4.5. The structure of LIN. TRANSF.

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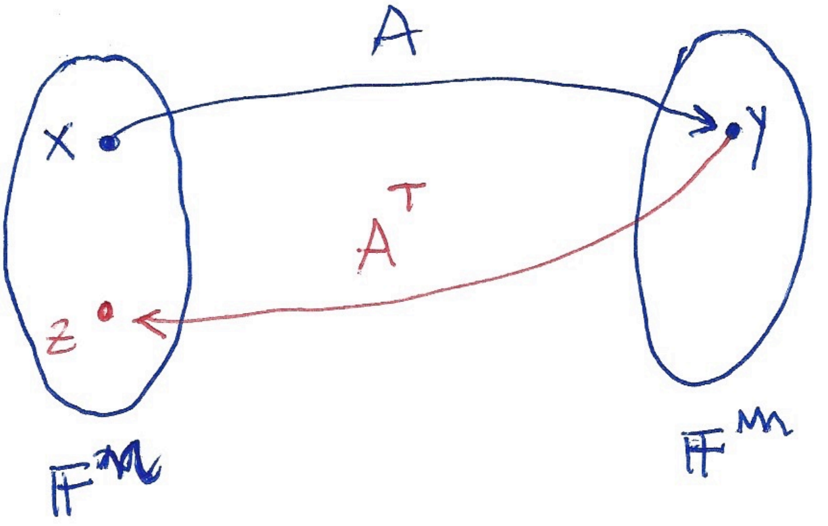
Any LT (matrix $A = \text{mat}\, L$) has an intrinsic structure that can be studied via the concepts of null spaces ($N(A)$), range spaces ($R(A)$), orthogonal complements and direct sum ($\oplus$), in terms of A and A^T (A^*), its transpose.

<u>Def</u>: Let $A: V \to W$ be a LT. Then its transpose is given by $A^T: W^* \to V^*$, in which $W^*: W \to \mathbb{F}$ is the dual space of W, and $V^*: V \to \mathbb{F}$ is the dual space of V. Formally, $A^T = \text{mat}\, L^T$, in which $L^T: W^* \to V^*$. It can be shown that if $[A_{ij}]: V \to W$ then $A^T = [A_{ji}] = \text{mat}\, L^T$. That is, the transpose is a "rotation" around the main diagonal of A.



$Y = Ax$

$z = A^T y$

$B_V = \{v_1, \dots, v_n\}$

$B_W = \{w_1, w_2, \dots, w_m\}$

<u>Example</u>: $A \in \mathbb{F}^{3\times 2}$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \cdot \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \subseteq \mathbb{F}^3$$

$$y = A \quad x$$

$A^T \in \mathbb{F}^{2\times 3}$

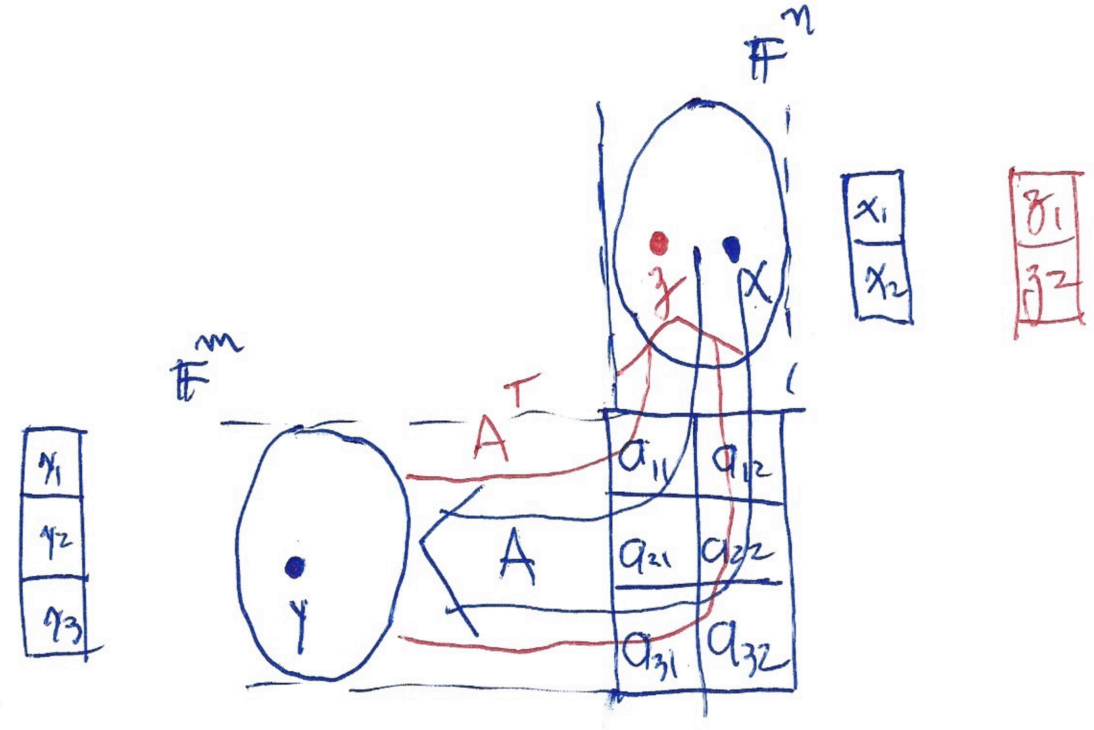
$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1 \cdot \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix} + y_2 \cdot \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix} + y_3 \cdot \begin{bmatrix} a_{31} \\ a_{32} \end{bmatrix} \subseteq \mathbb{F}^2$$

$$z = \quad A^T \quad y$$

<u>Question</u>: $y = Ax$, $z = A^T y$

$$A^T y = A^T A x \Rightarrow \boxed{z = A^T A x}$$

if $A^T A = I_2$, then $A^T = A^{-1}$?



Range and Null spaces

Let $A: \mathbb{F}^n \to \mathbb{F}^m$ be a LT.

The col space $R(A)$ is the space spanned by the columns of A.

$$R(A) = \{Ax \mid x \in \mathbb{F}^n\} \subseteq \mathbb{F}^m$$

(Also known as right range space)

the row space of A, $R(A^T)$, is the space spanned by the rows of A (also: left range)

$$R(A^T) = \{A^T y \mid y \in \mathbb{F}^m\} \subseteq \mathbb{F}^n$$

$(y^T A)^T = A^T y$

rows A = cols A^T

we work with col vecs usually

The right null space of A, $\mathcal{N}(A)$ is the solution set of $Ax=0$

$$\mathcal{N}(A) = \{x \mid Ax = 0, x \in \mathbb{F}^n\}$$

the left null space of A, $\mathcal{N}(A^T)$ is the sol set of $A^T y = 0$ (or $y^T A = 0$)

$$\mathcal{N}(A^T) = \{y \mid A^T y = 0, y \in \mathbb{F}^m\}$$

Such ~~four~~ subspaces allow for an important decomposition of A in its domain ($V = \mathbb{F}^n$) and codomain ($W = \mathbb{F}^m$)

INNER Product and Orthogonality

For vecs $x, z \in \mathbb{F}^n$, the usual inner product (euclidean) is defined as

$$\langle x, z \rangle \triangleq x^T z \in \mathbb{F}$$

From which we define an induced norm

$$\|x\|^2 = \langle x, x \rangle = x^T x \in \mathbb{F}$$

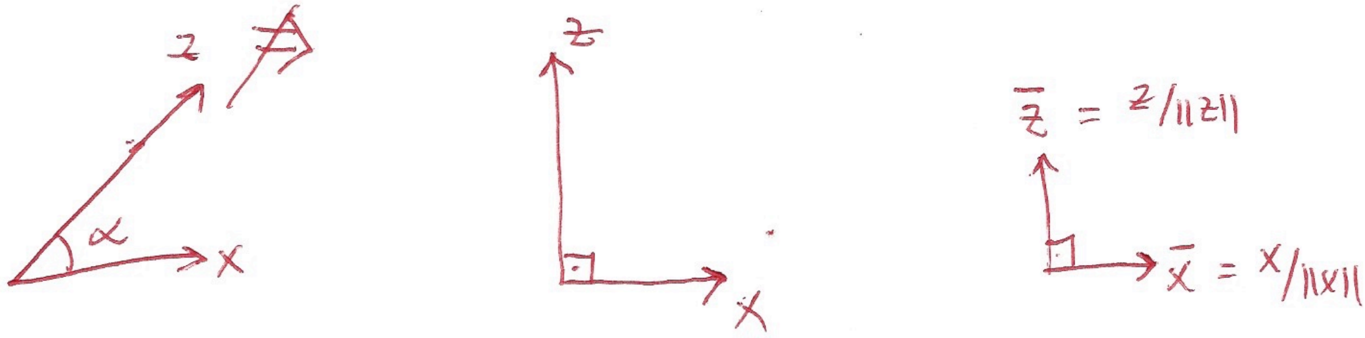
__Def__: two vecs x, z are __orthogonal__ when

$$\langle x, z \rangle = 0 \qquad (x \perp z)$$

__Def__: two vecs x, z are __orthonormal__ when

$$\langle x, z \rangle = 0 \quad \text{and} \quad \|x\| = \|z\| = 1$$

LI vecs $\Leftarrow$ Ortho vecs



LI   Orthogonal   Orthonormal

Orthogonal complement spaces (OCS)

Recall: $R, S \subseteq V = \mathbb{F}^n$

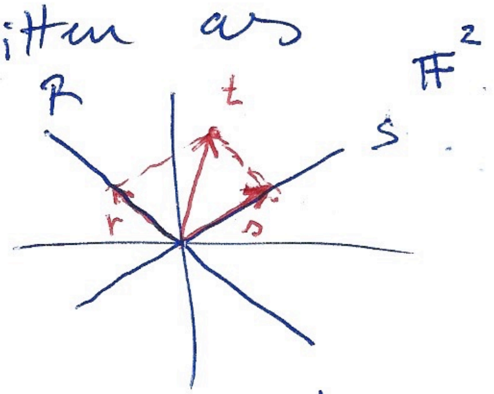
If $\begin{cases} R+S = V \\ R \cap S = \{0\} \end{cases}$ then R, S are complementary in V

Decomposition

$V = R \oplus S = \mathbb{F}^n$

Any $t \in V = \mathbb{F}^n$ is uniquely written as

$t = r + s, \quad r \in R, \quad s \in S,$



Analogy: R, S are "LI"

the OCS of $S \subseteq \mathbb{F}^n$ is denoted as $S^\perp$ and defined as

$$S^\perp = \{ x \in \mathbb{F}^n \mid \langle x, s \rangle = 0, \ s \in S \}$$

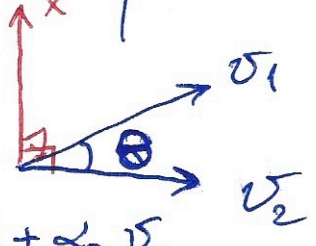
Analogy: $R \perp S$

Example:

$S = sp\left\{ \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \right\}$ then $S^\perp = sp\left\{ \begin{bmatrix} 2 \\ 31 \\ -23 \end{bmatrix} \right\}$

It follows from $\left.\begin{matrix} \langle v_1, x \rangle = 0 \\ \langle v_2, x \rangle = 0 \end{matrix}\right\} \begin{bmatrix} 3 & 5 & 7 \\ -4 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

then $\mathbb{R}^3 = S \oplus S^\perp$. Note that the subspaces are orthogonal, but their individual bases do not need to be



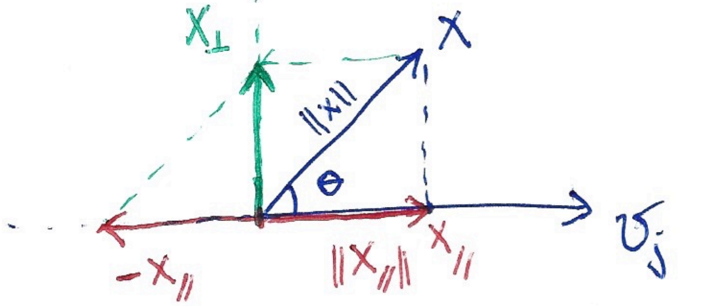
$x \perp \alpha_1 v_1 + \alpha_2 v_2$

thm: Let $S \subseteq \mathbb{F}^n$. then

1. $S^\perp \subseteq \mathbb{F}^n$
2. $S \oplus S^\perp = \mathbb{F}^n$
3. $(S^\perp)^\perp = S$

Proof of (2)

Part 1: projection of a vec X onto a vec v_j



$\|X_\parallel\| = \|X\| \cos\theta$

$\boxed{X_\parallel = \alpha\, v_j} \Rightarrow \|X_\parallel\| = |\alpha|\,\|v_j\|$

or $|\alpha| = \dfrac{\|X_\parallel\|}{\|v_j\|}$

(For simplicity, assume $|\theta| \leq 90^\circ$)

also: $\cos\theta = \dfrac{\langle X, v_j \rangle}{\|X\|\,\|v_j\|}$

then: $\alpha = \dfrac{\|X_\parallel\|}{\|v_j\|} = \dfrac{\|X\|\cos\theta}{\|v_j\|} = \dfrac{\cancel{\|X\|}\langle X, v_j\rangle}{\|v_j\|\,\cancel{\|X\|}\,\|v_j\|}$

$\alpha = \dfrac{\langle X, v_j\rangle}{\|v_j\|^2}$ then $\boxed{X_\parallel = \dfrac{\langle X, v_j\rangle}{\|v_j\|^2}\, v_j} = \langle X, v_j\rangle v_j$ if $\|v_j\| = 1$

Now: $X - X_\parallel = X_\perp$ or $\boxed{X = \underbrace{X_\parallel}_{\text{inline component}} + \underbrace{X_\perp}_{\text{ortho component}}}$

or $\mathbb{R}^2 = S \oplus S^\perp$

$X_\parallel \in S, \quad X_\perp \in S^\perp$

this simple (geometric) example can be extended to any $X \in \mathbb{F}^n$, and ~~~~ find Z orthogonal / ~~~~ to a set of vecs $\{v_j\}$

$$X - X_{|sp\{v_j\}} = Z$$

$X_{|sp\{v_j\}} =$ proj. of X on $sp\{v_j\}$

In which $Z \perp v_j \ \forall j$

or

$$Z \perp \sum_j \alpha_j v_j$$

<u>Part 2</u>: Let $B_k = \{v_1, \dots, v_k\}$ be a orthonormal basis for $S \subseteq \mathbb{F}^n$. Define $X \in \mathbb{F}^n$ and

$$\boxed{Y \triangleq \sum_{j=1}^{k} \langle X, v_j \rangle v_j} \text{ and } \boxed{Z = X - Y}.$$

then $\langle Z, v_i \rangle = Z^T v_i = (X-Y)^T v_i = X^T v_i - Y^T v_i$

$= X^T v_i - \left(\sum_{j=1} \langle X, v_j \rangle v_j\right)^T v_i$

$= X^T v_i - \sum_{j=1}^{k} \langle X, v_j \rangle \underbrace{v_j^T v_i}$

$v_j^T v_i = 0 \quad i \neq j$

$v_j^T v_i = 1 \quad i = j$

(orthonormal)

$= X^T v_i - \langle X, v_i \rangle \underbrace{\|v_i\|^2}_{1} = X^T v_i - X^T v_i = 0$

$$\boxed{Z \perp v_j} \ \forall j = 1, k \quad \text{also} \quad \boxed{Z \perp \sum_{\ell=1}^{k} \alpha_\ell v_\ell}$$

$$\langle z, \sum_\ell \alpha_\ell v_\ell \rangle = z^T \underbrace{\sum_{\ell=1}^{k} \alpha_\ell v_\ell}_{\text{generic vector in } S} = \sum_{\ell=1}^{k} \alpha_\ell z^T v_\ell = 0 \;//$$

or $\boxed{z \perp S}$

that is, for any $v = \sum_\ell \alpha_\ell v_\ell \in S$, we found a vector $z \mid \langle z, v \rangle = 0 \;\forall\, v \in S$.

this is precisely the definition of $S^\perp$:

$$S^\perp = \{ z \mid \langle z, v \rangle = 0 \;\forall\, v \in S \} \;\forall\, x \in \mathbb{F}^n$$

then $z \perp y \;\forall\, x \in \mathbb{F}^n$, with $y \in S$ and $z \in S^\perp$

recall $z = x - y \Leftrightarrow x = z + y$

so that $S + S^\perp = \mathbb{F}^n$. But $S \cap S^\perp = \{0\}$

therefore

$$\mathbb{F}^n = S \oplus S^\perp.$$
$$x = y + z$$

or $\underset{\text{inline}}{x = x_{\parallel}} + \underset{\text{orthog.}}{x_\perp}$

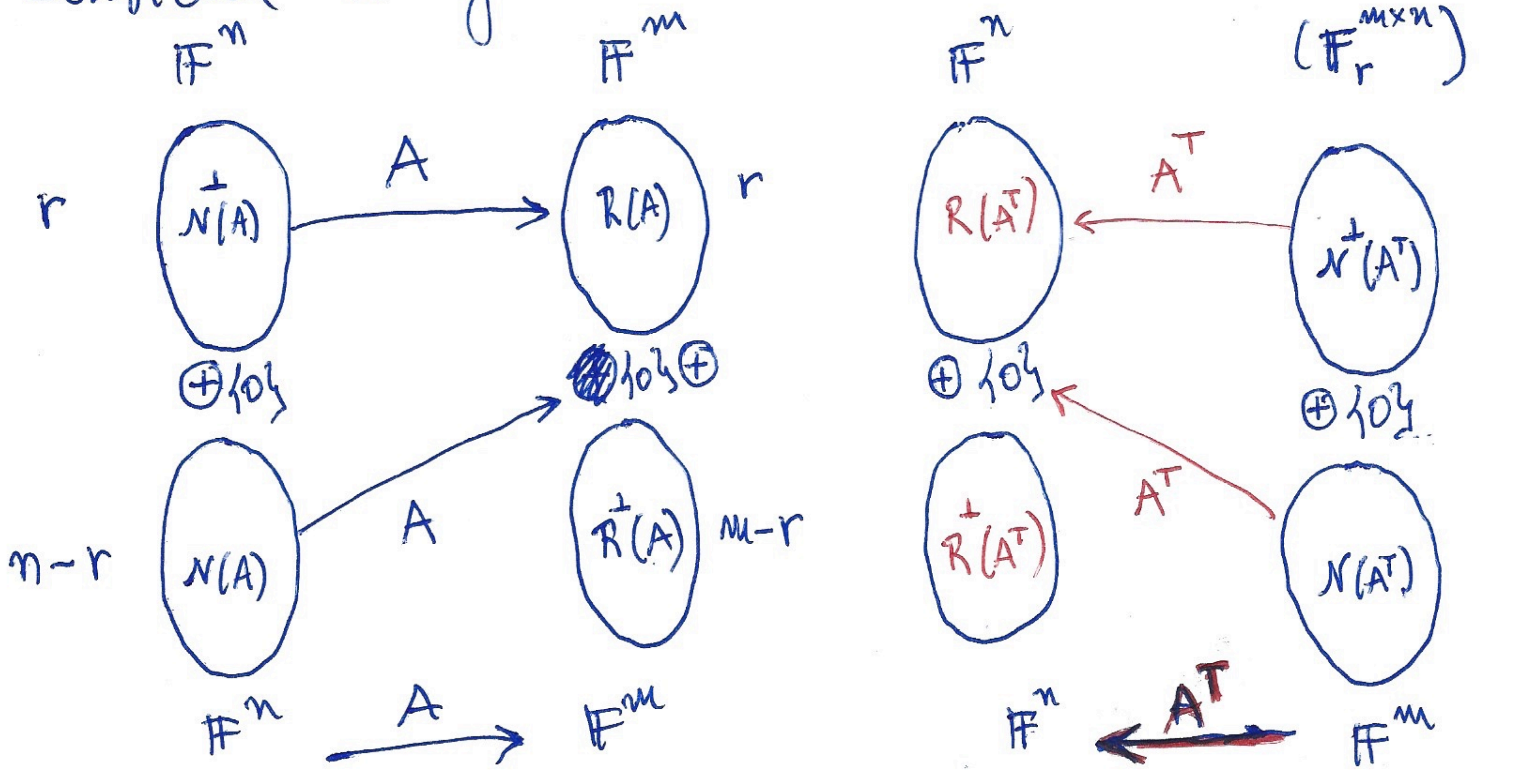
$$x_\parallel = y \in S$$

$$x_\perp = z \in S^\perp$$

4.6. The FOUR FUNDAMENTAL SPACES

the definitions of $R(A)$, $R(A^T)$, $N(A)$, $N(A^T)$ and the concept of Orthogonal Complement provide us with two results that allows for a very important decomposition of the domain $\mathbb{F}^n$ and the codomain $\mathbb{F}^m$ within a LT $A: \mathbb{F}^n \to \mathbb{F}^m$, together with $A^T: \mathbb{F}^m \to \mathbb{F}^n$.

Consider a general r-rank LT $A \in M^r_{m\times n}(\mathbb{F})$ $(\mathbb{F}_r^{m\times n})$



$$x = \underset{N_A^{\perp}}{u} + \underset{N_A}{v}$$

$$y = \underset{N_{A^T}^{\perp}}{p} + \underset{N_{A^T}}{q}$$

$$Ax = A(u+v)$$
$$= Au + Av$$
$$= Au \quad (Av = 0 \in \mathbb{F}^m)$$
$$= R(A) \subseteq \mathbb{F}^m$$

$$A^T y = A^T(p+q) = A^T p + A^T q$$
$$= A^T p \cong R(A^T) \subseteq \mathbb{F}^n$$

thm: Let $A : \mathbb{F}^n \to \mathbb{F}^m$ be a LT.

1. $\mathcal{N}^\perp(A) = R(A^T)$
2. $R^\perp(A) = \mathcal{N}(A^T)$

Remark with sets, $A = B$ means $A \subseteq B$ and $B \subseteq A$

Proof: LAUB proves 1, we prove 2

a) take an arbitrary $y \in \mathcal{N}(A^T)$

$$A^T y = 0 \in \mathbb{F}^n \quad \text{with } A^T \neq 0, \; y \neq 0$$

then $$x^T A^T y = x^T 0 = 0 \quad (\text{because } y \in \mathcal{N}(A^T))$$

or $$(\underbrace{Ax}_{\neq 0})^T y = 0 \quad \text{which means } y \perp Ax \; \forall x$$

(Recall $\mathbb{F}^m = R(A) \oplus R^\perp(A)$)

as $x \in \mathbb{F}^n$ varies, $Ax = R(A)$, or $y \perp R(A)$. or $y \in R^\perp(A)$

~~But $\mathbb{F}^m = R(A) \oplus R^\perp(A)$. then $y \in \mathcal{N}(A^T)$~~

~~$y \perp R(A)$ means $y \in R^\perp(A)$~~

but $y \in \mathcal{N}(A^T) \Rightarrow y \perp R(A) \equiv y \in R^\perp(A)$

$\forall \; y \in \mathcal{N}(A^T) \Rightarrow y \perp R(A)$ or $y \in R^\perp(A)$

or $$\boxed{\mathcal{N}(A^T) \subseteq R^\perp(A)}$$

b) Now take $Y \in R^{\perp}(A)$. Then $Y \perp R(A)$.

If we consider Ax generates $R(A)$ as x varies

then

$$(Ax)^T y = 0 \quad \forall\ x \in \mathbb{F}^n \text{ (because } y \perp R(A))$$

or

$$x^T \underbrace{A^T y}_{\hat{=}\, z} = 0 \iff x^T z = 0 \ \ \forall x$$

the only vector that is $\perp$ to any other vec x is the zero vec, i.e., $z = 0$

that is, $z = 0$ or $A^T y = 0$. $\forall y$

as $A^T \neq 0$, $Y \neq 0$, then $Y \in \mathcal{N}(A^T)$.

In other words

$$Y \in R^{\perp}(A) \Rightarrow Y \in \mathcal{N}(A^T)$$

or $\boxed{R^{\perp}(A) \subseteq \mathcal{N}(A^T)}$.

Together with $\mathcal{N}(A^T) \subseteq R^{\perp}(A)$, we conclude

$$\boxed{\mathcal{N}(A^T) = R^{\perp}(A)}$$

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Decomposition theorem

Let $A : \mathbb{F}^n \to \mathbb{F}^m$ be a LT.

1. Every $v \in \mathbb{F}^n$ (domain) can be written uniquely as

$$v = x + y \quad , \quad x \in \mathcal{N}(A),\ y \in \mathcal{N}^{\perp}(A) = \mathcal{R}(A^T)$$

or

$$\boxed{\mathbb{F}^n = \mathcal{N}(A) \oplus \mathcal{R}(A^T)}$$

2. Every $\omega \in \mathbb{F}^m$ (codomain) can be written uniquely as

$$\omega = z + t \quad , \quad z \in \mathcal{R}(A),\ t \in \mathcal{R}^{\perp}(A) = \mathcal{N}(A^T)$$

or

$$\boxed{\mathbb{F}^m = \mathcal{R}(A) \oplus \mathcal{N}(A^T)}$$

How to find bases for $\mathcal{R}(A)$, $\mathcal{R}^{\perp}(A)$, $\mathcal{N}(A^T)$, $\mathcal{N}^{\perp}(A^T)$??

Echelon forms on A and A^T.

Home Work 06

Oct/28/2021

1. Let $S \subseteq \mathbb{R}^n$. Prove the following results
   a) $S^{\perp} \subseteq \mathbb{F}^n$;
   b) $(S^{\perp})^{\perp} = S$.

2. Determine bases for the four fundamental spaces of the LT below

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 2 & 5 & 5 & 3 \end{bmatrix}$$

3. Consider a matrix $A_{3\times 3}$ such that

$$R = sp\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\} \quad \text{and} \quad S = sp\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\},$$

and consider a linear system $Ax = b$ in which $R(A) = R$ and $N(A) = S$, and $b^T = [1 \ -7 \ 0]$
   a) Explain why $Ax = b$ must be consistent;
   b) Explain why $Ax = b$ cannot have a unique solution.

4. Consider the matrices $A = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 0 & 6 \\ 1 & 2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -4 & 4 \\ 4 & -8 & 6 \\ 0 & -4 & 5 \end{bmatrix}$
   a) Do A and B have the same row space?
   b) Do A and B have the same col space?
   c) Do A and B have the same null space?
   d) Do A and B have the same left null space?