde{\Phi}(S) = \oint_S \widetilde{F}_{ij} \frac{\partial x^i}{\partial \sigma} \frac{\partial x^j}{\partial \tau} d\sigma \, d\tau = - \oint_S \vec{E} \cdot d\vec{\Sigma} = -\frac{Q}{\varepsilon_0}$$

$$\widetilde{F}_{ij} = -\varepsilon_{ijk} E_k \qquad \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

S with a time component



$$\Phi(\text{disc}) = \int_{\text{disc}} F_{ij} \frac{\partial x^i}{\partial \sigma} \frac{\partial x^j}{\partial \tau} \, d\sigma \, d\tau = -c \int_{\text{disc}} \vec{B} \cdot d\vec{\Sigma}$$

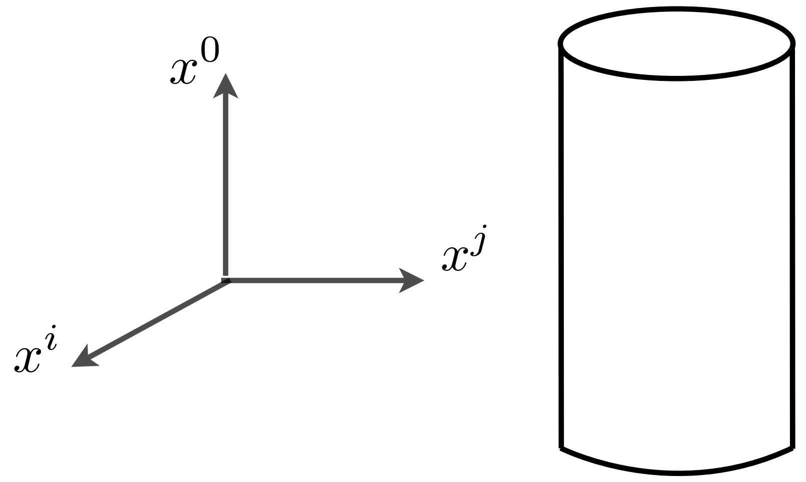
$$\Phi(\text{side}) = \int_{\text{side}} F_{k0} \frac{\partial x^k}{\partial \sigma} \frac{\partial x^0}{\partial \tau} \, d\sigma \, d\tau = -\int dx^0 \oint \vec{E} \cdot d\vec{l}$$

Then

$$\Phi(S) = 0 = -c \int_{\text{top disc}} \vec{B} \cdot d\vec{\Sigma} + c \int_{\text{botton disc}} \vec{B} \cdot d\vec{\Sigma} - \int dx^0 \oint \vec{E} \cdot d\vec{l}$$

In the limit $\delta x^0 \to 0$

$$\frac{d}{dt} \int \vec{B} \cdot d\vec{\Sigma} = -\oint \vec{E} \cdot d\vec{l} \qquad \rightarrow \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$



Analogously

$$
\begin{aligned}
\widetilde{\Phi}(S) &= -\int_{\text{top disc}} \vec{E}\cdot d\vec{\Sigma} + \int_{\text{botton disc}} \vec{E}\cdot d\vec{\Sigma} + c\int dx^0 \oint \vec{B}\cdot d\vec{l} \\
&= \frac{1}{c\,\varepsilon_0}\int dx^0 \int \vec{J}\cdot d\vec{\Sigma}
\end{aligned}
$$

In the limit $\delta x^0 \to 0$

$$\oint \vec{B}\cdot d\vec{l} - \varepsilon_0\,\mu_0 \frac{d}{d\,t}\int \vec{E}\cdot d\vec{\Sigma} = \mu_0 \int \vec{J}\cdot d\vec{\Sigma}$$

$$\vec{\nabla}\times\vec{B} - \mu_0\,\varepsilon_0\,\frac{\partial\,\vec{E}}{\partial\,t} = \mu_0\,\vec{J}$$

Summarizing

Maxwell's eqs. are equivalent to

$$\oint_S \left[\alpha\, F_{\mu\nu} + \beta\, \widetilde{F}_{\mu\nu}\right] \frac{\partial\, x^\mu}{\partial\, \sigma}\, \frac{\partial\, x^\nu}{\partial\, \tau}\, d\sigma\, d\tau = -\beta\, \varepsilon_0 \int_V \widetilde{J}_{\mu\nu\rho}\, \frac{\partial\, x^\mu}{\partial\, \sigma}\, \frac{\partial\, x^\nu}{\partial\, \tau}\, \frac{\partial\, x^\rho}{\partial\, \zeta}\, d\sigma\, d\tau d\zeta$$

V is the volume inside S

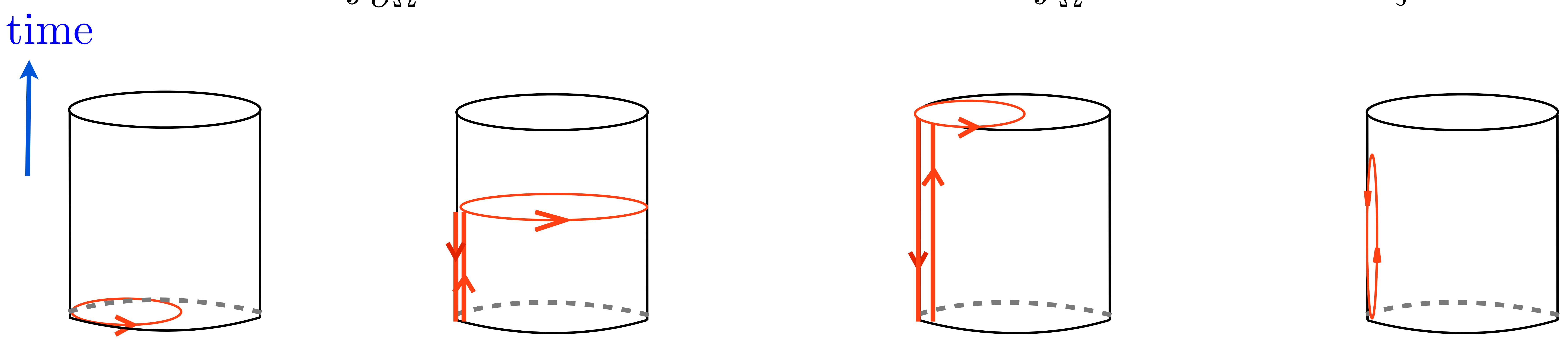
$$\left(\widetilde{J}_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma} j^\sigma\right)$$

Maxwell's eqs. are recovered in the limit where S is infinitesimal

Integral Equations and Conservation Laws

For a 3-volume Ω without border ($\partial\Omega = 0$)

$$0 = \int_{\partial\Omega} \left[\alpha\, F_{\mu\nu} + \beta\, \tilde{F}_{\mu\nu} \right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau} = \beta \int_{\Omega} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta}$$



$$\int_{\Omega_0} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta} \longrightarrow Q(0)$$

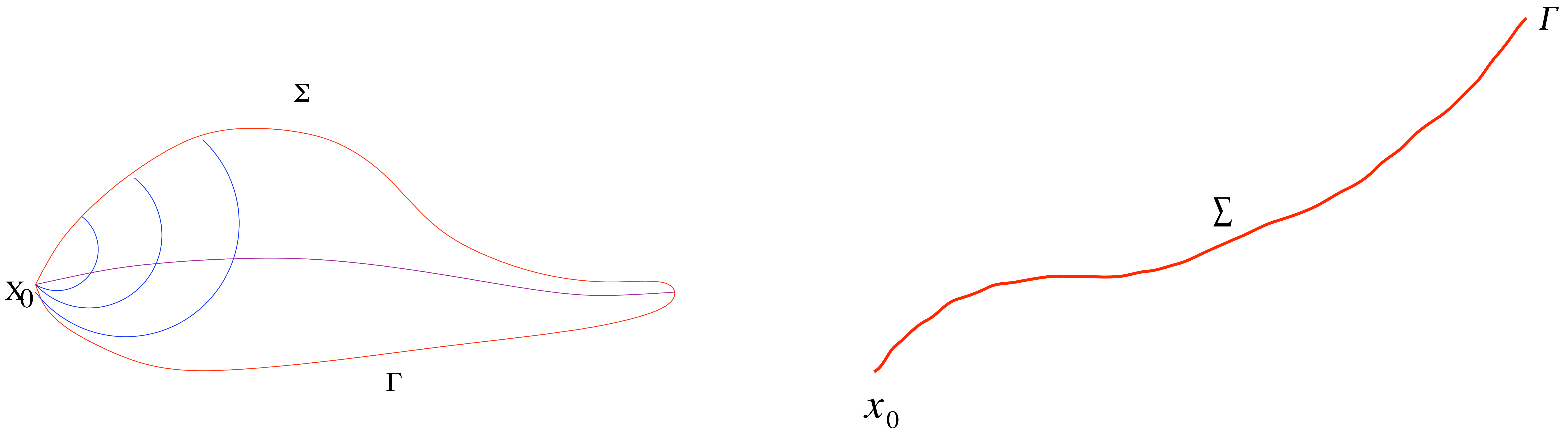
$$\int_{S^2_\infty \times I} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta} \longrightarrow 0$$

$$-\int_{\Omega_t} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta} \longrightarrow -Q(t)$$

$$\int_{S^2_0 \times I} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta} \longrightarrow 0$$

$$\boxed{\int_{\Omega} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta} = 0 \longrightarrow Q(t) = Q(0)}$$

Using Loop Spaces



space-time surface ⟷ path in loop space

Loop Space: $\Omega^{(1)} = \{ f : S^1 \to M \mid \text{north pole } \to x_0 \}$

Introduce a flat connection $\mathcal{A}$ in loop space

$$\mathcal{F} = \delta\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$$

Construct the charges using path independency!

Generalized Loop Spaces

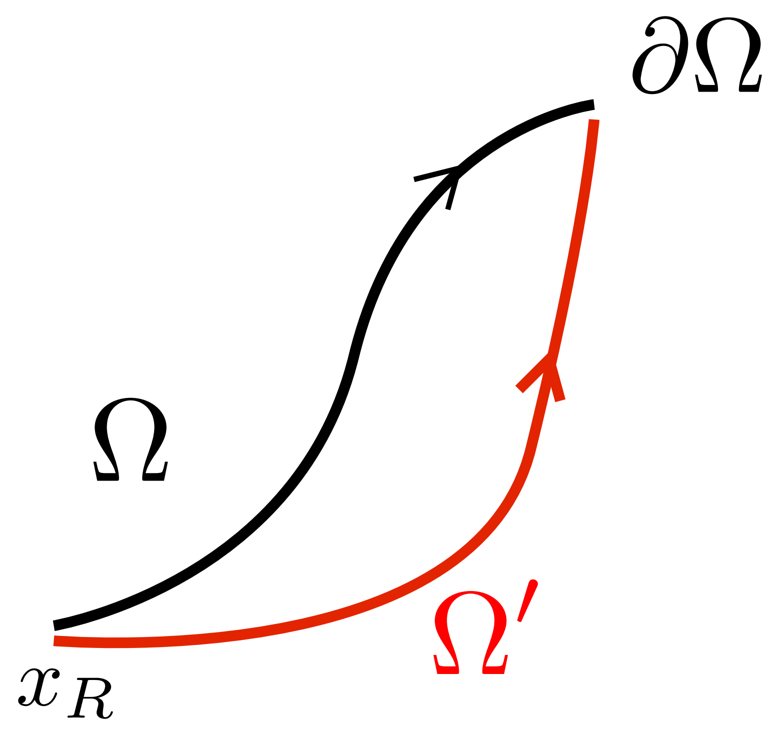
$$\Omega^{(2)} = \{f : S^2 \to M \mid \text{north pole} \to x_0\}$$

Volumes in M become paths in Loop Space $\Omega^{(2)}$

Faraday's Path Independency

$$\int_{\partial\Omega} \left[\alpha\, F_{\mu\nu} + \beta\, \tilde{F}_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau} = \beta \int_{\Omega} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta}$$

$$= \beta \int_{\Omega'} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta}$$



Ω and Ω': two 3-volumes with the same border

connection in loop space: $$\mathcal{A} \equiv \int_{\text{loop}} \tilde{j}_{\mu\nu\rho} \frac{d\,x^\mu}{d\,\sigma}\frac{d\,x^\nu}{d\,\tau}\,\delta x^\rho$$

$$\mathcal{F} = \delta\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = 0$$

$$\partial_\rho \tilde{F}_{\mu\nu} + \partial_\nu \tilde{F}_{\rho\mu} + \partial_\mu \tilde{F}_{\nu\rho} = \tilde{j}_{\rho\mu\nu}$$

$(d \wedge \tilde{j} = 0)$

abelian

The laws of Electromagnetism correspond to flat connections in loop space!!

Generalizing Maxwell: Yang-Mills theory

Yang-Mills equations $\quad D^{\mu} F_{\mu\nu} = J_{\nu} \qquad D^{\mu} \widetilde{F}_{\mu\nu} = 0$

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + i\, e\, [\, A_{\mu}\,,\, A_{\nu}\,] \qquad D_{\mu} * = \partial_{\mu} * + i\, e\, [\, A_{\mu}\,,\, *\,]$$

$$J_{\mu} = \bar{\psi}\, \gamma_{\mu}\, R(T_a)\, \psi\, T_a$$

Invariant under a gauge group G

The (text book) conserved charges

$$j_\nu \equiv \partial^\mu F_{\mu\nu} = J_\nu - i\,e\,[\,A_\mu\,,\,F_{\mu\nu}\,] \qquad \rightarrow \qquad \partial^\mu j_\mu = 0$$

$$\widetilde{j}_\nu \equiv \partial^\mu \widetilde{F}_{\mu\nu} = -i\,e\left[\,A_\mu\,,\,\widetilde{F}_{\mu\nu}\,\right] \qquad \rightarrow \qquad \partial^\mu \widetilde{j}_\mu = 0$$

Charges

$$Q = \int d^3x\,\partial^i F_{i0} = \int d^3x\,\vec{\nabla}\cdot\vec{E} = \int d\vec{\Sigma}\cdot\vec{E}$$

$$\widetilde{Q} = \int d^3x\,\partial^i \widetilde{F}_{i0} = -\int d^3x\,\vec{\nabla}\cdot\vec{B} = -\int d\vec{\Sigma}\cdot\vec{B}$$

Under a gauge transformation

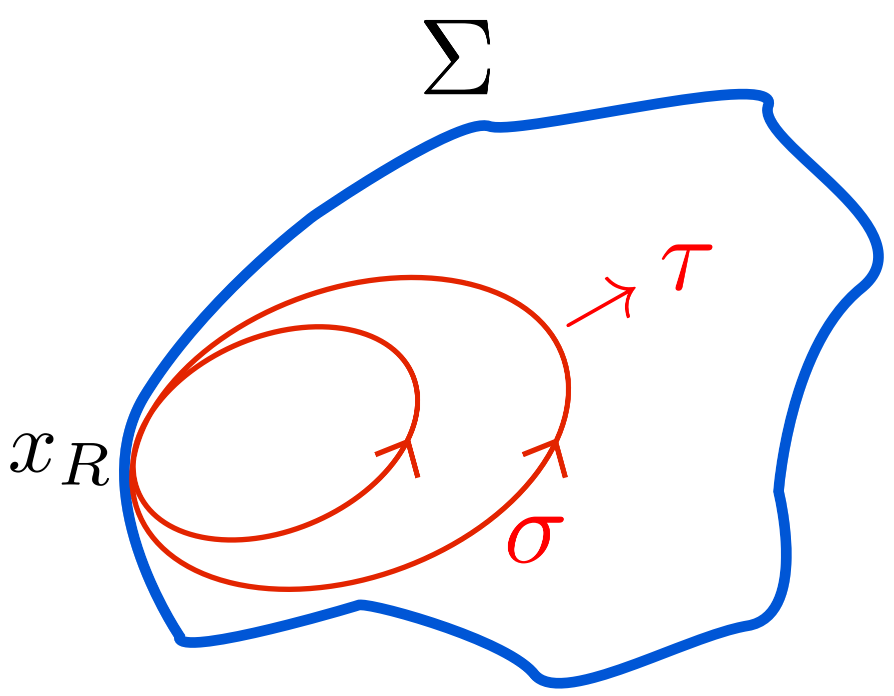
$$Q \rightarrow \int d\vec{\Sigma}\cdot g\,\vec{E}\,g^{-1} = g\,Q\,g^{-1} \longrightarrow$$

eigenvalues of Q are gauge invariant

not gauge invariant

if g is constant at infinity

Generalizing Faraday: Non-Abelian integrals



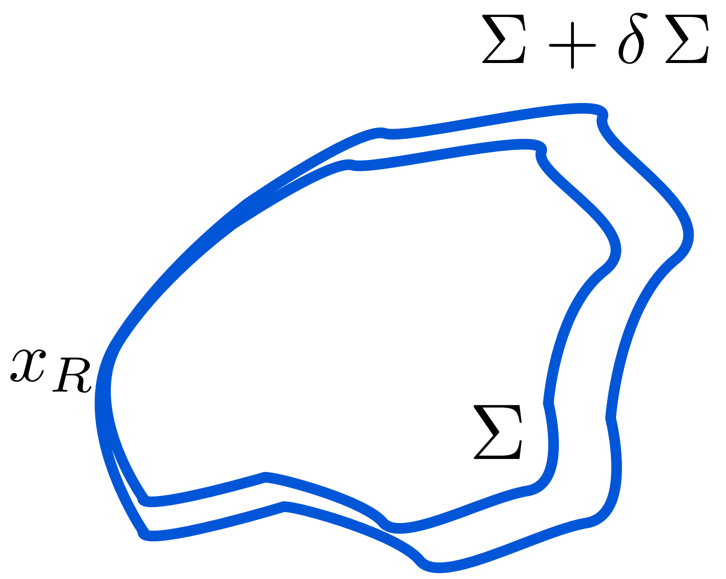
$$\frac{dV}{d\tau} - V\,T\,(A,B,\tau) = 0$$

$$T\,(B,A,\tau) \equiv \int_0^{2\,\pi} d\sigma\, W^{-1}\, B_{\mu\nu}\, W\, \frac{d\,x^\mu}{d\,\sigma}\, \frac{d\,x^\nu}{d\,\tau}$$

It is a surface ordered integral

$$V(\Sigma) = V_R\; P_2\, e^{\int_\Sigma d\sigma d\tau\; W^{-1}\, B_{\mu\nu}\, W\, \frac{d\,x^\mu}{d\,\sigma}\, \frac{d\,x^\nu}{d\,\tau}}$$

Vary Σ



$$\delta V\, V^{-1} \equiv \int_0^{2\,\pi} d\tau \int_0^{2\,\pi} d\sigma\, V\,(\tau)\, \{$$

$$W^{-1}\,\left[D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}\right]\, W \frac{d\,x^\mu}{d\,\sigma}\, \frac{d\,x^\nu}{d\,\tau}\, \delta x^\lambda$$

$$- \int_0^\sigma d\sigma'\, \left[\, B^W_{\kappa\rho}\,(\sigma') - ieF^W_{\kappa\rho}\,(\sigma')\,,\, B^W_{\mu\nu}\,(\sigma)\,\right] \frac{dx^\kappa}{d\sigma'}\, \frac{dx^\mu}{d\sigma}$$

$$\times \left(\frac{d\,x^\rho\,(\sigma')}{d\,\tau} \delta x^\nu\,(\sigma) - \delta x^\rho\,(\sigma')\, \frac{d\,x^\nu\,(\sigma)}{d\,\tau}\right)\Big\}\, V^{-1}\,(\tau)$$

The generalized non-abelian Stokes Theorem

$$\boxed{V_R\, P_2 e^{\int_{\partial\Omega} d\tau d\sigma W^{-1} B_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{d\,x^\nu}{d\,\tau}} = P_3 e^{\int_\Omega d\zeta\, \mathcal{K}}\, V_R}$$

$$\frac{d\,V}{d\,\tau} - V\,T\,(A,B,\tau) = 0$$

$$\frac{d\,V}{d\,\zeta} - \mathcal{K}\,V = 0$$

$$T\,(B,A,\tau) \equiv \int_0^{2\,\pi} d\sigma\; W^{-1}\, B_{\mu\nu}\, W\, \frac{d\,x^\mu}{d\,\sigma}\, \frac{d\,x^\nu}{d\,\tau}$$

$$\begin{aligned}
\mathcal{K} \equiv & \int_0^{2\,\pi} d\tau \int_0^{2\,\pi} d\sigma\, V\,(\tau)\, \{ \\
& W^{-1}\, [D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}]\, W \frac{d\,x^\mu}{d\,\sigma}\, \frac{d\,x^\nu}{d\,\tau}\, \frac{d\,x^\lambda}{d\,\zeta} \\
& - \int_0^\sigma d\sigma' \left[B^W_{\kappa\rho}\,(\sigma') - ieF^W_{\kappa\rho}\,(\sigma')\,,\, B^W_{\mu\nu}\,(\sigma) \right] \frac{dx^\kappa}{d\sigma'}\, \frac{dx^\mu}{d\sigma} \\
& \times \left(\frac{d\,x^\rho\,(\sigma')}{d\,\tau} \frac{d\,x^\nu\,(\sigma)}{d\,\zeta} - \frac{d\,x^\rho\,(\sigma')}{d\,\zeta} \frac{d\,x^\nu\,(\sigma)}{d\,\tau} \right) \Bigg\}\, V^{-1}\,(\tau)
\end{aligned}$$

O. Alvarez, L. A. Ferreira and J. Sanchez Guillen,
Nucl. Phys. B **529**, 689 (1998) [arXiv:hep-th/9710147].
Int. J. Mod. Phys. A **24**, 1825 (2009) [arXiv:0901.1654 [hep-th]]

The Integral Equations for Yang-Mills

$$\boxed{P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma\left[\alpha F^W_{\mu\nu}+\beta\widetilde{F}^W_{\mu\nu}\right]\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V\mathcal{J}V^{-1}}}$$

$$\mathcal{J} \equiv \int_0^{2\pi} d\sigma \left\{ ie\beta\widetilde{J}^W_{\mu\nu\lambda}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\zeta} + e^2\int_0^\sigma d\sigma' \right.$$
$$\times \left[\left(\left(\alpha-1\right)F^W_{\kappa\rho}+\beta\widetilde{F}^W_{\kappa\rho}\right)(\sigma'),\left(\alpha F^W_{\mu\nu}+\beta\widetilde{F}^W_{\mu\nu}\right)(\sigma)\right]$$
$$\left.\times\frac{dx^\kappa}{d\sigma'}\frac{dx^\mu}{d\sigma}\left(\frac{dx^\rho(\sigma')}{d\tau}\frac{dx^\nu(\sigma)}{d\zeta}-\frac{dx^\rho(\sigma')}{d\zeta}\frac{dx^\nu(\sigma)}{d\tau}\right)\right\}$$

$B_{\mu\nu} \to \alpha F_{\mu\nu} + \beta\widetilde{F}_{\mu\nu}$ $\quad D^\mu F_{\mu\nu} = J_\nu$ $\quad D^\mu \widetilde{F}_{\mu\nu} = 0$

$J^\mu = \frac{1}{3!}\varepsilon^{\mu\nu\rho\lambda}\,\widetilde{J}_{\nu\rho\lambda}$

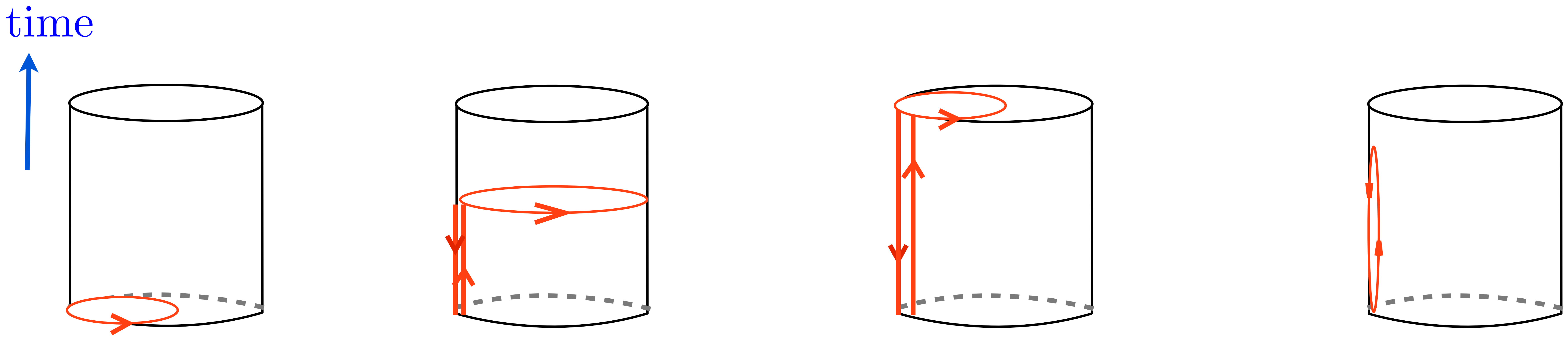
Direct consequence of Stokes theorem and Yang-Mills eqs.

Implies Yang-Mills eqs. in the limit $\Omega \to 0$

Conserved Charges: Quite Elementary Mr. Holmes

$$P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma \left[\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}$$

If Ω_c is a closed volume $(\partial\Omega_c = 0)$ $\longrightarrow$ $P_3 e^{\oint_{\Omega_c} d\zeta d\tau V \mathcal{J} V^{-1}} = \mathbb{1}$



$P_3 e^{\int_{\Omega_0} d\zeta d\tau V \mathcal{J} V^{-1}}$ $P_3 e^{\int_{S^2_\infty \times I} d\zeta d\tau V \mathcal{J} V^{-1}}$ $P_3 e^{\int_{\Omega_t^{-1}} d\zeta d\tau V \mathcal{J} V^{-1}}$ $P_3 e^{\int_{S^2_0 \times I} d\zeta d\tau V \mathcal{J} V^{-1}}$

Iso-spectral evolution: $\boxed{V(\Omega_t) = U(t) \cdot V(\Omega_0) \cdot U^{-1}(t)}$

Eigenvalues of $V(\Omega_t)$ are constant in time

Conserved charges are eigenvalues of the operator

$$\boxed{V(\Omega_t) = P_2\; e^{ie \int_{\mathcal{S}^{2,(t)}_\infty} d\tau\; d\sigma\; \left(\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right) \frac{dx^\mu}{d\sigma}\,\frac{dx^\nu}{d\tau}} = P_3\; e^{\int_{\Omega_t} d\zeta\; d\tau\; V \mathcal{J} V^{-1}}}$$

Conserved charges are:

1) Gauge invariant $\qquad V(\Omega_t) \to g_R\, V(\Omega_t)\, g_R^{-1}$

2) Independent of reference point

$$V_{x_R}(\Omega_t) \to W^{-1}(\widetilde{x}_R, x_R) V_{\widetilde{x}_R}(\Omega_t) W(\widetilde{x}_R, x_R)$$

3) Independent of parameterization

$$P_3\; e^{\int_{\Omega^{(t)}_\infty} d\zeta\; d\tau\; V \mathcal{J} V^{-1}} \quad \text{is path independent}$$

4) Gives non-trivial dynamical magnetic charges to monopoles

5) Relevant for the global aspects of Yang-Mills theory

Wu-Yang and 't Hooft-Polyakov monopoles

At spatial infinity they are the same:

$$A_i = -\frac{1}{e}\,\varepsilon_{ijk}\,\frac{\hat{r}_j}{r}\,T_k \qquad F_{ij} = \frac{1}{e}\,\varepsilon_{ijk}\,\frac{\hat{r}_k}{r^2}\,\hat{r}\cdot T \qquad [T_i\,,\,T_j] = i\,\varepsilon_{ijk}\,T_k$$

Important property:

$$\frac{d}{d\,\sigma}\left(W^{-1}\,\hat{r}\cdot T\,W\right) = W^{-1}\,D_i(\hat{r}\cdot T)\,W\,\frac{d\,x^i}{d\,\sigma} \xrightarrow{D_i(\hat{r}\cdot T)=0} W^{-1}\,\hat{r}\cdot T\,W = T_R$$

$$W^{-1}\,F_{ij}\,W = \frac{1}{e}\,\varepsilon_{ijk}\,\frac{\hat{r}_k}{r^2}\,T_R \qquad T_R \equiv (\hat{r}\cdot T)_{\text{at }x_R}$$

Charge operator:

$$Q_S = e^{i\,e\,\alpha\,\int_{S^2_\infty} d\sigma\,d\tau\,W^{-1}\,F_{ij}\,W\,\frac{dx^i}{d\sigma}\,\frac{dx^j}{d\tau}} = e^{-i\,e\,\alpha\,\int_{S^2_\infty} d\vec{\Sigma}\cdot\vec{B}^R} = e^{i\,4\,\pi\,\alpha\,T_R}$$

Conserved charges are the eigenvalues of Q_S

Text book charges: $$Q_{\text{old}} = \int_{S^2_\infty} d\sigma\,d\tau\,F_{ij}\,\frac{dx^i}{d\sigma}\,\frac{dx^j}{d\tau} = 0$$

Note however that one must have (for $\beta = 0$)

$$Q_S = P_2 e^{\alpha\, ie \int_{S^2_\infty} d\tau d\sigma F^W_{ij} \frac{dx^i}{d\sigma}\frac{dx^j}{d\tau}} = P_3 e^{e^2 \alpha(\alpha-1)\int_{R^3} d\zeta d\tau V \mathcal{C} V^{-1}}$$

with

$$\mathcal{C} \equiv \int_0^{2\pi} d\sigma \int_0^{\sigma} d\sigma' \left[F^W_{\kappa\rho}(\sigma'), F^W_{\mu\nu}(\sigma) \right] \frac{dx^\kappa}{d\sigma'}\frac{dx^\mu}{d\sigma}\left(\frac{dx^\rho(\sigma')}{d\tau}\frac{dx^\nu(\sigma)}{d\zeta} - \frac{dx^\rho(\sigma')}{d\zeta}\frac{dx^\nu(\sigma)}{d\tau} \right)$$

density of magnetic charge

Integral Bianchi identity implies: $\quad Q_S^{(\alpha=1)} = e^{-ie\int_{S^2_\infty} d\vec{\Sigma}\cdot\vec{B}^R} = \mathbb{1}$

eigenvalues of $\int_{S^2_\infty} d\vec{\Sigma}\cdot\vec{B}^R = \dfrac{2\pi n}{e}$

For 'tHooft-Polyakov monopole both sides match.
Commutator plays the role of density of magnetic charge

For Wu-Yang monopole $\mathcal{C} = 0$
One needs a source of magnetic charge

$$\vec{D}\cdot\vec{B} = -\frac{1}{e}\,\hat{r}\cdot\vec{T}\,\frac{\delta(r)}{r^2}$$

It satisfies Bianchi identity in the sense of distribution theory

C.P. Constantinidis, LAF, G. Luchini; JPA 52, 155202 (2019)

One can expand both sides of the integral equation in powers of α

$$Q_S = P_2 e^{\alpha\, ie \int_{S^2_\infty} d\tau d\sigma F_{ij}^W \frac{dx^i}{d\sigma}\frac{dx^j}{d\tau}} = P_3 e^{e^2 \alpha(\alpha-1)\int_{R^3} d\zeta d\tau V \mathcal{C} V^{-1}}$$

It implies that

$$1 + \alpha \int_0^\tau d\tau' T(\tau') + \alpha^2 \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' T(\tau'')\, T(\tau') + \ldots$$
$$= 1 + \alpha(\alpha-1) \int_0^\zeta d\zeta' K(\zeta') + \alpha^2(\alpha-1)^2 \int_0^\zeta d\zeta' \int_0^{\zeta'} d\zeta'' K(\zeta')\, K(\zeta'') + \ldots$$

with $\quad T \equiv ie \int_0^{2\pi} d\sigma W^{-1} F_{\mu\nu} W \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau} \qquad K \equiv e^2 \int_0^{2\pi} d\tau V \mathcal{C} V^{-1}$

The 'tHooft-Polyakov monopole satisfies it for any α (checked up to second order)

C.P. Constantinidis, LAF, G. Luchini, [1710.03359], PRD 97 (085006) (2018)

To think further...

- Yang-Mills is equivalent to the flatness condition on loop space $\mathcal{L}^{(2)}\ (S^2 \to M)$

$$\delta\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = 0$$

$$\begin{aligned}\mathcal{A} \equiv & \int_0^{2\pi} d\tau \int_0^{2\pi} d\sigma\, V\, \Big\{ ie\beta \widetilde{J}^W_{\mu\nu\lambda} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\delta x^\lambda \\ & + e^2 \int_0^\sigma d\sigma' \left[\left((\alpha-1)\, F^W_{\kappa\rho} + \beta \widetilde{F}^W_{\kappa\rho} \right)(\sigma') \, , \left(\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right)(\sigma) \right] \\ & \times \frac{d\,x^\kappa}{d\,\sigma'}\frac{d\,x^\mu}{d\,\sigma}\left(\frac{d\,x^\rho(\sigma')}{d\tau}\delta x^\nu(\sigma) - \delta x^\rho(\sigma')\frac{d\,x^\nu(\sigma)}{d\,\tau}\right)\Big\} V^{-1}\end{aligned}$$

(it solves parameterization problem!)

- The hidden symmetries are the gauge transformations on loop space $\mathcal{L}^{(2)}$

$$\mathcal{A} \to g\,\mathcal{A}\,g^{-1} + \delta g\, g^{-1}$$

$$g = P_2 e^{\int d\sigma\, d\tau\ \mathcal{W}^{-1}\beta_{\mu\nu}\mathcal{W}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}}$$

$$\frac{d\mathcal{W}}{d\sigma} + \alpha_\mu \frac{dx^\mu}{d\sigma}\mathcal{W} = 0$$

- The conserved charges are eigenvalues of the holonomy

$$\mathcal{Q} = \mathcal{P}e^{\int_{\rm space}\mathcal{A}}$$

- It connects to integrable field theories