1 Basis for plane wave solutions of Dirac equation

Consider the Dirac equation

$$(\gamma^\mu\, p_\mu - m\, c)\, \psi = 0 \tag{1.1}$$

with

$$p_\mu = i\, \hbar\, \partial_\mu \tag{1.2}$$

and

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{1.3}$$

and

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.4}$$

We have four linear independent plane wave solutions given by

$$\psi^u_r = u_r\,(\vec{p})\; e^{-i\, p\cdot x/\hbar} \qquad \psi^v_r = v_r\,(\vec{p})\; e^{i\, p\cdot x/\hbar} \qquad r = 1, 2 \tag{1.5}$$

where the spinors satisfy the algebraic equations

$$(\gamma^\mu\, p_\mu - m\, c)\, u_r\,(\vec{p}) = 0 \qquad (\gamma^\mu\, p_\mu + m\, c)\, v_r\,(\vec{p}) = 0 \tag{1.6}$$

and where

$$c\, p_0 = E = \sqrt{\vec{p}^2\, c^2 + m^2\, c^4} \tag{1.7}$$

A suitable basis for the spinors is

$$u_1\,(\vec{p}) \;=\; N_- \begin{pmatrix} \frac{p_3+|\vec{p}|}{p_1+ip_2} \\ 1 \\ \frac{c|\vec{p}|(p_3+|\vec{p}|)}{(p_1+ip_2)(E+m\,c^2)} \\ \frac{c|\vec{p}|}{E+m\,c^2} \end{pmatrix} \qquad u_2\,(\vec{p}) = N_+ \begin{pmatrix} \frac{p_3-|\vec{p}|}{p_1+ip_2} \\ 1 \\ \frac{c|\vec{p}|(|\vec{p}|-p_3)}{(p_1+ip_2)(E+m\,c^2)} \\ -\frac{c|\vec{p}|}{E+m\,c^2} \end{pmatrix} \tag{1.8}$$

$$v_1\,(\vec{p}) = N_+ \begin{pmatrix} \frac{c|\vec{p}|(|\vec{p}|-p_3)}{(p_1+ip_2)(E+m\,c^2)} \\ -\frac{c|\vec{p}|}{E+m\,c^2} \\ \frac{p_3-|\vec{p}|}{p_1+ip_2} \\ 1 \end{pmatrix} \qquad v_2\,(\vec{p}) \;=\; N_- \begin{pmatrix} \frac{c|\vec{p}|(p_3+|\vec{p}|)}{(p_1+ip_2)(E+m\,c^2)} \\ \frac{c|\vec{p}|}{E+m\,c^2} \\ \frac{p_3+|\vec{p}|}{p_1+ip_2} \\ 1 \end{pmatrix} \tag{1.9}$$

with

$$N_\pm = \frac{1}{2}\sqrt{\left(1 \pm \frac{p_3}{|\,\vec{p}\,|}\right)\,\left(1 + \frac{E}{m\,c^2}\right)} \tag{1.10}$$

They satisfy

$$u^\dagger_r\,(\vec{p})\; u_s\,(\vec{p}) = v^\dagger_r\,(\vec{p})\; v_s\,(\vec{p}) = \frac{E}{m\,c^2}\;\delta_{r,s} \tag{1.11}$$

and

$$u_r^\dagger(\vec{p})\; v_s(-\vec{p}) = 0 \tag{1.12}$$

In addition, they are eigenstates of the helicity

$$\frac{\vec{\Sigma}\cdot\vec{p}}{|\,\vec{p}\,|}\,u_1(\vec{p}) = u_1(\vec{p}) \qquad \frac{\vec{\Sigma}\cdot\vec{p}}{|\,\vec{p}\,|}\,v_1(\vec{p}) = -v_1(\vec{p}) \tag{1.13}$$

and

$$\frac{\vec{\Sigma}\cdot\vec{p}}{|\,\vec{p}\,|}\,u_2(\vec{p}) = -u_2(\vec{p}) \qquad \frac{\vec{\Sigma}\cdot\vec{p}}{|\,\vec{p}\,|}\,v_2(\vec{p}) = v_2(\vec{p}) \tag{1.14}$$

with

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{1.15}$$

and so

$$\vec{\sigma}\cdot\vec{p} = \begin{pmatrix} p_3 & p_1 - i\,p_2 \\ p_1 + i\,p_2 & -p_3 \end{pmatrix} \tag{1.16}$$

So, we can write

$$\frac{\vec{\Sigma}\cdot\vec{p}}{|\,\vec{p}\,|}\,u_r(\vec{p}) = (-1)^{r+1}\,u_r(\vec{p}) \qquad \frac{\vec{\Sigma}\cdot\vec{p}}{|\,\vec{p}\,|}\,v_r(\vec{p}) = (-1)^{r}\,v_r(\vec{p}) \tag{1.17}$$

One can also check that

$$\sum_{r=1}^{2} \bar{u}_r(\vec{p})_\alpha\; u_r(\vec{p})_\beta = \frac{1}{2\,m\,c}\,\left[\gamma^\mu\,p_\mu + m\,c\,\mathbb{1}\right]_{\alpha\,\beta} \tag{1.18}$$

and

$$\sum_{r=1}^{2} \bar{v}_r(\vec{p})_\alpha\; v_r(\vec{p})_\beta = \frac{1}{2\,m\,c}\,\left[\gamma^\mu\,p_\mu - m\,c\,\mathbb{1}\right]_{\alpha\,\beta} \tag{1.19}$$

2 Helicity and Chirality

[1]

We have seen the concept of helicity: if the spin of a particle is in the direction of its motion the helicity is right-handed or positive, and if it is contrary to the direction of motion is left-handed or negative. However, the helicity is not a Lorentz invariant quantity. Indeed, if the particle has mass and so is traveling with a velocity smaller than that of light then an observer can overtake the particle and will see its velocity in the opposite direction but not its spin. So, the value of helicity depends on the reference frame. On the other hand, if the

[1]See section II.1 of A. Zee, Quantum Field Theory in a Nutshell, and chapter 2 of Itzykson and Zuber, Quantum Field Theory.

particle is massless it is bound to travel with the speed of light and no observer can overtake it. So, its helicity is independent of the reference frame.

In order to discuss chirality let us introduce the matrix

$$\gamma_5 = \gamma^5 = i\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 \tag{2.1}$$

Using the basis (1.3) one gets

$$\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{2.2}$$

An important property of such a matrix is that it commutes with the generators of the Lorentz group in the spinor representation. Indeed, a Dirac spinor transforms as

$$\psi \to S\,\psi \qquad\qquad S^{-1}\,\gamma^\mu\,S = \Lambda^\mu_\nu\,\gamma_\nu \tag{2.3}$$

where Λ is the matrix of Lorentz transformation in the vector representation

$$x'^\mu = \Lambda^\mu_\nu\,x^\nu \tag{2.4}$$

We have that

$$S = e^{-\frac{i}{4}\,\varepsilon_{\mu\nu}\,\sigma^{\mu\nu}} \qquad\qquad \sigma^{\mu\nu} = \frac{i}{2}\,[\,\gamma^\mu\,,\,\gamma^\nu\,] \tag{2.5}$$

Since, γ^5 anti-commutes with all γ-matrices, i.e.

$$\left\{\gamma^5\,,\,\gamma^\mu\right\} = 0 \tag{2.6}$$

It follows that

$$\left[\gamma^5\,,\,\sigma^{\mu\nu}\right] = 0 \qquad \to \qquad \left[\gamma^5\,,\,S\right] = 0 \tag{2.7}$$

None of the four γ^μ matrices have such a property. In addition, one has that

$$\left(\gamma^5\right)^2 = \mathbb{1} \tag{2.8}$$

Therefore, the eigenvalues of γ^5 are ± 1, and such eigenvalues are Lorentz invariant. Such eigenvalues are the chirality of a spin 1/2 particle. Contrary to helicity they do not depend upon the reference frame. Given a Dirac spinor ψ, one can split it into eigenvectors of the chirality using projectors, i.e.

$$\psi_R = \frac{1}{2}\left(1+\gamma^5\right)\,\psi \qquad\qquad \psi_L = \frac{1}{2}\left(1-\gamma^5\right) \tag{2.9}$$

and

$$\gamma^5\,\psi_R = +\,\psi_R \qquad\qquad \gamma^5\,\psi_L = -\,\psi_L \tag{2.10}$$

Due to (2.7) such eigenvectors transform independently under Lorentz. From (2.3) one has

$$\psi_R \to S\,\psi_R \qquad\qquad \psi_L \to S\,\psi_L \tag{2.11}$$

The spinors ψ_R and ψ_L are called Weyl spinors.

In the case of massless spinors chirality and helicity coincide. Indeed, the Dirac equation for a massless spinor is

$$\gamma^\mu \, p_\mu \, \psi = 0 \tag{2.12}$$

Multiplying such an equation by $\gamma^5 \, \gamma^0 = -i \, \gamma^1 \, \gamma^2 \, \gamma^3$, gives

$$\left(\gamma^5 \, p_0 + i \, \gamma^2 \, \gamma^3 \, p_1 - i \, \gamma^1 \, \gamma^3 \, p_2 + i \, \gamma^1 \, \gamma^2 \, p_3 \right) \, \psi = 0 \tag{2.13}$$

or

$$\vec{\Sigma} \cdot \vec{p} \, \psi = \gamma^5 \, p_0 \, \psi \tag{2.14}$$

with $\vec{\Sigma}$ given in (1.15). Since $m = 0$ we have $E = p_0 \, c = \mid \vec{p} \mid \, c$, and so

$$\frac{\vec{\Sigma} \cdot \vec{p}}{\mid \vec{p} \mid} \, \psi = \gamma^5 \, \psi \tag{2.15}$$

and so the eigenvalues of the helicity operator $\frac{\vec{\Sigma} \cdot \vec{p}}{|\vec{p}|}$ and the eigenvalues of the chirality operator γ^5 are indeed the same when $m = 0$.

Note that if a Dirac spinor ψ satisfies (2.12) so does $\gamma^5 \, \psi$, since γ^5 anti-commutes with γ^μ. Therefore, the Weyl spinors satisfy the massless Dirac equations

$$\gamma^\mu \, p_\mu \, \psi_R = 0 \qquad\qquad \gamma^\mu \, p_\mu \, \psi_L = 0 \tag{2.16}$$

For the case $m \neq 0$ the equations get coupled. Indeed, if ψ satisfies the massive Dirac equation

$$\left(\gamma^\mu \, p_\mu \, - m \, c \right) \psi = 0 \tag{2.17}$$

then

$$\gamma^\mu \, p_\mu \, \psi_L = m \, c \, \psi_R \qquad\qquad \gamma^\mu \, p_\mu \, \psi_R = m \, c \, \psi_L \tag{2.18}$$

The Dirac Lagrangian becomes

$$\mathcal{L} = \bar{\psi} \left(i \, \hbar \, \gamma^\mu \, \partial_\mu - m \, c \right) \, \psi = \bar{\psi}_L \, i \, \hbar \, \gamma^\mu \, \partial_\mu \psi_L + \bar{\psi}_R \, i \, \hbar \, \gamma^\mu \, \partial_\mu \psi_R - m \, c \, \left(\bar{\psi}_L \, \psi_R + \bar{\psi}_R \, \psi_L \right) \tag{2.19}$$

Note that the Dirac Lagrangian is invariant under the (global) phase transformation

$$\psi \rightarrow e^{i \, \theta} \, \psi \qquad\qquad \psi_R \rightarrow e^{i \, \theta} \, \psi_R \qquad\qquad \psi_L \rightarrow e^{i \, \theta} \, \psi_L \tag{2.20}$$

and the corresponding Noether conserved current is

$$J^\mu = \bar{\psi} \, \gamma^\mu \, \psi \tag{2.21}$$

For the case $m = 0$ the Dirac Lagrangian is also invariant under

$$\psi \rightarrow e^{i \, \theta \, \gamma^5} \, \psi \qquad\qquad \psi_R \rightarrow e^{i \, \theta} \, \psi_R \qquad\qquad \psi_L \rightarrow e^{-i \, \theta} \, \psi_L \tag{2.22}$$

and the corresponding Noether conserved current is

$$J_5^\mu = \bar{\psi} \, \gamma^\mu \, \gamma^5 \, \psi \tag{2.23}$$

3 Parity

The spatial parity transformation is

$$x^{\mu} \rightarrow x'^{\mu} = \left(x^0\,,\, -\vec{x}\right) \tag{3.1}$$

Then multiplying (2.17) by γ^0 one gets

$$\gamma^0 \left(i\,\hbar\,\gamma^{\mu}\,\partial_{\mu}\, - m\,c\right)\psi = \left(i\,\hbar\,\gamma^{\mu}\,\partial'_{\mu}\, - m\,c\right)\gamma^0\,\psi = 0 \tag{3.2}$$

So, $\gamma^0\,\psi$ satisfies the transformed equation and so

$$\psi'\left(x'\right) = \gamma^0\,\psi\left(x\right) \tag{3.3}$$

Note that

$$\begin{aligned} \bar{\psi}'\left(x'\right)\,\psi'\left(x'\right) &= \bar{\psi}\left(x\right)\,\psi\left(x\right) \\ \bar{\psi}'\left(x'\right)\,\gamma^5\,\psi'\left(x'\right) &= -\bar{\psi}\left(x\right)\,\gamma^5\,\psi\left(x\right) \end{aligned} \tag{3.4}$$

Since both are Lorentz scalars we say that $\bar{\psi}\left(x\right)\,\psi\left(x\right)$ is a scalar and $\bar{\psi}\left(x\right)\,\gamma^5\,\psi\left(x\right)$ is a pseudoscalar.

4 Coupling to the electromagnetic field

The coupling to the electromagnetic field is made by the so-called minimal coupling which is an infinitesimal version of Weyl's gauge principle. One replaces the ordinary derivative by the covariant derivative

$$\partial_{\mu} \qquad \rightarrow \qquad D_{\mu} \equiv \partial_{\mu} - i\,e\,A_{\mu} \tag{4.1}$$

where A_{μ} is the four potential of Maxwell theory. The Dirac Lagrangian and Dirac equation become

$$\mathcal{L} = \bar{\psi}\left(i\,\hbar\,\gamma^{\mu}\,D_{\mu} - m\,c\right)\,\psi \qquad\qquad \left(i\,\hbar\,\gamma^{\mu}\,D_{\mu} - m\,c\right)\,\psi = 0 \tag{4.2}$$

Now, the global phase transformation (2.20) can be made local

$$\psi \rightarrow e^{i\,\theta(x)}\,\psi \qquad\qquad A_{\mu} \rightarrow A_{\mu} + \frac{1}{e}\,\partial_{\mu}\theta \tag{4.3}$$

So, the covariant derivative of the field transforms the same way as the field

$$D_{\mu}\psi \rightarrow e^{i\,\theta(x)}\,D_{\mu}\psi \tag{4.4}$$

5 Charge Conjugation

The Dirac equation (4.2) coupled to the electromagnetic field is

$$\left(i\,\hbar\,\gamma^{\mu}\,\left(\partial_{\mu}-i\,e\,A_{\mu}\right)-m\,c\right)\,\psi=0 \tag{5.1}$$

Taking the complex conjugate we get

$$\left(-i\,\hbar\,\gamma^{\mu\,*}\,\left(\partial_{\mu}+i\,e\,A_{\mu}\right)-m\,c\right)\,\psi^{*}=0 \tag{5.2}$$

But

$$\{\gamma^{\mu}\,,\,\gamma^{\nu}\}=2\,g^{\mu\nu}\,\mathbb{1} \qquad \rightarrow \qquad \{-\gamma^{\mu\,*}\,,\,-\gamma^{\nu\,*}\}=2\,g^{\mu\nu} \tag{5.3}$$

and so there must exist a matrix C such that

$$-\gamma^{\mu\,*}=C^{-1}\,\gamma^{\mu}\,C \tag{5.4}$$

Denoting

$$\psi_{c}\equiv C\,\psi^{*} \tag{5.5}$$

the equation (5.2) becomes

$$\left(i\,\hbar\,\gamma^{\mu}\,\left(\partial_{\mu}+i\,e\,A_{\mu}\right)-m\,c\right)\,\psi_{c}=0 \tag{5.6}$$

So, ψ_c satisfies the Dirac equation with a charge of opposite sign to that satisfied by ψ. So, ψ_c should be associated to the anti-particle associated to ψ. For the basis (1.3) we have that γ^2 is the only pure imaginary γ-matrix, since the other are real. Therefore, C should commute with γ^2. In fact, $C=\gamma^2$, and $C^{-1}=-\gamma^2$, and so

$$\psi_{c}=\gamma^{2}\,\psi^{*} \tag{5.7}$$

Note that

$$\psi_{R}^{c}=\gamma^{2}\,\psi_{R}^{*}=\gamma^{2}\,\frac{1}{2}\left(1+\gamma^{5}\right)\,\psi^{*}=\frac{1}{2}\left(1-\gamma^{5}\right)\,\psi_{c} \tag{5.8}$$

and

$$\psi_{L}^{c}=\frac{1}{2}\left(1+\gamma^{5}\right)\,\psi_{c} \tag{5.9}$$

So, the charge conjugate of a left handed field is right handed and vice-versa.

From (2.5) and (5.4) we have that

$$\sigma^{\mu\nu\,*}=-\frac{i}{2}\left[\,\gamma^{\mu\,*}\,,\,\gamma^{\nu\,*}\,\right]=-C^{-1}\,\sigma^{\mu\nu}\,C \tag{5.10}$$

Therefore, from (2.5)

$$\psi^{\prime}=S\,\psi \qquad \rightarrow \qquad \psi_{c}^{\prime}=C\,e^{\frac{i}{4}\,\varepsilon_{\mu\nu}\,\sigma^{\mu\nu\,*}}\,\psi^{*}=S\,\psi_{c} \tag{5.11}$$

So, ψ_c transform under the Lorentz group in the same way as ψ.

6 The Majorana equation

Since ψ_c transforms as a spinor, Majorana noted that the following equation

$$i\,\hbar\,\gamma^\mu\,\partial_\mu\psi = m\,c\,\psi_c \tag{6.1}$$

is also invariant under the Lorentz group. That is the Majorana equation. Note that by complex conjugating (6.1) and multiplying by C one gets

$$i\,\hbar\,\gamma^\mu\,\partial_\mu\psi_c = m\,c\,\psi \tag{6.2}$$

In addition, acting with $i\,\hbar\,\gamma^\nu\,\partial_\nu$ on (6.1) and using (6.2) one gets

$$\partial^2\psi + \frac{m^2\,c^2}{\hbar^2}\,\psi = 0 \tag{6.3}$$

and so m^2 is indeed a mass, and it is the Majorana mass. The Majorana equation can be obtained from the Majorana Lagrangian

$$\mathcal{L} = \bar{\psi}\,i\,\hbar\,\gamma^\mu\,\partial_\mu\psi - \frac{1}{2}\,m\,\left(\psi^T\,C\,\psi + \bar{\psi}\,C\,\bar{\psi}^T\right) \tag{6.4}$$

Note that since ψ and ψ_c carry opposite charges, the Majorana equation can only be applied to electrically neutral particles. Indeed, the Majorana equation is not invariant under the phase transformation

$$\psi \to e^{i\,\theta}\,\psi \qquad\qquad \psi \to e^{-i\,\theta}\,\psi_c \tag{6.5}$$

and so the electric charge would not be conserved.

Note that multiplying (6.1) by $\frac{1}{2}\,(1+\gamma^5)$ and using (5.8) and (5.9) we get that

$$i\,\hbar\,\gamma^\mu\,\partial_\mu\psi_L = m\,c\,\psi_L^c \tag{6.6}$$

and similarly multiplying (6.1) by $\frac{1}{2}\,(1+\gamma^5)$ we get

$$i\,\hbar\,\gamma^\mu\,\partial_\mu\psi_R = m\,c\,\psi_R^c \tag{6.7}$$

So, contrary to the Dirac equation (see (2.18)), the Majorana equation preserves chirality, and so it is tailor made for the neutrino.

The spinors satisfying

$$\psi = \psi_c \tag{6.8}$$

are called Majorana spinors.