PROBLEMS

10.1. **(a)** Calculate $\langle|\mathcal{M}|^2\rangle$ for $\nu_\mu + e^- \to \mu^- + \nu_e$ using the more general coupling $\gamma^\mu(1 + \epsilon\gamma^5)$. Check that your answer reduces to equation (10.11) in the case $\epsilon = -1$.

$$\left[\textit{Answer: } \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{2}\left(\frac{g_w}{M_W c}\right)^4 [(1 - \epsilon^2)^2(p_1 \cdot p_4)(p_2 \cdot p_3) + (1 + 6\epsilon^2 + \epsilon^4)(p_1 \cdot p_2)(p_3 \cdot p_4)]\right]$$

(b) Let $m_e = m_\mu = 0$, and calculate the *CM* differential scattering cross section. Also, find the *total* cross section.

(c) If you had accurate experimental data on this reaction, how could you determine ϵ?

10.2. Calculate the lifetime of the τ lepton. Compare the experimental result. (Assume that the muon mass can be neglected, in comparison with m_τ. Do the experimental data support this approximation?)

10.3. Suppose the weak interaction were *pure vector* (as Fermi supposed). Would you still get the graph shown in Figure 10.1?

10.4. Using the coupling $\gamma^\mu(1 + \epsilon\gamma^5)$ for $n \to p + W$, but $\gamma^\mu(1 - \gamma^5)$ for the leptons, calculate the spin-averaged amplitude for neutron beta decay. Show that your result reduces to equation (10.43) when $\epsilon = -1$.

$$\left[\textit{Answer: } \langle|\mathcal{M}|^2\rangle = \frac{1}{2}\left(\frac{g_w}{M_W c}\right)^4 [(p_1 \cdot p_2)(p_3 \cdot p_4)(1 - \epsilon)^2 + (p_1 \cdot p_4)(p_2 \cdot p_3)(1 + \epsilon)^2 - (1 - \epsilon^2)m_p m_n c^2(p_2 \cdot p_4)]\right]$$

10.5. **(a)** Derive equation (10.55). **(b)** Derive equation (10.61).

10.6. In the text I said that electron energies in neutron decay range up to about $(m_n - m_p)c^2$. This is not *exact,* since it ignores the *kinetic* energy of the proton and the neutrino. What kinematic configuration gives the maximum electron energy? Apply conservation of energy and momentum to determine the *exact* maximum electron energy.

[*Answer:* $(m_n^2 - m_p^2 + m_e^2)c^2/2m_n$.]

How far off is the approximate answer (give the percent error)?

10.7. **(a)** Integrate equation (10.62) to get equation (10.63).

(b) Approximate as suitable for $m_e \ll \Delta m = (m_n - m_p)$. Note that m_e now drops out.

10.8. Obtain equation (10.65).

10.9. Find the minimum de Broglie wavelength ($\lambda = h/p$) of the W in neutron decay, and compare it with the diameter of the neutron ($\sim 10^{-13}$ cm). [*Answer:* maximum

$|\mathbf{p}| = 1.18$ MeV/c, occurring when p and e emerge back to back, so the minimum $\lambda = 10^{-10}$ cm]

10.10. Analyze π^- decay as a scattering process, using the methods of Example 7.8 and Section 9.3. Calculate the decay rate, and, by comparing your answer with the one in the text, obtain the formula for f_π in terms of $|\psi(0)|^2$. Assume $m_u = m_d = m$.

$$\left[\textit{Answer: } f_\pi^2 = \frac{4\hbar^3 m}{m_\pi^2 c} \cos^2\theta_C |\psi(0)|^2\right]$$

10.11. Show that if $mc^2 \ll E$

$$\gamma^5 u \cong \begin{pmatrix} \sigma\cdot\hat{p} & 0 \\ 0 & \sigma\cdot\hat{p}\end{pmatrix} u$$

where u is a particle spinor satisfying the Dirac equation:

$$u = \begin{pmatrix} u_A \\ \dfrac{c(\mathbf{p}\cdot\sigma)}{E + mc^2}\, u_A \end{pmatrix}$$

with $E > 0$ [eq. (7.36)]. Show therefore that the *projection matrix*

$$P_\pm \equiv \tfrac{1}{2}(1 \pm \gamma^5)$$

picks out the helicity $\pm$ 1 component of u:

$$\boldsymbol{\Sigma}\cdot\hat{p}(P_\pm u) = \pm(P_\pm u)$$

10.12. Calculate the ratio of the decay rates $K^- \to e^- + \bar{\nu}_e$ and $K^- \to \mu^- + \bar{\nu}_\mu$. The observed K^- lifetime is 1.2×10^{-8} sec, and 64% of all K^- particles decay by the $\mu^- + \bar{\nu}_\mu$ route. Estimate the kaon decay constant f_K.

10.13. Calculate decay rates for the following processes: (a) $\Sigma^0 \to \Sigma^+ + e + \bar{\nu}_e$, (b) $\Sigma^- \to \Lambda + e + \bar{\nu}_e$, (c) $\Xi^- \to \Xi^0 + e + \bar{\nu}_e$, (d) $\Lambda \to p + e + \bar{\nu}_e$, (e) $\Sigma^- \to n + e + \bar{\nu}_e$; (f) $\Xi^0 \to \Sigma^+ + e + \bar{\nu}_e$. Assume the coupling is always $\gamma^\mu(1 - \gamma^5)$—that is, ignore the strong interaction corrections to the axial coupling—but do not forget the Cabibbo factor. Compare the experimental data, where available.

10.14. **(a)** Show that as long as the KM matrix is *unitary* ($U^{-1} = U^\dagger$), the GIM mechanism for eliminating $K^0 \to \mu^+\mu^-$ works for three (or any number of) generations.

(b) How many independent real parameters are there in the general 3×3 unitary matrix? How about $n \times n$?

We are free to change the *phase* of each quark wave function (normalization of u really only determines $|N|^2$; see Problem 7.3), so $2n$ of these parameters are arbitrary—or rather, $(2n - 1)$, since changing the phase of *all* quark wave functions by the same amount has no effect on U. *Question:* Can we thus reduce the KM matrix to a *real* matrix (if it is real and unitary, then it is *orthogonal:* $U^{-1} = \tilde{U}$).

(c) How many independent real parameters are there in the general 3×3 (real) orthogonal matrix? How about $n \times n$?

(d) So, what is the answer? *Can* you reduce the KM matrix to real form? How about for only two generations ($n = 2$)?

10.15. Show that the KM matrix (10.90) is unitary for any (real) numbers θ_1, θ_2, θ_3, and δ.

10.16. Suppose you started with a T^+ meson ($t\bar{d}$). Given equation (10.91), what is the most likely sequence of decays? [*Answer:* Leaving out pions or leptons, we expect $T^+ \to B^0 \to D^+ \to \bar{K}^0 \to \pi^+$.]

10.17. Using the value of the Fermi constant G_F [eq. (10.40)] and of θ_w [eq. (10.95)], "predict" the mass of the $W^\pm$ and the Z^0, in GWS theory. Compare the experimental values.

10.18. In Example 10.4 I used *muon* neutrinos, rather than *electron* neutrinos. As a matter of fact, ν_μ and $\bar{\nu}_\mu$ beams are easier to produce then ν_e and $\bar{\nu}_e$, but there is also a *theoretical* reason why $\nu_\mu + e^- \to \nu_\mu + e^-$ is simpler than $\nu_e + e^- \to \nu_e + e^-$ or $\bar{\nu}_e + e^- \to \bar{\nu}_e + e^-$. Explain.

10.19. **(a)** Calculate the differential and total cross section for $\bar{\nu}_\mu + e^- \to \bar{\nu}_\mu + e^-$ in the GWS model.
[*Answer:* Same as equation (10.103), only with the sign of $c_A c_V$ reversed; see Halzen and Martin, ref. 12, eq. 13.49.]

(b) Find the ratio $\sigma(\bar{\nu}_\mu + e^- \to \bar{\nu}_\mu + e^-)/\sigma(\nu_\mu + e^- \to \nu_\mu + e^-)$. Assume the energy is high enough that you can set $m_e = 0$.

10.20. **(a)** Calculate the decay rate for $Z^0 \to f + \bar{f}$, where f is any quark or any lepton. Assume f is so light (compared to the Z) that its mass can be neglected.

$$\left[\textit{Answer: } \Gamma(Z^0 \to f + \bar{f}) = \frac{g_z^2 M_Z c^2}{48\pi\hbar}(|c_V^f|^2 + |c_A^f|^2).\right]$$

(b) Assuming these are the dominant decay modes, find the branching ratio for each species of quark and lepton (remember that the quarks come in three colors). Assume that $2m_t < M_Z$, and that the approximation in (a) is valid even for t.
[*Answer:* 3% each for e, μ, τ; 6% each for ν_e, ν_μ, ν_τ; 10% each for u, c, t; 14% each for d, s, b.]

(c) Calculate the lifetime of the Z^0. How would it change if there exists a fourth generation? (Notice that an accurate measurement of the Z^0 lifetime will tell us how many quarks and leptons there can be with masses less than 45 GeV/c^2.)

10.21. Estimate R (the total ratio of quark pair production to muon pair production in e^+e^- scattering), when the process is mediated by Z^0. For the sake of argument assume the top quark is light enough so that equation (10.112) can be used. Don't forget color.

10.22. Graph the ratio, equation (10.116) as a function of total energy ($2E$), using 2 for the expression in brackets, $M_Z c^2 = 90$ GeV, and $\hbar\Gamma_Z = 2.5$ GeV.

10.23. Derive equation (10.120), using equation (7.36). Also derive equation (10.124).

10.24. **(a)** If $u(p)$ satisfies the Dirac equation (7.34), show that u_L and u_R (Table 10.2) do *not* (unless $m = 0$).

(b) Find the eigenvalues and eigenspinors of the matrices $P_\pm \equiv \frac{1}{2}(1 \pm \gamma^5)$.

(c) Can there exist spinors that are simultaneously eigenstates of P_+ (say) *and* of the Dirac operator ($\not{p} - mc$)?
[*Answer:* No; these operators do not commute.]

10.25. Work out the weak isospin currents $j_\mu^\pm$ and j_μ^3 for the light quark doublet u and d'. Also, construct the electromagnetic current (j_μ^{em}) and the weak hypercharge current (j_μ^Y). (Leave your answers in terms of d'.)

10.26. From expression (10.159), determine the vector and axial vector couplings in Table 10.1.