Principles of High-Level Net Theory

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Abstract. The paper gives an introduction to fundamentals and recent trends in the theory of high-level nets. High-level nets are first formally derived from low-level nets by means of a quotient construction. Based on a linear-algebraic representations, we develop an invariant calculus that essentially corresponds to the algebraic core of the well-known coloured nets. We demonstrate that the modelling power of high-level nets stems from the use of expressive symbolic annotation languages, where as a typical model we consider predicate-transition nets, both concrete models and net-schemes. As examples of specific high-level analysis-tools we discuss symbolic place-invariants and reachability-trees.

1 Introduction

Large scale modelling with nets would be impossible without the conceptual and notational convenience offered by high-level descriptions. The characteristic feature of high-level nets is that tokens represent individual data items (as opposed to boolean conditions in elementary nets or "dimensionless" multiplicities in place-transition nets). The use of structured tokens permits concise net representations of complex systems, ranging from distributed data bases to communication networks and flexible manufacturing systems. Throughout the years an ever increasing number of different high-level models has been proposed and put into use.

From a mathematical point of view, high-level models are essentially a concise representation of low-level nets, obtained by a suitable folding-operation. This approach has the advantage, that well-understood low-level concepts such as the semantics of non-sequential processes, but also analysis tools such as invariant-calculus can be lifted canonically from low-level to high-level models. We shall discuss the precise relationship in the first part of the paper.

However, the main interest in high-level nets for system modelling is due to the expressiveness of powerful description languages and specific analysis tools based on these. We shall develop and illustrate some of the central ideas, using *predicate-transition nets* as a paradigmatic example.

The paper is organized as follows. Section 2 contains standard mathematical prerequisites. Section 3 provides (or recalls) the basic concepts of *place-transition nets* and *elementary net systems*. In Section 4 we introduce the fundamental concepts leading from low- to high-level nets and the relationship between the levels. The key idea is folding of system representation along certain equivalence relations. In the following Section 5 we develop a linear-algebraic representation and a corresponding invariant-calculus. This leads to an approach similar to the original version of the well-known *coloured nets*. Section 6 gives a precise definition of *predicate-transition nets*, arguably *the* classical high-level model. Sections 7 and 8 are concerned with analysis methods. Among the topics covered are *invariants*, in particular *place-invariants*, and *reachability analysis*. In Section 9 we consider *abstract* and *algebraic predicate-transition nets*, where the logic symbols are not a priori bound to a fixed interpretation. Section 10 presents an overview over recent trends.

The paper is essentially self-contained and does in particular not require any formal knowledge of net theory. However for motivation purposes, some basic familiarity with modelling with nets will certainly be useful.

2 Prerequisites

We use standard notations for handling sets. In particular we write $A \subseteq B$ if A is a subset of B. The complement of B in A will be denoted by $A - B$. The cardinality of a set A will be denoted by $|A|$.

The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. We use $\mathbb{Z}$ as symbol for the integers.

A *multiset* over a set A is a mapping from A to $\mathbb{N}$. The collection of all multisets over A is denoted by $\mathcal{M}(A)$.

For a binary relation R we usually write $x\,R\,y$ instead of $(x, y) \in R$.

Let A be a set, and $n, m \geqslant 1$. An $n \times m$*-matrix in* A is a matrix M with n rows and m columns, where each entry M_{ij} is an element from A. If $(A, +, \cdot)$ is a (non-necessarily commutative) ring over A, we assume the ring-operations to be continued to matrix operations as usual.

We shall often have use for expressions denoting *symbolic sums* over certain sets, formally defined as follows. For a set X the *free abelian group* over X is the pair $(\mathfrak{G}(X), +)$ where $\mathfrak{G}(X)$ the set of functions $f: X \to \mathbb{Z}$ such that $\{x \in X \mid f(x) \neq 0\}$ is finite. Addition between functions is defined componentwise as usual, as is scalar multiplication of the form $m \cdot f$ for $m \in \mathbb{Z}$. Identifying each element $x \in X$ with the characteristic function $\chi_x: x \mapsto 1$ and $\chi_x: y \mapsto 0$ for $y \neq x$, we may interpret any expression of the form $m_1 \cdot x_1 + \cdots + m_n \cdot x_n$, with $x_i \in X, m_i \in \mathbb{Z}$, as an element of the group $\mathfrak{G}(X)$.

3 Low-level net systems

This section gives a short introduction to the notions and concepts of low-level nets that will be needed in this paper. For a more detailed treatment we refer to [22,25] and the corresponding articles in this volume.

Example 3.1. As a running example we consider the notoriously known system of five philosophers. The philosophers are sitting around a table, and between any two neighbouring philosophers there is one fork. Each philosopher may assume one of two states, thinking or eating. For eating he needs both his left and the right fork. The situation for one philosopher is illustrated by the *net-system* in Fig. 1. (Formal definitions will be given below.)

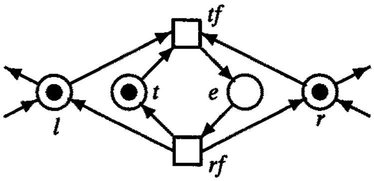

Fig. 1. Part of an elementary net model of the five-philosophers system

The philosopher is presently in the state of thinking (represented by the black token within the circle t). The tokens in the circles l and r indicate that the left and right forks are available. When the philosopher starts eating, he takes both forks, which in the figure would be represented by a removal of the tokens from t, l, r, and marking e with a token instead. When he has finished, he releases the forks (represented by the box rf) and returns to thinking. The additional arrows indicate that his neighbour philosophers share a common fork with him. If on of them picks up one of the shared forks, the transition from thinking to eating tf is disabled. If all of the possible actions and states of all philosophers are connected via the "interface-arrows" to the left of l and right of r, we arrive at a net-representation of the complete five-philosophers system, more precisely at an elementary-net representation.

3.1 Formal definitions

Definition 3.2. By a *net* we shall mean a triple $N := (P, T, F)$ satisfying the following properties:

(i) $P \cap T = \emptyset$,
(ii) $F \subseteq (P \times T) \cup (T \times P)$.

The elements of P and T are called *places* and *transitions*, respectively. The relation F is called *flow relation*.

As illustrated in Example 3.1, we follow the usual graphical conventions for representing nets: places are drawn as circles or ellipses, transitions as boxes and the flow relation F is indicated by appropriately directed arcs.

We use X to denote the *set of elements* $P \cup T$ of a net N. For an element x of X, the set ${}^\bullet x := \{y \in X \mid y\,F\,x\}$ is the *pre-set* of x, and $x^\bullet := \{y \in X \mid x\,F\,y\}$ is the *post-set*. The union of ${}^\bullet x$ and $x^\bullet$ is denoted by $\mathrm{vic}(x)$.

The net N is said to be *pure*, if ${}^\bullet x \cap x^\bullet = \emptyset$ for all $x \in X$.

We come to net *semantics*. For reasons of technical convenience we start with *place-transition nets*, PT-nets for short. Elementary net systems, though conceptually more fundamental, are defined as special PT-nets.

Definition 3.3. A *place-transition net* $\mathrm{PT} = (N, W, K, M_{in})$ consists of a net N, a *weight function* $W: F \to \mathbb{N}^+$, a *partial* function K from P to $\mathbb{N}^+$, called *capacity constraint*, and a distinguished *initial marking* M_{in}, where in general a marking is a mapping from P to $\mathbb{N}$ such that $M(p) \leqslant K(p)$ whenever $K(p)$ is defined.

For technical convenience we usually assume the weight function to be trivially continued to all of $(P \times T) \cup (T \times P)$ by putting $W(x, y) = 0$ if $(x, y) \notin F$. Note that the flow-relation may be then be deducted from W by $x\ F\ y \Leftrightarrow W(x, y) \neq 0$.

A transition t is *enabled* in a marking M, in symbols $M \models t$,[1] iff for all $p \in {}^\bullet t$ we have $W(p, t) \leqslant M(p)$, and for all $p \in t^\bullet$ where K is defined, $M(p) + W(t, p) \leqslant K(p)$.

If t is enabled in M, *firing* of t yields the *follower marking* M', denoted $M[t\rangle M'$, given by $M' : p \mapsto M(p) - W(p, t) + W(t, p)$. If there is no need to mention the specific transition transforming M into M' we write $M \mathbf{r} M'$.

A marking M' is *reachable* from another marking M, in symbols $M' \in [M\rangle$, iff it can be reached from M by a finite sequence $M = M_0 \mathbf{r} M_1 \mathbf{r} \cdots \mathbf{r} M_n = M'$ of transition-firings.

Note that we use the letters PT as abbreviation for 'place-transition' and also as a symbol for actual PT-nets. This will usually not lead to any confusion. In fact, we implicitly adopt similar conventions also for other net classes.

We now define an elementary net system as a *strict* PT-net, i.e. a net where numeric values are limited to 0 or 1:

Definition 3.4. An *elementary net system*, EN-system or ENS for short, is a PT-net (N, W, K, M_{in}), where arc weights $W(x, y)$ are 1 for all arcs $(x, y) \in F$, the capacity $K(p)$ for each place is 1, and consequently all reachable markings are $\{0, 1\}$-valued. Due to their constant nature in EN-systems, the functions W and K are usually not mentioned. A $\{0, 1\}$-valued marking M is identified with the set $\{p \in P \mid M(p) = 1\}$, which is then again written as M.

Remark 3.5. Sometimes *isolated elements*, i.e. elements x such that $\text{vic}(x) = \emptyset$, are excluded from elementary net systems. This means in particular that every occurrence of a transition must have some visible effect.

Example 3.6. Disregarding the extra arrows connected to l and r, Fig. 1 in Example 3.1 shows an elementary net system.

3.2 Incidence matrix and invariants

In this paragraph, we assume nets to be *finite*, and the sets of places and transitions to be of the form $P = \{p_1, \dots, p_n\}$ and $T = \{t_1, \dots, t_m\}$ for some arbitrary but fixed order.

Definition 3.7. Let PT be a finite place-transition net as above. The *incidence matrix* of PT is the $n \times m$-matrix PT in $\mathbb{N}$ with entries

$$\mathsf{PT}_{ij} = W(t_j, p_i) - W(p_i, t_j).$$

The incidence matrix is the basis for *invariant-computations*, more precisely *place-invariants* and *transition-invariants*. In this paper we shall mainly be concerned with place-invariants, as these are by far the most relevant in theory and applications.

[1] This relation is often also written as $M[t\rangle$.

Definition 3.8. A *place-invariant* is an n-vector i in $\mathbb{N}$, written as a column-vector, such that

$$i^T \cdot \mathsf{PT} = \mathbf{0}, \tag{3.1}$$

where $\mathbf{0}$ denotes the 0-vector of length m.

The system-property characterized by such a place-invariant i is that

$$i^T \cdot M = i^T \cdot M_{\text{in}} \quad \text{for every } M \in [M_{\text{in}}\rangle, \tag{3.2}$$

where we assume markings to be written as column vectors with reference to the ordering $p_1, \ldots, p_n$ of P.

The proof of (3.2) is by induction using the following observation: Described in vector notation, the firing of a transition $t \in T$ in a marking M corresponds to the addition of the t-column from PT to M. Multiplication with i^T distributes over the sum, where $i^T \cdot t$ cancels out because of (3.1), and we are left with $i^T \cdot M$.

Place-invariants i can be used to show that a marking M is *not* reachable from the initial marking. A sufficient condition for this is that $i^T \cdot M \neq i^T \cdot M_{\text{in}}$.

A (trivial) example is the demonstration that not both of t and e in Example 3.1 can be marked at the same time. We leave this to the interested reader and return to place-invariants in more detail in the context of high-level nets below.

The notion of *transition-invariant* is defined dually to place-invariants: A transition-invariant is an m-vector $i = (i_1, \ldots, i_m)$ such that

$$\mathsf{PT} \cdot i = \mathbf{0},$$

where $\mathbf{0}$ now denotes the 0-vector of length n. The characteristic property of such a transition-invariant is that the effect of an i_k-fold occurrence of all transitions t_k for $k = 1, \ldots, m$ would reproduce the initial marking. Transition-invariants have attracted much less attention in the literature than place-invariants.

4 Quotient nets and basic high-level nets

After these preliminaries we now come to our main topic, high-level nets. The term is usually used rather informally for a whole class of nets, where the common idea is that a high-level net can be unfolded into a low-level net system, and thus conversely may be conceived as a condensed representation of the latter. This is also the point of view we shall adhere to in this paper.

In this section we show how high-level representations can in fact formally be derived from the structure and semantics of low-level nets. The central concept is *factorization* with respect to suitable *equivalence relations*, resulting in certain *net-quotients*. In the "condensed net", structure and behaviour of the original net is retained by the introduction of *individual* – instead of indistinguishable black – tokens and the distinction between different *firing-modes* for transitions. The development is a generalization of ideas from [29,28].

All definitions and constructions are formulated for place-transition nets. However, main emphasis will be put on elementary net systems *without multiplicities*. In particular, most examples will be of this type.

4.1 Quotient nets

We start with in informal example, which illustrates the main ideas.

Example 4.1. Consider the ENS defined by Fig. 2(a). The net (b) shows a quotient representation of (a). The "macro"-elements in the underlying net are sets of elements from the ENS. The black token on the place a_1 is translated into an individual token, again denoted a_1, on the macro-place $p_1 = \{a_1, a_2\}$. The macro-transitions may occur in various modes, corresponding to the original low-level transitions they contain. The specific effect of the original transitions does not appear in the figure, it will have to be represented by some additional annotation in the system description.

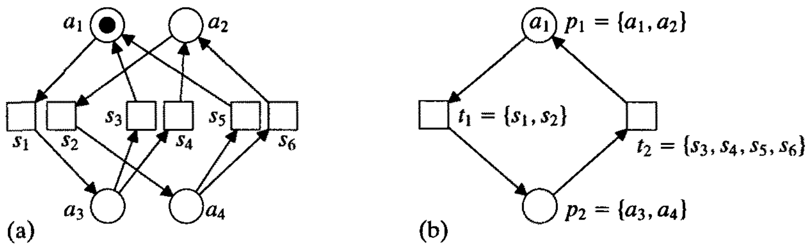

Fig. 2. An ENS (a) and a quotient representation (b).

Formalizing the idea behind Example 4.1, we shall now introduce high-level nets as quotient systems, derived from certain equivalence relations on low-level nets. Semantic notions will be represented in a condensed form as sets (or in general rather multisets) of low-level items.

The following definition collects the necessary notions concerning equivalences and quotient constructions. A more extensive treatment of quotient nets can be found in [29,27]. For the algebraic foundations of quotient constructions in general we refer to [7].

Definition 4.2. Let ρ be a fixed equivalence on the domain $X = P \cup T$ of a net N.

(i) For $x \in X$ let $\bar{x}$ denote the equivalence class $\{y \in X \mid x \rho y\}$ of x.
(ii) For $Y \subseteq X$ let $\bar{Y} := \{\bar{x} \mid x \in Y\}$.
(iii) ρ is *sort-respecting*, if $\rho \cap (P \times T) = \emptyset$.
(iv) The relation $\bar{F}$ on $\bar{X}$ is defined by

$$\bar{x} \bar{F} \bar{y} :\Leftrightarrow \exists x' \in \bar{x} \exists y' \in \bar{y} : x F y.$$

(v) We denote the triple $(\bar{P}, \bar{T}, \bar{F})$ by $\bar{N}$.

The definition of $\bar{F}$ ensures that arcs in $\bar{N}$ are inherited from N, no new arcs are introduced.

Clearly, *sort-respecting* means that each equivalence class contains *either* places or transitions, but not both. We are particularly interested in sort-respecting equivalences because of the following

Proposition 4.3. *Let ρ be an equivalence in a net N. The triple $\bar{N}$ is a net iff ρ is sort-respecting.*

Proof. It is sufficient to observe that $\bar{P} \cap \bar{T} = \emptyset \Leftrightarrow \rho \cap (P \times T) = \emptyset$. □

Definition 4.4. If ρ is sort-respecting, we call $\bar{N}$ a *quotient net*, more precisely the *quotient of N with respect to ρ*.

We shall now show how also the concept of marking and transition-occurrence may be suitably folded along with the net. We give the definition for general PT-nets, and then indicate more convenient representations for EN-systems.

Let N be a net, ρ a sort-respecting equivalence in N, and let $\bar{N}$ be the quotient of N with respect to ρ.

We first indicate how *markings* of N may be folded canonically to markings of $\bar{N}$. Recall that a marking of N is a map $M: P \to \mathbb{N}$, i.e. a multiset over P, symbolically $M \in \mathcal{M}(P)$. For such a marking we denote by $\bar{M}$ the map with domain $\bar{P}$ and as values multisets $\bar{M}(\bar{p}) \in \mathcal{M}(\bar{p})$, given by

$$\bar{M}(\bar{p})(q) = M(q). \tag{4.3}$$

By the same token, the capacity constraint K is translated into a mapping $\bar{K}$, the only difference being that K may be undefined for some values, which then also carries over to $\bar{K}$.

For the arc-weight function W we define $\bar{W}$ as the mapping with domain $\bar{F}$ and values $\bar{W}(\bar{x}, \bar{y}) \in \mathcal{M}(\bar{x} \times \bar{y})$ such that

$$\bar{W}(\bar{x}, \bar{y})(u, v) = W(u, v). \tag{4.4}$$

Transitions in $\bar{N}$ are *sets of transitions* from N. For a T-element $\bar{t}$ and $s \in \bar{t}$, we say that in a marking $\bar{M}$, $\bar{t}$ is *enabled in mode s*, if s is enabled in M, symbolically:

$$\bar{M} \models \bar{t}(s) :\Leftrightarrow M \models s. \tag{4.5}$$

In that case an *occurrence of $\bar{t}$ in mode s* yields the follower marking $\bar{B}$, where $B = A[s]$. Hence all behavioural notions are carried over canonically without the need for any specific new definition.

Putting all of the above together we arrive at the following definition of a quotient system:

Definition 4.5. Let $\mathrm{PT} = (N, W, K, M_{\mathrm{in}})$ be a PT-net, and ρ a sort-respecting equivalence in N. Then the system $\overline{\mathrm{PT}} := (\bar{N}, \bar{W}, \bar{K}, \bar{M}_{\mathrm{in}})$ is called the *quotient of* PT *with respect to ρ*. The *behaviour* of $\overline{\mathrm{PT}}$ is defined via the behaviour in PT according to (4.5).

Remark 4.6. Some of the behavioural notions may be simplified if we start out with an elementary net-system $\mathrm{ENS} = (N, M_{\mathrm{in}})$.

(i) Markings of ENS can be conceived as sets. Correspondingly, a marking $\bar{M}$ of the quotient can be conceived as a map with domain $\bar{P}$ and values $M(\bar{x}) \subseteq P$, such that

$$\bar{M}(\bar{x}) = \bar{x} \cap M. \tag{4.6}$$

(ii) The capacity constraint $\bar{K}$ does not have to be mentioned explicitly, since it always assumes the constant value 1 anyway; we usually write the quotient-representation in the form

$$\overline{\text{ENS}} = (\bar{N}, \bar{W}, \bar{M}_{\text{in}}). \tag{4.7}$$

(iii) The weight function $\bar{W}$ is related to the F-relation in N by

$$\bar{W}(\bar{x}, \bar{y})(u, v) = \begin{cases} 1, & \text{if } u\ F\ v \\ 0, & \text{otherwise.} \end{cases}$$

4.2 Basic high-level nets

From a mathematical point of view, Definition 4.5 already contains the quintessence of high-level nets. However, it is obviously not very convenient to work directly with net-quotients. It is usually sufficient to know that a high-level net *may be conceived* as such. To this end we introduce a notation of high-level nets, where all traces of quotient-origin have been discarded. We then show that it nonetheless does represent the same concept as the rigorous construction above.

The central idea is:

(i) places may be marked with *individual tokens*,
(ii) transitions fire in different *modes*, removing tokens from some places and adding tokens to some, where the only a-priori restriction is a *locality constraint*, namely that a transition in any mode may only remove tokens from its own pre-places and likewise add tokens only to its own post-places.

We formalize this as follows:

Definition 4.7. A *basic high-level system*, BHL-net or BHL for short, is a tuple

$$\text{BHL} = (N, D, \Phi, W, K, M_{\text{in}}),$$

where $N = (P, T, F)$ is a net and the remaining components are mappings explained in the following.

(i) D associates to each $p \in P$ a non-empty *domain* $D(p)$. i.e. a set of *individual tokens*.
(ii) A *marking* of BHL is a map M with domain P and values $M(p) \in \mathcal{M}(D(p))$.
(iii) The *capacity constraint* K is formally similar to a marking, except that $K(p)(a)$ may be undefined for some a in some $D(p)$. For all legal markings we require $M(p)(a) \leqslant K(p)(a)$, whenever the latter is defined.
(iv) M_{in} denotes a distinguished legal marking.

We now come to the meaning of Φ and W.

(v) Φ is a map which associates to each transition t a non-empty collection of *modes* $\Phi(t)$,
(vi) W is a map with domain F and values $W(p, t)$ in $\mathcal{M}\big(D(p) \times \Phi(t)\big)$ resp. $W(t, p)$ in $\mathcal{M}\big(\Phi(t) \times D(p)\big)$.

The *behaviour* of BHL is now defined as follows:

(vii) In a marking M a transition t is *enabled in a mode* $m \in \Phi(t)$ iff

$$\begin{aligned} &\forall p \in {}^\bullet t\ \forall a \in D(p): W(p,t)(a,m) \leqslant M(p)(a) \\ \text{and} \\ &\forall p \in t^\bullet\ \forall a \in D(p): M(p)(a) + W(t,p)(m,a) \leqslant K(p)(a), \end{aligned} \tag{4.8}$$

where again the capacity constraint is only relevant if $K(p)(a)$ is defined.

(viii) If the enabling condition is satisfied, a *firing* of t *in mode* m transforms the marking M into the follower marking M', where $M'(p) = M(p)$ for $p \notin \text{vic}(t)$ and for $p \in \text{vic}(t)$ and $a \in D(p)$ we have:

$$M'(p)(a) = M(p)(a) - W(p,t)(a,m) + W(t,p)(m,a). \tag{4.9}$$

Symbolically we use the notations $M \models t(m)$ and $M[t(m)\rangle M'$ to denote enabling, and transformation of markings through firing, respectively. Given these basic concepts, the definition of derived notions such as the *reachability class* $[M_{\text{in}}\rangle$ etc. is now straightforward.

The general definition may be simplified if item-multiplicities are excluded:

Definition 4.8. A basic high-level net BHL $= (N, D, \Phi, W, K, M_{\text{in}})$ is called *strict*, if all mappings involved are $\{0,1\}$-valued, i.e.

(i) for all markings M the values $M(p)(a)$ are in $\{0,1\}$, hence the $M(p)$ may be conceived as subsets of $D(p)$,
(ii) for every $p \in P$ and $a \in D(p)$, always $K(p)(a) = 1$, hence there is no need to mention the capacity constraint explicitly,
(iii) all $W(x,y)(u,v)$ belong to $\{0,1\}$, and may therefore be identified with the set $\{(u,v) \mid W(x,y)(u,v) = 1\}$.

We leave it to the reader to formulate simplifications for the general enabling rule and occurrence relation.

Remark 4.9. In Remark 3.5 we mentioned that in elementary net systems it is commonly required that each transition-occurrence must have some visible effect. The corresponding property for modes in $\Phi(t)$ is that for all $m \in \Phi(t)$ there is some $p \in {}^\bullet t$ or $p \in t^\bullet$, and some a in $D(p)$ such that $W(p,t)(a,m) \neq 0$ (resp. $W(t,p)(m,a) \neq 0$).

Example 4.10. Recall the elementary net model of the five-philosophers system in Example 3.1. We represent it as a (strict) basic high-level net BHL_{5Phil} as follows:

The underlying net consists of three places *think*, *eat*, and *avail*, denoting the currently thinking and eating philosophers, and available forks, respectively. We introduce two transitions *tf* "take forks" and *rf* "release forks" to describe the possible system dynamics when a philosopher decides to change his state between thinking and eating. The structure of the net (and the initial marking) is shown in Fig. 3.

We put $D(think) = D(eat) = \{ph_0, \ldots, ph_4\}$ and $D(avail) = \{fk_0, \ldots, fk_4\}$.

A marking of a place is a mapping which associates philosophers or forks to the respective places. Since we are dealing with strict nets, markings may be conceived as

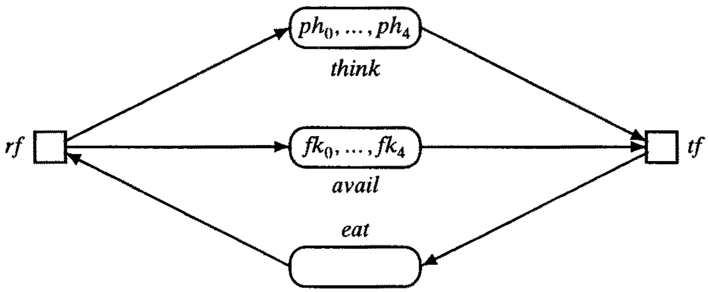

Fig. 3. The net N underlying a strict basic high-level model BHL_{5Phil} of the five-philosophers system, and the initial marking M_{in}, shown in set-representation.

subsets of the place-domains. The initial marking, where all philosophers are thinking and thus all forks are available, is given by $M_{\mathrm{in}}(think) = \{ph_0, \ldots, ph_4\}$, $M_{\mathrm{in}}(eat) = \emptyset$ and $M_{\mathrm{in}}(avail) = \{fk_0, \ldots, fk_4\}$.

Both of the transitions may fire in five modes, one for each philosopher; we put $\Phi(tf) = \Phi(rf) = \{m_0, \ldots, m_4\}$. A priori, there is no relationship between names of individual tokens and names of transition-modes. However, in concrete models it is often convenient to indicate intended correspondence by suitable names for transition-modes. In the present example, a natural choice of mode-sets is $\Phi(tf) = \Phi(rf) = Phil$ instead of the anonymous m_i.

We come to the *weight function* W, which describes the effect of firing-modes. In mode ph_i, the transitions represent the change of state for philosopher i. We thus define $W(think, tf)(ph_i ph_j) = 1$ if $i = j$, and 0 otherwise. Similarly, we define $W(avail, tf)(fk_i, ph_j) = 1$ if $i = j$ or $i = j \oplus 1$, and 0 otherwise. (The operator $\oplus$ denotes addition modulo 5.) We leave the definition of the remaining W-values to the reader. Since W is $\{0, 1\}$-valued, it may be identified with the values mapped to 1, and we may for instance write $W(think, tf) = \{(ph_0, ph_0), \ldots, (ph_4, ph_4)\}$.

4.3 From quotients to BHL-nets

We shall show that basic high-level nets and quotient systems are essentially the same; the difference lies only in the *description* of tokens and firing-modes. To this end, we show that (1) quotients may be conceived as BHL-nets, and conversely, that (2) BHLs may be obtained as quotients from low-level nets, such that (3) both operations are inverse to each other.

Consider first a quotient $\overline{\mathrm{PT}} = (\bar{N}, \bar{K}, \bar{W}, \bar{M}_{\mathrm{in}})$, derived from a place-transition net $\mathrm{PT} = (N, W, K, M_{\mathrm{in}})$ via a sort-respecting equivalence. We wish to translate it into a BHL-net. But this is straightforward. We *forget* that places and transitions in quotient nets are *sets* containing the essential information about the systems's behaviour; in the BHL that information is then instead made explicit in the mappings D and Φ:

For $\bar{p} \in \bar{P}$ and $\bar{t} \in \bar{T}$ we set

$$D(\bar{p}) := \bar{p} \quad \text{and} \quad \Phi(t) := \bar{t}. \tag{4.10}$$

It is now easily verified that markings $\bar{M}$, the capacity constraint $\bar{K}$ as well as the weight-function $\bar{W}$ in the quotient $\overline{\mathrm{PT}}$, defined according to (4.3) and (4.4) in Section 4.1, also satisfy the specifications for BHLs according to Definition 4.7 above. We show the claim for markings and leave the rest to the reader.

Now, a marking $\bar{M}$ of the quotient $\overline{\mathrm{PT}}$ maps a $\bar{p}$ to a multiset over $\bar{p}$. On the other hand, in the BHL-interpretation a marking is a map of the place $\bar{p}$ to a multiset over $D(\bar{p})$. But, by definition, $D(p) = \bar{p}$, hence both characterizations coincide.

Similarly, it can easily be verified that transition-enabling (4.8) and -occurrence (4.9) according to the BHL-semantics coincide with the quotient semantics according to (4.5).

We summarize these observations in the following

Definition 4.11. Let $\overline{\mathrm{PT}} = (\bar{N}, \bar{K}, \bar{W}, \bar{M}_{\text{in}})$ be a quotient system. The basic high level net $(\bar{N}, D, \Phi, \bar{W}, \bar{K}, \bar{M}_{\text{in}})$, where D and Φ are defined according to (4.10), is called the BHL *associated with* $\overline{\mathrm{PT}}$.

For the special case of elementary net-systems, the BHL resulting from the quotient is obviously strict in the sense of Definition 4.8.

4.4 Unfolding BHL- to PT-nets

We turn to the converse and show that *any BHL can be conceived as a quotient.* To this end, we first canonically unfold a given BHL- into a PT-net. We then show that the original BHL may be regained from the unfolded net by a suitable factorization. Moreover, unfolding and factorization are inverse to each other, such that both transformations induce a correspondence between system behaviour.

Let BHL $= (N, D, \Phi, W, K, M_{\text{in}})$ be a basic high-level net. We wish to transform it into a PT-net with corresponding behaviour.

We start by unfolding the net $N = (P, T, F)$ to a net $\tilde{N} = (\tilde{P}, \tilde{T}, \tilde{F})$ as follows. First put

$$\tilde{P} := \{(p,a) \mid p \in P, a \in D(p)\}, \quad \tilde{T} := \{(t,m) \mid t \in T, m \in \Phi(t)\}. \tag{4.11}$$

$\tilde{P}$ is essentially the disjoint union of the domains $D(p)$. If these sets happen to be disjoint already, as for instance if they are derived directly from a quotient, there is no need for the extra tag p in (p,a), the a alone would then be sufficient.

$\tilde{T}$ consists of all firing-modes of BHL together with the transitions they belong to. Here we always have to be careful since two different modes of one transition can have the same effect, or conversely two transitions may share a common mode.

The flow-relation $\tilde{F}$ between elements $\tilde{p} := (p,a) \in \tilde{P}$ and $\tilde{t} := (t,m) \in \tilde{T}$ is then given by

$$\tilde{p}\,\tilde{F}\,\tilde{t} :\Leftrightarrow p\;F\;t \wedge W(p,t)(a,m) \neq 0,$$

and

$$\tilde{t}\,\tilde{F}\,\tilde{p} :\Leftrightarrow t\;F\;p \wedge W(t,p)(m,a) \neq 0. \tag{4.12}$$

This takes care of the underlying net $\tilde{N}$.

We come to the transformation of markings, capacity constraint and the weight-function. A marking M of BHL is translated into the marking $\tilde{M}$ of $\tilde{N}$, such that

$$\tilde{M}(\tilde{p}) := M(p)(a), \tag{4.13}$$

and similarly for the capacity constraint K. The weight function of $\tilde{N}$ is given by

$$\tilde{W}(\tilde{p}, \tilde{t}) := W(p,t)(a,m) \quad \text{and} \quad \tilde{W}(\tilde{t}, \tilde{p}) := W(t,p)(m,a). \tag{4.14}$$

It is again straightforward to verify that transition-enabling and -occurrence are conserved in this transformation, i.e. we have:

Proposition 4.12. *Let m be a mode of $t \in T$, and let M be a marking of* BHL. *Then with $\tilde{t} = (t, m)$:*

$$M \models t(m) \Leftrightarrow \tilde{M} \models \tilde{t}$$

and

$$M[t(m)\rangle M' \Leftrightarrow \tilde{M}[\tilde{t}\rangle \tilde{M}'.$$

We summarize the construction in the following

Definition 4.13. The *unfolded system associated* with a basic high level net BHL is the PT-net $\widetilde{\text{BHL}} = (\tilde{P}, \tilde{T}, \tilde{F}, \tilde{W}, \tilde{K}, \tilde{M}_{\text{in}})$ as defined above.

It is immediately verified that a *strict* BHL gives rise to an *elementary net system.*

4.5 From BHL- to PT-nets and back

We finally show that folding of PT- into BHL-nets, and the unfolding of BHL into PT-nets are inverse to each other. In particular no behaviour-information is lost in either. More precisely we show that the transformation sequences

$$\text{(i)} \quad \text{PT} \to \overline{\text{PT}} \to \widetilde{\overline{\text{PT}}} \qquad \text{and} \qquad \text{(ii)} \quad \text{BHL} \to \widetilde{\text{BHL}} \to \overline{\widetilde{\text{BHL}}}$$

according to Definitions 4.5, 4.11, and 4.13 return the original PT- resp. BHL-net, when the quotient in (ii) is taken with respect to a canonic equivalence relation on $\widetilde{\text{BHL}}$.

Proposition 4.14.

(i) *Let $\overline{\text{PT}}$ be a quotient of a PT-net* PT. *Let $\widetilde{\overline{\text{PT}}}$ be the unfolding of $\overline{\text{PT}}$ according to Definition 4.13. Then, except for names of elements,* PT *and $\widetilde{\overline{\text{PT}}}$ are identical. In particular, transition-enabling and -occurrence in both systems coincide.*

(ii) *Let $\widetilde{\text{BHL}}$ be the PT-net obtained from a basic high-level net* BHL *according to Definition 4.13. Let $\overline{\widetilde{\text{BHL}}}$ be the quotient of $\widetilde{\text{BHL}}$ with respect to the equivalence ρ given by*

$$(x, y)\; \rho\; (x', y') :\Leftrightarrow x = x'.$$

Then the systems BHL *and $\overline{\widetilde{\text{BHL}}}$ are identical, except for names of elements. In particular, transition-enabling and -occurrence coincide.*

Proof. (i). It is easily checked that the map $\varphi : x \mapsto (\bar{x}, x)$ is a bijection between the elements of PT and $\widetilde{\overline{\mathrm{PT}}}$, such that moreover $x\ F\ z \Leftrightarrow \varphi(x)\ \bar{\bar{F}}\ \varphi(z)$. Hence the structure of the underlying nets coincide.

It remains to show that markings, capacity constraint and weight-function, as well as transition-enabling and -occurrence carry over. We compute markings and leave the rest to the reader:

For a place p we get, evaluating according to (4.13) and (4.3) in that order:

$$\bar{\bar{\tilde{M}}}(\varphi(p)) = \bar{\bar{\tilde{M}}}((\bar{p}, p)) = \bar{M}(\bar{p})(p) = M(p).$$

(ii). For every $x \in X$ chose y_x in $D(x)$ resp. $\Phi(x)$. (This is possible, because these sets are non-empty by definition.) We then put $\psi(x) := \overline{(x, y_x)}$. It is easily verified that ψ defines a bijection between the elements in BHL and $\widetilde{\overline{\mathrm{BHL}}}$, such that also $x\ F\ z \Leftrightarrow \psi(x)\ \bar{\bar{F}}\ \psi(z)$. This shows that the structure of the underlying nets coincide.

For $p \in P$, we have the bijection $\psi_p : a \mapsto (p, a)$ between the domains $D(p)$ and $D(\psi(p))$. For $t \in T$, the bijection ψ_t is defined analogously. We have to show that markings, capacity constraint and weight function, as well as transition-enabling and -occurrence are respected. This is again routine, as an example we establish the relationship for markings:

For $p \in P$ and $a \in D(p)$ we get, using (4.3) and (4.13):

$$\bar{\bar{\tilde{M}}}(\psi(p))(\psi_p(a)) = \tilde{M}(\psi_p(a)) = \tilde{M}((p, a)) = M(p)(a). \qquad \square$$

4.6 Lifting of low-level semantics: an example

Interdefinability between low- and high-level nets in particular also implies that behavioural properties carry over. We give an example which illustrates how the notion of *non-sequential process* may be lifted. Readers not acquainted with processes may skip the example, processes will not be mentioned any further in the sequel.

Example 4.15. We return to the philosopher system BHL_{5Phil} from Example 4.10. Fig. 4 shows a possible description of the process in which philosophers 0 and 2 start to eat concurrently, and once both have finished, philosopher 1 takes his turn.

What makes this description high-level specific is that each process-element carries *two* labels, the outer (e.g. *avail* or *tf*) indicates which system element is represented, and the inner (e.g. ph_0 or fk_1), which individual token (or transition-mode) is concerned.

Remark 4.16. Example 4.15 also illustrates that the concept of *concurrent behaviour* carries over to high-level nets. It should be noted, however, that concurrency in BHLs must be considered as a relation between transitions *plus* specific firing-modes, not between transitions per se.

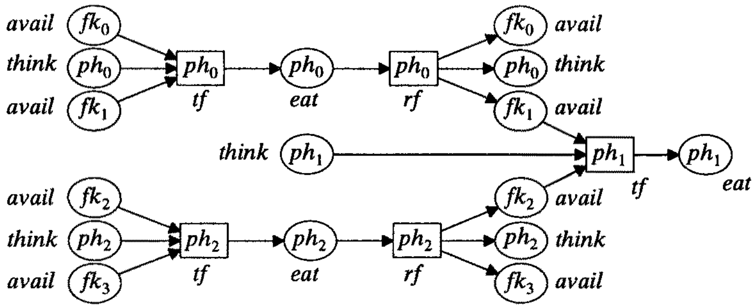

Fig. 4. A process of BHL_{5Phil}

5 Matrix-labelled nets

Until now we have not made any assumption on cardinality, all definitions so far apply to finite as well as to infinite nets. In this section we shall specifically consider BHLs, where the underlying net and the sets $D(p)$ and $\Phi(t)$ are all *finite*. This makes it possible to represent systems and their behaviour in form of finite matrices and corresponding linear-algebraic operations. In particular, it permits a precise invariant-calculus. The model we arrive at is essentially the original version of so-called *coloured Petri nets*, proposed by K. JENSEN around 1980 [13]. In order to avoid confusion with later definitions of coloured nets, we shall, however, prefer the term *matrix-labelled nets* from [19]. This section may be read independently from the technical development in Section 4.

We start with an informal example, which illustrates the main ideas:

Example 5.1. Recall the five-philosophers system BHL_{5Phil} from Example 4.10. All sets involved are finite, the underlying net as well as token-domains and mode-sets. In a matrix-labelled net we write place-markings $M(p)$ as (column)-vectors with reference to an arbitrary but fixed order on the domains $D(p)$. For instance, the marking $M(think) = \{ph_1, \cdots, ph_4\}$ is naturally represented by its characteristic $|D(p)|$-vector in $\{0, 1\}$, namely $(0, 1, 1, 1, 1)^T$.

Transition-firing may then be represented by vector addition. To illustrate, assume the initial situation M_{in} where all philosophers are thinking. Firing *tf* for ph_0 transfers ph_0 from *think* to *eat*, coincidently removing fk_0 and fk_1 from *avail*. In vector notation this corresponds to a

subtraction of $(1, 0, 0, 0, 0)^T$ from $M_{\text{in}}(think) = (1, 1, 1, 1, 1)^T$ yielding $(0, 1, 1, 1, 1)^T$,
subtraction of $(1, 1, 0, 0, 0)^T$ from $M_{\text{in}}(avail) = (1, 1, 1, 1, 1)^T$ yielding $(0, 0, 1, 1, 1)^T$,
and the addition of $(1, 0, 0, 0, 0)^T$ to $M_{\text{in}}(eat)$ yielding $(1, 0, 0, 0, 0)^T$.

Generally speaking, for every arc (x, y) with $\{x, y\} = \{p, t\}$ between a place p and a transition t in a matrix-labelled net there is one such $|D(p)|$-vector $\mathbf{v}_m$ *for each* $m \in \Phi(t)$. The $\mathbf{v}_m$, $m \in \Phi(t)$ belonging to the same arc (x, y) may then be conceived as *column* vectors in a $|D(p)| \times |\Phi(t)|$-matrix $W(x, y)$. Firing t in mode m then amounts to the subtraction (resp. addition) of the m-column from each $W(p,t)$ (resp. $W(t,p)$) belonging to arcs surrounding t.

In the general definition, we consider also the possibility of *multiple* occurrences of identical tokens. To illustrate, assume for the moment that philosopher ph_0 has *two* right forks fk_1 and fk_1', indistinguishable with respect to system behaviour. In vector notation the initial marking of the place *avail* may then be described by the vector $(1,2,1,1,1)$. Moreover, assume that ph_0 always uses both fk_1 and fk_1' when eating. In matrix notation this may be represented by replacing the ph_0-column, i.e. column 1, by $(1,2,0,0,0)$ in the matrix $W(avail, tf)$.

5.1 Formal definitions

Note that the following definition is in accordance with the general Definition 4.7, it merely reformulates some basic concepts in terms of matrix language.

Definition 5.2. A *matrix-labelled net* $\mathrm{ML} = (N, D, \Phi, W, K, M_{\mathrm{in}})$ consists of a net N, where we assume the sets of places and transitions are finite of the form $P = \{p_1, \dots, p_n\}$ and $T = \{t_1, \dots, t_m\}$. The other components are explained in the following.

(i) The *domain-function* D is a mapping from P into finite non-empty sets, similarly the *mode-function* Φ is a map from T to non-empty sets. We assume each set $D(p)$ and $\Phi(t)$ to be enumerated in an arbitrary but fixed order.

(ii) W is a mapping with domain $(P \times T) \cup (T \times P)$, such that if $\{x,y\} = \{p,t\}$, then $W(x,y)$ is a $|D(p)| \times |\Phi(t)|$-matrix in $\mathbb{N}$. Moreover, for $(x,y) \notin F$ we let $W(x,y)$ be the 0-matrix of the appropriate dimension.

(iii) A *marking* of ML is a mapping M with domain P, such that $M(p)$ is a $|D(p)|$-vector in $\mathbb{N}$, written as a column-vector. M_{in} is the *initial marking*.

(iv) The capacity constraint K is again formally like a marking, except that it may be a partial function, and when defined always has values $\geqslant 1$.

(v) A transition t is *enabled in mode* $m \in \Phi(t)$ in a marking M iff for each $p \in {}^\bullet t$ and for each individual token $a_p \in D(p)$ the a_p-entry in $M(p)$ is $\geqslant$ the (a_p, m)-entry in $W(p,t)$, and whenever K is defined, the a_p-entry in $M(p)$ *plus* the (a_p, m)-entry in $W(t,p)$ does not exceed the a_p-entry in $K(p)$.

(vi) In that case, a *firing of t in mode m* leads to the follower marking M' obtained from M by subtracting the m-column of $W(p,t)$ from $M(p)$ for each $p \in {}^\bullet t$, and adding the corresponding m-column of $W(t,p)$ to $M(p)$ for each $p \in t^\bullet$.

(vii) We call the system *unbounded*, if K is nowhere defined.

(viii) It is *strict*, if all numeric values appearing in places and arc-matrices W are limited to 0 and 1, and the capacity constraint has the constant value 1.

Remark 5.3. If the function K is clear from the context, it is usually not mentioned explicitly. This may be the case for two diametrically opposite reasons: (1) either if the system is unbounded, (2) or if is strict.

5.2 The invariant-method

The ML-representation is the base for the following *incidence matrix*, which is particularly useful in system analysis:

Definition 5.4. The *incidence matrix* ML of a ML-net is the $|P| \times |T|$-matrix with entries

$$\mathsf{ML}_{ij} = W(t_j, p_i) - W(p_i, t_j).$$

The matrix representation usually gives a complete characterization, with one notable exception, namely when $W(t_j, p_i)_{kl} \neq 0 \neq W(p_i, t_j)_{kl}$ for some indices. In all examples we shall discuss here, the underlying net is pure, and we are therefore certainly on the safe side.

Example 5.5. We reformulate the system in Fig. 2 as an ML-net, where for simplicity we assume $D(p_1) = D(p_2) = \{a, b\}$. As transition-modes it appears natural to use the words aa, bb, ab, ba, such that $\Phi(t_1)$ becomes $\{aa, bb\}$ and $\Phi(t_2) = \{aa, bb, ab, ba\}$. Disregarding the columns M and **i** for a moment, the ML-net is then given by Fig. 5, where the four 'sub-matrices' separated by the dashed lines represent the arc-weights in the form $W(t, p)$ and $-W(p, t)$, respectively.

		t_1		t_2						
		aa	bb	aa	ab	ba	bb	M_{in}	M	**i**
p_1	a	-1		1		1		1	0	1
	b		-1		1		1	0	1	1
p_2	a	1		-1	-1			0	0	1
	b		1			-1	-1	0	0	1

Fig. 5. The ML-net corresponding to Fig. 2(b)

We come to one of the main features of ML-nets, a powerful linear-algebraic invariant-calculus based on the incidence matrix ML.

In the following, we use the term *P-vector* for (column)-vectors $\mathbf{v}$ of the form

$$\mathbf{v}^T = (v_1, \ldots, v_n), \quad \text{where each entry } v_i \text{ is a } |D(p_i)|\text{-vector in } \mathbb{N}.$$

It is tedious but straightforward to verify, that for a P-vector $\mathbf{v}$, "dimension-compatibility" permits the definition of $\mathbf{v}^T \cdot \mathsf{ML}$ and $\mathbf{v}^T \cdot M$ for markings M, where the multiplication as usual in matrices reduces to a sum of products of the components. These in turn are easily defined, since the components themselves are matrices in $\mathbb{N}$.

We may now define a place-invariant as follows:

Definition 5.6. A *place-invariant* in an ML-net with incidence matrix ML is a P-vector $\mathbf{i}^T = (i_1, \dots, i_n)$, such that

$$\mathbf{i}^T \cdot \mathsf{ML} = \mathbf{0}, \tag{5.15}$$

where $\mathbf{0}$ denotes the appropriate 0-vector, i.e. the vector $(0_1, \dots, 0_m)$, $m = |T|$, such that each of the 0_j is the 0-vector in $\mathbb{N}$ of length $|C(t_j)|$.

Similar to equation (3.2) in Section 3 and Theorem 7.7 we have:

Theorem 5.7. *Let* ML *be a matrix-labelled net. If* $\mathbf{i}$ *is an invariant according to Definition* 5.6, *then*

$$\mathbf{i}^T \cdot M = \mathbf{i}^T \cdot M_{\text{in}}$$

for every reachable marking $M \in [M_{\text{in}}\rangle$.

Proof. If $M \in [M_{\text{in}}\rangle$, then by definition there is a sequence $t_1, \dots, t_k$ of transitions and transition-modes $m_i \in \Phi(t_i)$, $1 \leqslant i \leqslant k$, such that M results from M_{in} by addition of the m_i-columns in the $W(t_i, p)$ (resp. subtraction of the m_i-columns in $W(p, t_i)$). Multiplication with $\mathbf{i}$ distributes over that sum, and because of (5.15) the factors not involving M_{in} evaluate to 0. □

Example 5.8. We continue with Fig. 5 and verify that the vector

$$\mathbf{i}^T = (i_1, i_2) = \big((1,1), (1,1)\big)$$

is an invariant according to Definition 5.6:

We have to show that $\mathbf{i}^T \cdot t = 0$ for $t = t_1, t_2$. Now, for t_1 we get

$$\begin{aligned}
\mathbf{i}^T \cdot t_1 &= i_1 \cdot W(p_1, t_1) + i_2 \cdot W(p_2, t_2) \\
&= (1,1) \cdot \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + (1,1) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= (-1,-1) + (1,1) \\
&= (0,0).
\end{aligned}$$

Similarly, it is easily verified that for t_2 we have $\mathbf{i}^T \cdot t_2 = (0,0,0,0)$.

To illustrate Theorem 5.7, we check that $\mathbf{i}^T \cdot M_{\text{in}} = \mathbf{i}^T \cdot M$ for the (obviously reachable) marking M in Fig. 5:

$$\begin{aligned}
\mathbf{i}^T \cdot M_{\text{in}} &= i_1 \cdot M_{\text{in}}(p_1) + i_2 \cdot M_{\text{in}}(p_2) \\
&= (1,1) \cdot (1,0)^T + (1,1) \cdot (0,0)^T \\
&= 1,
\end{aligned}$$

and an evaluation of $\mathbf{i}^T \cdot M_{\text{in}}$ clearly leads to the same result.

This last computation may also serve to illustrate the following: The product of an invariant with a marking is always an element of $\mathbb{N}$. The reason for this is quite simple: Essentially, what we do in the invariant-computation is to unfold the ML-net into a place-transition net PT, compute the invariants in PT by the usual method of solving a homogeneous linear equation in the integers, and finally rearrange the solutions into ML-representation.

For example, reconsider the incidence matrix in Fig. 5 above. If we disregard the dashed lines altogether, we are left with the incidence matrix PT of a place-transition net.

Likewise, the markings M_{in} and M are automatically translated into usual PT-markings. The same holds for the invariant $\mathbf{i}$, which becomes a vector $\mathbf{i}'$ in $\mathbb{N}$ of length $\sum_{i=1}^{n} |D(p_i)|$, such that $\mathbf{i}'^T \cdot \mathsf{PT}$ evaluates to the 0-vector of length $\sum_{j=1}^{m} |\Phi(t_j)|$ in $\mathbb{N}$.

Conversely, we may define an ML-net as a place-transition net, which for convenience is partitioned into suitable "compartments". The walls can at liberty be withdrawn or inserted without any essential change to the corresponding mathematics.

In fact, an arbitrary place-transition net (where the sets of places and transitions are finite) can be turned into an ML-net at will, simply by a renumbering of the lines and columns of the incidence matrix, followed by an *arbitrary* insertion of border lines, and a suitable renaming of the compartments. Note that this transformation of a PT-net into an ML-net is again only a reformulation of the general quotient construction discussed in Section 4.

5.3 Lifting of low-level analysis: an example

The fact that ML-nets can be conceived as condensed PT-nets, has the positive consequence that all of the analysis methods developed for the latter may be lifted to ML-nets. A good example is the method of *stubborn sets* developed by A.VALMARI, see e.g. [31]. Stubborn sets are used to construct a reduced reachability tree, which however still allows to determine dead states where no transition is enabled, or to find an infinite occurrence sequence if there is one. We illustrate the idea behind this approach.

For simplicity we give the definition for elementary net systems. In this case a set S of transitions is *stubborn in a marking* M, if the following three conditions hold:

(i) At least one transition $t \in S$ is enabled in M,
(ii) if $t \in S$ and t' is another transition with ${}^\bullet t \cap {}^\bullet t' \neq \emptyset$, i.e. such that t could be disabled by the firing of t', then also $t' \in S$,
(iii) if t is not enabled in M, then there is a $p \in {}^\bullet t$ such that $M(p) = 0$ and ${}^\bullet p \subseteq S$, i.e. t cannot be enabled without a direct predecessor that also belongs to S.

The reduced reachability tree is now inductively constructed as follows: For a marking M select a stubborn set S and include only follower-markings of M into the tree that result from occurrences of transitions in S. It is not hard to see that this construction in fact ensures that every dead marking will be reached, and conversely an infinite sequence will be generated if there is one.

Once the stubborn sets have been defined for low-level nets, it is then easy to lift the method. To this end it is essentially only necessary to replace "transition" with "pair of transition *plus* firing-mode". We refer to [31] for details.

6 Predicate-transition nets

The high-level models considered so far were derived directly by folding of low-level nets. This immediate approach does, however, not yet exploit the full possibilities of high-level nets. Their real advantage in modelling and analysis is invariably due to specific *languages* used in the characterization of individual tokens and firing modes. The rest of the paper is concerned with that aspect. It may be read independently from the technical development in the previous sections.

For definiteness, we shall discuss one particular model, the so-called *predicate-transition nets*, PrT-nets for short, introduced by H. GENRICH and K. LAUTENBACH in the late seventies [9]. It continues to be one of the central paradigms, and may moreover serve to illustrate ideas common to a wide variety of other approaches. We concentrate on strict nets without item-multiplicities, and only briefly indicate extensions to the non-strict case.

6.1 Introduction

PrT-nets combine dynamic system behaviour with the expressiveness of (first order) predicate logic. They are based on the observation that places in elementary nets are associated with changeable truth values, hence may be seen as a model in the realm of *dynamic propositional logic*. Similar to the extension from propositional logic to predicate logic, PrT-nets are built around *dynamically changing predicates* or *relations* over some universe. The process of change may be controlled by the evaluation of logic formulas associated with the transitions.

We start with an informal example:

Example 6.1. Fig. 6 shows a simple PrT net.

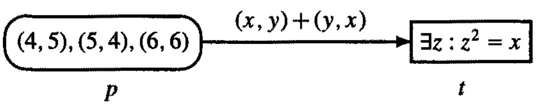

Fig. 6. A PrT-net

We assume that the place p may contain pairs of natural numbers, presently $(4,5)$, $(5,4)$, and $(6,6)$. The transition t is enabled for a binding of two number-pairs to the variable-pairs (x,y) and (y,x), for which additionally the condition $\exists z : z^2 = x$ holds. For example, if the number 4 is assigned to x and 5 to y, the pair $(4,5)$ is matched with (x,y) and $(5,4)$ with (y,x). The transition-constraint is then also satisfied. Now t may occur, withdrawing $(4,5)$ and $(5,4)$. This results in a new marking of p that contains only the pair $(6,6)$.

6.2 Predicate logic

We briefly recall the basic concepts from predicate logic needed for the definition of PrT-nets. For an exhaustive treatment the reader is referred to [6,30] or to any other standard text in logic.

Definition 6.2. We define a *language* to be a collection $\mathcal{L}$ of relation symbols, function symbols and (individual) constant symbols. Each relation symbol *Rel* is assumed to be an n-placed relation for some integer $n \geqslant 1$, depending on *Rel*. Similarly, each function symbol *func* of $\mathcal{L}$ denotes an m-placed function, where $m \geqslant 1$ and m depends on *func*.

We assume an infinite supply *Var* of individual variables plus the standard logical connectors and quantifiers.

Terms and *formulas* over $\mathcal{L}$ are defined as usual. We let *Term* denote the set of terms.

Definition 6.3. A *model* for a language $\mathcal{L}$ is a pair $\mathfrak{A} = (A, I)$ consisting of a *universe* A and an *interpretation* function I mapping the symbols of $\mathcal{L}$ to corresponding relations, functions and constant elements of A. We shall usually identify a symbol with its interpretation according to I.

An *assignment* is a mapping $\alpha\colon Var \to A$.

The *value* $\theta[\alpha]$ of a term θ for an assignment α is defined inductively as usual. We extend this notion to *tuples* $\tau = (\theta_1, \ldots, \theta_n)$ of terms, setting $\tau[\alpha] := (\theta_1[\alpha], \ldots, \theta_n[\alpha])$. Finally, the definition $\mathfrak{A} \models \varphi[\alpha]$, meaning that the formula φ is satisfied in $\mathfrak{A}$ for α, is now straightforward.

6.3 The definition of predicate-transition nets

After these preliminaries, we come to the central definition of this section. We formulate it here for *strict* nets, and indicate extensions to the non-strict case later.

Definition 6.4. A *predicate-transition net* is a tuple $\mathrm{PRT} := (N, \mathcal{L}, \lambda, \mathfrak{A}, M_{\mathrm{in}})$, where N is a net (P, T, F), such that

(i) P is a set of relation symbols, with given arity,
(ii) $\mathcal{L}$ is a language disjoint from P,
(iii) λ is a (partial) mapping with domain $T \cup F$, inscribing
   (a) some $t \in T$ (not necessarily all) with $\mathcal{L}$-formulas φ_t, called *transition-guards*,
   (b) each arc $(x, y) \in F$ with a finite set $\lambda(x, y) = \{\tau_1, \ldots, \tau_m\}$ of n-tuples of $\mathcal{L}$-*terms*, where n is the arity of the place belonging to the arc,
(iv) $\mathfrak{A}$ is a model for $\mathcal{L}$,
(v) M_{in} is a distinguished marking called the *initial* marking, where in general a *marking* is a mapping M which interprets each $p \in P$ as a relation over A with the corresponding arity.

For technical reasons we always require $\mathcal{L}$ to contain at least one constant symbol (this is needed to ensure the existence of variable-free terms). Usually there is no need to mention the language $\mathcal{L}$, as it may be assumed to be implicitly given by $\mathfrak{A}$ and λ.

Note that each marking M extends the model $\mathfrak{A}$ to a model $\mathfrak{A}_M = (A, \mathcal{I} \cup M)$ for the extended language $\mathcal{L} \cup P$.

We usually denote arc-inscriptions $\lambda(x, y)$ as symbolic sums $\tau_1 + \cdots + \tau_m$, i.e. as particular elements of the free group $\mathfrak{G}(Term^k)$, where k is the arity of the place $p \in \{x, y\}$. We also use the notion $-\lambda(x, y)$ for the expression $-\tau_1 - \cdots - \tau_m$. We use 0 as a symbol for the 0-element in $\mathfrak{G}(Term^k)$. Occasionally, also markings will be denoted in the form of symbolic sums.

Example 6.5. We return to the five-philosophers system. Fig. 7 shows a PrT-model.

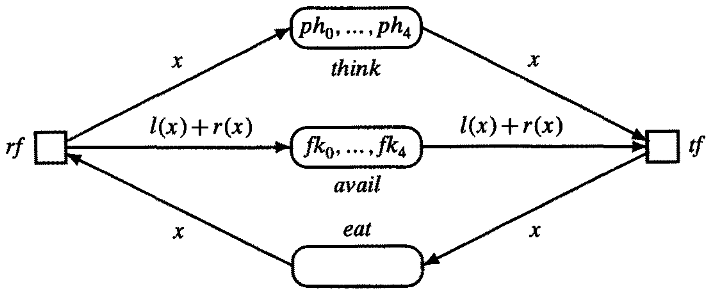

Fig. 7. PrT-model PRT_{5phil} of the five-philosophers system

The universe of discourse A is the union $Phil \cup Forks$, where $Phil = \{ph_0, \ldots, ph_4\}$ is the set of philosophers, and $Forks = \{fk_0, \ldots, fk_4\}$ denotes the set of forks.

The underlying net consists of three unary predicates *think*, *eat*, and *avail* with variable extension, denoting the currently thinking and eating philosophers, and available forks, respectively. The transitions *tf* "take forks" and *rf* "release forks" describe the possible system dynamics when a philosopher decides to change his state between thinking and eating. Fig. 7 shows the system in its initial state M_{in}, where all philosophers are thinking. So far the description corresponds directly to the basic model BHL_{5Phil} in Example 4.10.

What is different, is the use of arc-inscriptions to denote firing-modes and the weight-function: There are two functions l and r in $\mathcal{L}$, assigning a left fork $l(ph_i) = fk_i$ and right fork $r(ph_i) = fk_{i \oplus 1}$ to each philosopher (where again $\oplus$ denotes addition modulo 5). There are no relations or constants in $\mathcal{L}$.

We come to the *behaviour* of PrT-nets. It is based on the firing of transition-*instances* $t[\alpha]$, characterized by value-assignments α to arc-inscriptions. Undesired instances may be prevented from firing by means of transition-guards φ_t. Formally:

Definition 6.6. Let $\mathrm{PRT} := (N, \lambda, \mathfrak{A}, M_{\text{in}})$ be a PrT-net. An assignment $\alpha\colon Var \to A$ is called *feasible* for a transition t, if

(i) for any two different τ_1, τ_2 belonging to a common arc connected to t we have $\tau_1[\alpha] \neq \tau_2[\alpha]$,

(ii) $\mathfrak{A} \models \varphi_t[\alpha]$.

Note that *feasibility* expresses a *static*, i.e. marking-independent condition.

An *instance of a transition t* – also called a *firing-mode* – is a pair (t, α), denoted $t[\alpha]$, consisting of t *plus* a feasible assignment.

A transition-instance $t[\alpha]$ is *enabled* (equivalently we say that t is *α-enabled*) in a marking M, in symbols $M \models t[\alpha]$, iff

(iii) $\tau[\alpha] \in M(p)$ for all $p \in {}^{\bullet}t$ and $\tau \in \lambda(p,t)$,

(iv) $\tau[\alpha] \notin M(p)$ for all $p \in t^{\bullet}$ and $\tau \in \lambda(t,p)$.

If $M \models t[\alpha]$, a *firing* or *occurrence* of $t[\alpha]$, *α-occurrence of t* for short, leads to the follower marking

$$M' : p \mapsto \big(M(p) - \{\tau[\alpha] \| \tau \in \lambda(p,t)\}\big) \cup \{\tau[\alpha] \| \tau \in \lambda(t,p)\}. \tag{6.16}$$

The abbreviation $M[t[\alpha]\rangle M'$ means that M' results from M according to (6.16). Again we write $M \mathbf{r} M'$ if reference to the particular $t[\alpha]$ is not of interest. The *reachability class* $[M\rangle$ of a marking M is defined as the set of all M' reachable by a sequence $M = M_0 \mathbf{r} M_1 \mathbf{r} \cdots \mathbf{r} M_n = M'$.

Example 6.7. Returning to the philosophers in Example 6.5, consider the assignment $\alpha : x \mapsto ph_0$. It is readily verified that $M_{\text{in}} \models tf[\alpha]$ and $M_{\text{in}}[tf[\alpha]\rangle M'$, where

$$M' : think \mapsto \{ph_1, \ldots, ph_4\},\ avail \mapsto \{fk_2, fk_3, fk_4\},\ eat \mapsto \{ph_0\}. \tag{6.17}$$

In this new situation M', either of the following transition-instances may occur: $rf[\alpha]$, meaning that ph_0 puts his forks down and returns to thinking, or $tf[\beta]$ where β is one of the assignments $x \mapsto ph_2$ or $x \mapsto ph_3$.

Remark 6.8. Examples 6.5 and 6.7 should suffice to convince oneself that the behaviour of PrT-nets according to Definition 6.6 is compatible with the general basic high-level semantics in Section 4. A rigorous verification is straightforward but tedious. We leave the details to the interested reader.

Transition-guards

In the dynamics of the philosopher system in Example 6.5, we did not have to check condition (ii) in Definition 6.6, for the obvious reason that the system does not contain any transition-guards. To illustrate the meaning of transition-guards, assume for the moment that the system includes the formula $x \neq ph_0$ as a guard φ_{rf} for the transition "release forks". This guard would inhibit the return of ph_0 from eating to thinking.

Weak interpretation of PrT-nets

As mentioned, Definition 6.6 follows the *strict* interpretation of nets, where item-multiplicities are excluded. Sometimes, however, it is desirable to permit multiple appearances of tokens. This is possible in the *weak interpretation* of PrT-nets [10].

For the sake of argument, assume that there are *two* forks $fk_{i\oplus 1}$ and $fk'_{i\oplus 1}$ on the table between any two philosophers ph_i, $ph_{i\oplus 1}$. Assume moreover, that every philosopher uses *one left* fork and *two right* forks for eating. To capture this in our (strict) model, we augment the set of forks by $Forks' := \{fk'_0, \ldots, fk'_4\}$, such that the initial marking of $M_{\text{in}}(avail)$ now becomes $Forks \cup Forks'$. Additionally the expressions on the arcs connected to *avail* are replaced by, say, $l(x)+r(x)+r'(x)$.

Now, with respect to *system behaviour* the forks in each pair fk_i, fk'_i are actually indistinguishable from one another. It therefore appears as an overspecification to insist on their individuality within the model. Following the *weak interpretation* of PrT-nets, which permits token-multiplicities, we could model the situation by setting $M_{\text{in}}(avail) = \{2fk_0, \ldots, 2fk_4\}$, and labelling the arcs with $l(x)+2r(x)$. With regards to net behaviour this means that e.g. an occurrence of the *tf*-instance $tf[x \mapsto ph_0]$ in M_{in} would withdraw one left and both right forks, resulting in the follower marking $M'(avail) = \{fk_0, 2fk_2, 2fk_3 2fk_4\}$.

From a formal point of view there seems to be no need for multiplicities, as any n-placed predicate where we allow multiple appearances, may be turned into an $(n+1)$-placed predicate where distinguishing 'tags' are included in the additional component.

Moreover, sometimes it is not easy to get rid of the remainders of individuality. As a trivial but typical example, consider the arc function l above. It takes 'left fork' as value. But which one? If l is to remain a function, the value has to be unique. This means that we would have to make a choice among apparently indistinguishable forks after all.

6.4 Incidence matrix

As in PT- and ML-nets, it is often useful to represent a PrT-net by its *incidence matrix*. To this end we make use of the fact that arc inscriptions $\lambda(x, y)$ may be interpreted as elements of the free group $\mathfrak{G}(Term^k)$, where k denotes the arity of the place $p \in \{x, y\}$. Moreover, we trivially extend the domain of the mapping λ from F to all of $(P \times T) \cup (T \times P)$ by putting $\lambda(x, y) = 0$ for $(x, y) \notin F$.

Definition 6.9. Let PRT be a finite PrT-net, and assume $P = \{p_1, \ldots, p_n\}$ and $T = \{t_1, \ldots, t_m\}$. Then the *incidence matrix* of PRT is the $n \times m$-matrix PRT with entries

$$\mathsf{PRT}(p_i, t_j) := \lambda(t_j, p_i) - \lambda(p_i, t_j).$$

Example 6.10. Fig. 8 shows the dining philosophers system in matrix representation together with the marking (6.17) in Example 6.7.

Remark 6.11. Under certain conditions the structure of a PrT-net PRT is uniquely determined by its incidence matrix. This is the case if PRT has no transition-guards and, moreover, for all loops $(t, p), (p, t) \in F$ we have $\lambda(t, p) \cap \lambda(p, t) = \emptyset$. This last condition is necessary, since terms common to both label sets would be canceled out in the sum representation. In *pure* nets, for instance, we are on the safe side.

	tf	rf	M_{in}	M'
think	$-x$	x	$p_0+\cdots+ph_4$	$ph_1+\cdots+ph_4$
avail	$-r(x)-l(x)$	$r(x)+l(x)$	$fk_0+\cdots+fk_4$	$fk_3+fk_4+fk_5$
eat	x	$-x$		ph_0

Fig. 8. Incidence matrix and two markings of PRT_{5phil} in Fig. 7

7 Place-invariants

In this section we show how particular properties of the PrT-language may be used in system analysis. As a typical example we discuss the concept of symbolic invariants. We restrict the discussion to *place-invariants*, leaving aside the dual notion of *transition-invariants*.

The first approach to invariants in PrT-nets was proposed by GENRICH and LAUTENBACH in [9]. It is based on unification of terms by suitable variable-assignments. This method is not without problems, but as it continues to be in use – albeit in modified forms – we shall briefly discuss it. We then develop an approach proposed by J. VAUTHERIN and W. REISIG [24,23], based on symbolic term substitution.

All PrT-nets considered in this section are assumed to be finite.

7.1 The unification method

In the following let PRT be a finite PrT-net without transition-guards. For simplicity we assume the net to be pure. Assume that all places are unary predicates, and, moreover, that each arc-label consists of a *single variable*. Let PRT denote the incidence matrix of PRT according to Definition 6.9.

For the present discussion, a *P-vector* will be a (column) vector $\mathbf{v}^T = (v_1, \ldots, v_n)$ with components v_i from $\mathfrak{G}(Term)$.

Assume we have a commutative product "$\cdot$" between elements of $(\mathfrak{G}(Term), +)$, distributive over addition, such that for any $v_1, v_2, w_1, w_2 \in \mathfrak{G}(Term)$:

$$v_1 \cdot v_2 = w_1 \cdot w_2 \Rightarrow (v_1, v_2) = (w_1, w_2) \vee (v_1, v_2) = (w_2, w_1).$$

Formally, this product can be defined by extending the free group $\big(\mathfrak{G}(Term), +\big)$ to the ring $\big(\mathfrak{R}(Term), +, \cdot\big)$ of *formal polynomials* over *Term*. This is a standard construction, similar to the construction of the free group itself. We shall not bother with the details.

The ring-product may now as usual be extended canonically to cover also products $\mathbf{v}^T \cdot \mathbf{w}$ between P-vectors. Note in particular that markings M of PRT and the columns of PRT are P-vectors. Hence also the products $\mathbf{v}^T \cdot M$ and $\mathbf{v}^T \cdot \mathsf{PRT}$are defined.

Definition 7.1. As in PT-nets, a P-vector $\mathbf{i}$ is called a *place-invariant* iff

$$\mathbf{i}^T \cdot \mathsf{PRT} = \mathbf{0}, \tag{7.18}$$

where $\mathbf{0}$ is the zero-vector of length $|T|$ in $\mathfrak{R}(Term)$.

Recall that place-invariants in PT-nets were used to establish the invariance-equation (3.2). This naturally raises the question, whether a similar statement holds also for PrT-invariants $\mathbf{i}$ as in (7.18). In other words: can we prove that

$$\mathbf{i}^T \cdot M_{\text{in}} = \mathbf{i}^T \cdot M \tag{7.19}$$

for every reachable marking M? Unfortunately, this is not true in general. In order to make sense at all, (7.19) must be carefully interpreted. In the following three examples we shall discuss some of the possibilities and difficulties involved. In all examples we assume the underlying universe of discourse to be $A = \{a, b\}$.

Example 7.2. Consider the system Σ_1 given by the following table:

	t	M_{in}	M	$\mathbf{i}$
p_1	$-x$	a	0	y
p_2	y	0	b	x

(7.20)

In addition to the incidence matrix $\underline{\Sigma_1}$ (first column) and the initial marking we have included another marking M and a vector $\mathbf{i}$ in the table. Clearly, M is reachable from M_{in}, more precisely $M_{\text{in}}[t\,[x \mapsto a, y \mapsto b]\rangle M$. The vector $\mathbf{i}$ is a place-invariant according to (7.18), because

$$\mathbf{i}^T \cdot \underline{\Sigma_1} \;=\; \mathbf{i}^T \cdot t \;=\; -y \cdot x + y \cdot x \;=\; 0.$$

However, an immediate computation shows that (7.19) is *not* satisfied:

$$\mathbf{i}^T \cdot M_{\text{in}} \;=\; a \cdot y \;\neq\; b \cdot x \;=\; \mathbf{i}^T \cdot M. \tag{7.21}$$

It is, however, possible to *deduct* a "real" invariance-equality from (7.21). This is accomplished by a suitable *unification* of the expressions $a \cdot y$ and $b \cdot x$: Replacing the variable x by the constant a, and y by b, we obtain a P-vector $\mathbf{i}'$, which now satisfies (7.18) *as well as* (7.19).

The successful unification in (7.21) suggests the following re-interpretation of the invariance-equation (7.19):

For every reachable marking M there is a unifying assignment α of constants to the variables of $\mathbf{i}$, *such that* $\mathbf{i}[\alpha]^T \cdot M_{\text{in}} = \mathbf{i}[\alpha]^T \cdot M$. (7.22)

Indeed, (7.22) holds true whenever one and only one transition is involved. This is due to the fact that what the unification determines, is basically the variable-assignment needed for the transformation of M_{in} into the follower-marking M. The unification/assignment $x \mapsto a, y \mapsto b$ in Σ_1 illustrates this.

Unfortunately, (7.22) fails in general. Actually it may fail already in case of *two* transitions, as the following example shows.

Example 7.3. Consider the PrT-net Σ_2 given by the incidence matrix $\underline{\Sigma_2}$ in the following table:

$$\begin{array}{c|cc|c|c|c} & t_1 & t_2 & M_{\text{in}} & M & \mathbf{i} \\ \hline p_1 & -x & x & a & b & y \\ p_2 & y & -y & 0 & 0 & x \end{array} \tag{7.23}$$

It is again easily verified that $\mathbf{i}^T \cdot \underline{\Sigma_2} = \mathbf{0}$. Moreover, the marking M is obviously reachable from M_{in}. But a multiplication of $\mathbf{i}$ with the markings M_{in} and M yields

$$\mathbf{i}^T \cdot M_{\text{in}} = a \cdot y \neq b \cdot y = \mathbf{i}^T \cdot M. \tag{7.24}$$

Now, in contrast to (7.21), in (7.24) there cannot possibly exist any unification of the variables which would turn it into an equality, as such a unification would have to map y to a as well as to b!

The deeper reason for the failure of (7.22) in Example 7.3 is that *different* constants may be assigned to the variables in successive transition-firings, whereas such a dynamically changing assignment is not reflected in the "static" expression (7.19).

As an attempt to cope with dynamically changing assignment one could demand that every transition-environment (i.e. each column in the incidence matrix) should use a private set of variables. This remedy, however, would not help to deal with cases where the *same* transition successively occurs under two different assignments.

On the other hand, keeping variables local to transitions may lead to a converse problem since there will then usually not be sufficiently many identical variables to allow cancellation. In these cases, the unification approach may fail to find any invariant at all, even if there are obvious candidates. This is illustrated in the following

Example 7.4. Fig. 9 shows the PrT-version Σ_3 of the system in Figs. 2(b) and 5.

$$\begin{array}{c|cc|c|c|c} & t_1 & t_2 & M_{\text{in}} & M & \mathbf{i} \\ \hline p_1 & -x & z & a & b & v \\ p_2 & x & -y & 0 & 0 & w \end{array}$$

Fig. 9. A PrT-net Σ_3, an additional marking M, and a P-vector $\mathbf{i}$

In Example 5.8 we have already established, that there *is* an obvious invariant in the system, which *intuitively* may be described as follows. In every reachable marking of the system – as for instance in $M = (b, 0)$ in Fig. 9 – there is exactly one of a or b on exactly one of the places p_1 or p_2.

Unfortunately that invariant is not reflected in any place-invariant of Σ_3 according to (7.18): Now, if there were a vector $\mathbf{i} = (v, w)$ for some $v, w \in \mathfrak{G}(Term)$ satisfying (7.18), this would imply that both

$$-x \cdot v + x \cdot w = 0 \quad \text{and} \quad z \cdot v - y \cdot w = 0,$$

which would only be possible if $v = w$, and hence also $y = z$! But this is not the case.

7.2 The substitution method

We shall now develop an invariant-calculus, which does not depend on ex-post interpretations of the solutions. The method is based on the fact that transition-occurrence corresponds to function-application, mapping markings to follower markings. Consequently, firing-*sequences* may be conceived as function-*composition*. Since transitions operate via term-evaluation in PrT-nets, this suggests to use *term substitution* as a basis for invariant-computation. Moreover, as the 'occurrence-functions' are *additive* by nature, term-*substitution* may naturally be distributed over term-*addition*.

The method is applicable to PrT-nets in general, but for simplicity we shall here only consider *strict PrT-nets with unary places*. In particular this implies that all tuples appearing in arc-inscriptions consist of single terms. We also assume that system behaviour is not controlled by *transition-guards*. As before we concentrate on *place-invariants*, and only briefly mention the dual notion of *transition-invariants*.

Definition 7.5. Let PRT be a PrT-net as specified above. For terms θ, η in the language $\mathcal{L}$ of PRT, we define the *product* $\theta \cdot \eta$ to be the term $\theta[\eta]$ resulting from θ by a replacement of all variables of θ by η. Note that this multiplication is of course *not* commutative.

It is then straightforward to extend this operation to $\mathfrak{G}(\mathit{Term})$ by distributing multiplication into sums of terms, putting

$$\sum_i \theta_i \cdot \sum_j \eta_j = \sum_{ij} \theta_i \cdot \eta_j. \tag{7.25}$$

In particular, for the 0-element of $\mathfrak{G}(\mathit{Term})$ (the "empty sum") we have $\theta \cdot 0 = 0$ and $0[\eta] = 0$ for any terms θ, η.

We assume the operations to be continued to vector- and matrix-operations as usual.

By a *P-vector* we understand a (column) vector $\mathbf{v}^T = (v_1, \ldots, v_n)$ where each v_i is an element in $\mathfrak{G}(\mathit{Term})$.

Based on this product we can define *place-invariants* as follows:

Definition 7.6. Let PRT denote the incidence matrix of of a PrT-net PRT according to Definition 6.9. A P-vector $\mathbf{i}$ is a *place-invariant* of PRT iff

$$\mathbf{i}^T \cdot \mathsf{PRT} = \mathbf{0}, \tag{7.26}$$

where $\mathbf{0}$ denotes the 0-vector in $\mathfrak{G}(\mathit{Term})$ of appropriate length.

By essentially the same argument as in PT- and ML-nets we obtain:

Theorem 7.7. *Let* $\mathbf{i}$ *be a place-invariant (according to Definition 7.6) of a PrT-net* PRT *with initial marking* M_{in}. *Then*

$$\mathbf{i}^T \cdot M_{\text{in}} = \mathbf{i}^T \cdot M \tag{7.27}$$

for all reachable markings $M \in [M_{\text{in}}\rangle$.

Proof. By induction, using the following argument: Assume the marking M' results from another marking M by an α-occurrence of a transition t. In vector-notation this can be written as $M' = M + t[\alpha]$, where we identify t with the corresponding column in PRT. By multiplication with $\mathbf{i}^T$ we obtain $\mathbf{i}^T \cdot M + \mathbf{i}^T \cdot t[\alpha]$, where the second summand can be further evaluated to $(\mathbf{i}^T \cdot t)[\alpha]$ by associativity of substitution, and thus finally to $0[\alpha] = 0$, where again 0 is the 0-element in $\mathfrak{G}(\mathit{Term})$. □

Example 7.8. For illustration, we return to the dining philosophers. Consider the following P-vector $\mathbf{i}$ of the system PRT_{5phil} in Example 6.10:

$$\mathbf{i}^T = \big(r(y)+l(y), -y, 0\big). \tag{7.28}$$

We show that $\mathbf{i}$ is an invariant according to Definition 7.6, by verifying that both $\mathbf{i}^T \cdot tf = 0$ and $\mathbf{i}^T \cdot rf = 0$: Now,

$$\begin{aligned}\mathbf{i}^T \cdot tf &= \big(l(y)+r(y)\big)\cdot(-x)+(-y)\cdot\big(-l(x)-r(x)\big)\\ &= \big(-l(x)-r(x)\big)-\big(-l(x)-r(x)\big)\\ &= 0.\end{aligned}$$

The other equality can be treated similarly.

The system-invariant characterized by $\mathbf{i}$ is computed from the initial marking as follows:

$$\begin{aligned}\mathbf{i}^T \cdot M_{\mathrm{in}} &= (l(y)+r(y))\cdot\textstyle\sum ph_i - y\cdot\sum fk_i\\ &= \textstyle\sum l(ph_i)+\sum r(ph_i)-\sum fk_i\\ &= \textstyle\sum fk_i,\end{aligned}$$

where the last equality holds, because each of the two first summands evaluate to the expression $\sum fk_i$.

Application of invariants
As in PT- and ML-nets, place-invariants may be applied to establish that certain situations in a net-system are *not* reachable. As an example we use the invariant $\mathbf{i}$ in Example 7.8 to prove that not all philosophers may refrain from thinking at the same time. Suppose to the contrary, that M is a marking with $M(\mathit{think}) = \emptyset$. Then $\mathbf{i}^T \cdot M$ evaluates to a negative expression $-y \cdot M(\mathit{avail})$ and therefore certainly *not* to $\sum fk_i$.

Computation of invariants
At present the invariant-method is essentially limited to *verification* of invariants (as in Example 7.8). There is no systematic method known to *solve* the invariant-equation (7.26) in general.

In some cases, however, solutions can be found by straightforward reasoning. We give a simple example.

Example 7.9. Consider be the PrT-net Σ_3 defined by Fig. 9 above.

Let $\mathbf{i} = (v, w)$ be a vector, where v, w are sums of terms. If $\mathbf{i}$ is an invariant, we must have $\mathbf{i}^T \cdot t_1 = 0$ and $\mathbf{i}^T \cdot t_2 = 0$. From the first equality we deduce that both components of $\mathbf{i}$ are equal, such that $v = w$. From this we get $v \cdot z = v \cdot y$ by the second equality. Since z and y are different variables, this is true iff the value of v does not depend on the variables. Hence if for instance v is a constant c, we obtain the solution $\mathbf{i}^T = (c, c)$ for the invariant-equation (7.26).

Using the invariant $\mathbf{i}^T = (c, c)$, we can apply Theorem 7.7 to deduct the corresponding system-property: In every reachable marking – as for instance in $M = (b, 0)$ in Fig. 9 – there is exactly one of a or b on exactly one of the places p_1 or p_2.

Place-invariants in nets with transition-guards

In the discussion above we assumed the nets under consideration to be free from transition-guards. However, it is clear that Theorem 7.7 holds a fortiori if PRT *does* contain transition-guards, since these guards only *reduce* the class of reachable markings.

Transition-invariants

In Section 3 we briefly discussed the notion of transition-invariants in PT-nets. It is technically straightforward to define *transition-invariants* also for PrT-nets, namely as solutions to the equation $\mathsf{PRT} \cdot \mathbf{i} = \mathbf{0}$, where multiplication is again term-substitution according to Definition 7.5. At present, however, these transition-invariants do not play any significant role in system analysis.

Generalized place-invariants

The crucial property for the application of place-invariants is the invariance-equation (7.27). Closer analysis of the proof of Theorem 7.7 shows that (7.27) depends essentially only on the fact that the product is distributive over transition-occurrences; term-substitution is therefore only one possible approach (though probably the most natural one).

Another possibility to obtain more general invariants is to interpret the equality in (7.26) and (7.27) not in the restricted sense of "true" equality, but rather as a *congruence relation* compatible with addition. This could for instance be discussed in the context of term-equivalence in algebraic specifications.

8 Reachability analysis

Besides invariant-computation, reachability analysis is probably the most versatile analysis method in net theory. In its simplest form, reachability analysis is based on the generation of the complete state space, which of course tends to be intractably large in real modelling. This has lead to the investigation of various techniques of state space reduction. We shall take a closer look at the method of *equivalent markings* [12], which is probably the one most used. As an alternative approach we then briefly discuss the method of *parameterized markings* [20,21].

For simplicity we shall again only be concerned with strict interpretations of PrT-nets.

Reduction by equivalent markings

Definition 8.1. The reachability tree for a PrT-net PRT is a labelled tree $\mathcal{T}$ inductively constructed as follows:

(i) The root of $\mathcal{T}$ is labelled by the initial marking M_{in}.
(ii) If $\mathcal{T}$ contains a node n_M labelled by a marking M, then for each immediate follower marking M' of M a node $n_{M'}$ labelled by M' is included as successor node to n_M.

Sometimes it is also useful to assume the edges to be labelled by the transition-instances $t[\alpha]$ connecting the corresponding node labels.

The building of the reachability tree serves to determine all reachable markings. Hence if a node label appears repeatedly, only one of the occurrences has to be continued, as all subtrees with the same root label will be identical. The idea behind reduction by equivalent markings is an extension of cutting the tree at duplicate markings. It is based on the exploitation of certain symmetries in the system.

Definition 8.2. Let PRT be a PrT-net over the universe A. A bijection (or permutation) π on A is an *automorphism* iff

(i) $\pi(M_{\text{in}}) = M_{\text{in}}$, and
(ii) $M \,\mathbf{r}\, M' \Leftrightarrow \pi(M) \,\mathbf{r}\, \pi(M')$ for all markings M, M' of PRT.

Here the notion $\pi(M)$ stands for the marking obtained from M by renaming each constant a appearing in $M(p), p \in P$, by $\pi(a)$. The relation $\mathbf{r}$ again denotes "reachable by *one* transition-occurrence".

Example 8.3. Recall that in the five-philosophers system PRT_{5phil} in Example 6.5, the universe of discourse is $A = \mathit{Phil} \cup \mathit{Forks}$. Let π be the permutation $ph_i \mapsto ph_{i\oplus 1}$, $fk_i \mapsto fk_{i\oplus 1}$, $i = 0, \ldots, 4$. Both of the conditions (i) and (ii) in Definition 8.2 are easily verified.

To illustrate (i) we compute $\pi(M_{\text{in}})(\mathit{think})$:

$$\begin{aligned}\pi(M_{\text{in}})(\mathit{think}) &= \pi(M_{\text{in}}(\mathit{think}))\\ &= \{\pi(ph_0), \ldots, \pi(ph_4)\}\\ &= M_{\text{in}}(\mathit{think}).\end{aligned}$$

To illustrate (ii), consider the assignment $\alpha{:}x \mapsto ph_0$. The transition tf is α-enabled in M_{in}, and an occurrence of $tf[\alpha]$ leads to a situation M where ph_0 is eating.

We show that also $\pi(M_{\text{in}}) \,\mathbf{r}\, \pi(M)$: The difference between $\pi(M)$ and M is that ph_1 is eating in $\pi(M)$ instead of ph_0 With the assignment $\beta := \pi \circ \alpha$, which maps x to ph_1, and observing (i) we immediately get $\pi(M_{\text{in}}) = M_{\text{in}}[tf[\beta]\rangle\pi(M)$.

The following proposition is easily proved by induction:

Proposition 8.4. *Let π be an automorphism of* PRT.

(i) *Then for every marking M of* PRT, *M is reachable from M_{in} if and only if the marking $\pi(M)$ is.*

(ii) *Moreover, subtrees $\mathcal{T}_M$ and $\mathcal{T}_{\pi(M)}$ of $\mathcal{T}$ with respective root labels M and $\pi(M)$ are structurally isomorphic, such that each node label in $\mathcal{T}_{\pi(M)}$ is the π-image of the corresponding label in $\mathcal{T}_M$.*

Proposition 8.4 guarantees the correctness of the following algorithm (called "reduction by equivalent[2] markings") for the generation of the reachability class $[M_{\text{in}}\rangle$:

Algorithm 8.5 (Reduction by equivalent markings).

(i) Put $Reach := \{M_{\text{in}}\}$.
(ii) Then proceed to generate the tree $\mathcal{T}$. For each node label M encountered, check if it is already in *Reach* or if it is the π-image of a label in *Reach*. If either is the case, do not process that node any further. Otherwise include M into *Reach*.
(iii) Continuing in this way, we finally obtain $[M_{\text{in}}\rangle = Reach \cup \{\pi(M) | M \in Reach\}$.

Example 8.6. As an example of the effect of reduction by equivalent markings, we mention that in the five-philosophers system, reduction with respect to the rotation $ph_i \mapsto ph_{i\oplus 1}$ results in $|Reach| = 6$, whereas simple reduction by duplicate markings leaves 11 markings to be considered.

A crucial question for the application of the reduction algorithm is of course how to find suitable automorphisms. For an arbitrary PrT-net, this is probably a hopeless undertaking. Moreover, if automorphisms are to serve as a tool for simplified generation of the reachability tree, it appears paradox that condition (ii) in Definition 8.2 already depends on the prior knowledge of the whole reachability class.

In practical applications, however, most high-level models are usually already *developed* along some symmetry, which can then profitably be exploited.

In fact, this often results in a considerable number of usable automorphisms. In such cases, all of the available automorphisms may be applied to reduce the tree $\mathcal{T}$ even further. In the philosopher system, for instance, all rotations π_j mapping the ph_i to $ph_{i\oplus j}$ and rotating the forks correspondingly, may be used together in the algorithm above, reducing the number of non-equivalent markings further to $|Reach| = 4$.

In many cases the design behind a system implicitly relies on different *types* of individuals in the net's universe A, which are internally correlated by obvious bijections. Such knowledge often yields simplified definitions of usable automorphisms. For instance, in case of the philosophers, the set *Fork* is related to *Phil* via the bijection $l: ph_i \mapsto fk_i$. By means of l, every permutation π' of *Phil* has a unique extension to a permutation π of A, putting $\pi: fk_i \mapsto (l \circ \pi' \circ l^{-1})(fk_i)$. We refer to [12] for details.

Parameterized markings

An alternative method to reduce reachability trees is due to M. Lindqvist [20,21]. It is based on *symbolic* transition-occurrences, representing a whole class of concrete

[2] actually the relation between a marking M and its image $\pi(M)$ based on a single permutation is not an equivalence in the strict sense of the word.

firing-modes. Such a symbolic (or *parameterized*) occurrence results in a symbolic follower marking from which all corresponding "real" markings may be obtained by the assignment of appropriate values to the symbolic parameters. To give a (simplified) idea of the construction, in the philosopher system one would for instance fire the transition *tf* "for philosopher *i*" without binding *i* to a concrete value. In this way all markings and the whole reachability tree can be generated in parameterized form, which may lead to a considerable reduction in size. Unfortunately the necessary technical machinery turns out to be quite intricate. The method has recently been revised by K. Schmidt [26].

Partial analysis of reachability trees
Both methods discussed above, reduction by equivalent markings, and evaluation of parameterized markings, essentially construct the whole reachability tree, albeit in a condensed form. For special purposes it is often useful and sufficient to generate only a part of the reachability tree. For example, in the special case that the problem is whether a system will run into a deadlock where no transition is enabled, one may try the method of *stubborn sets*, mentioned at the end of Section 5.

9 Abstract PrT-nets and algebraic nets

Often it is convenient to deal with a whole class of structurally related systems within one generic framework instead of repeating similar arguments anew for each instance. For illustration, passing from 5 philosophers to an arbitrary number n, there are general statements which will still hold, for example it will still certainly be impossible for all philosophers to eat at the same time.

In PrT-nets, however, such a generalization within *one* representation is not immediately possible. According to Definition 6.4, every PrT-net is based on a *fixed* model $\mathfrak{A}$, hence for each n we would have to introduce a PrT-model of its own.

A promising approach to overcome this limitation is the introduction of *abstract PrT-nets* or net-*schemes* [8], where actual models $\mathfrak{A}$ are replaced by a set of *axioms* characterizing the whole class of intended models. Individual elements in the universe A are replaced by the notion of *variable-free* terms, as in the Herbrand theory of logic programming. We illustrate the basic concepts, where again we concentrate on strict nets.

9.1 Abstract PrT-nets

We use the notion $Ax \vdash \varphi$ to denote *derivability* of a formula φ from a set of (first order) formulas Ax. Again the details can be found in [6,30] or in any other standard text on logic.

Definition 9.1. By an *abstract predicate-transition net* we understand a tuple APRT $:= (N, \mathcal{L}, \lambda, Ax)$, where N, $\mathcal{L}$ and λ are as in Definition 6.4, and again without loss of generality we assume $\mathcal{L}$ to contain at least one constant symbol. Ax is a set of (first order) $\mathcal{L}$-formulas.

Note that in Definition 9.1 there is no reference to any initial marking. The reason is that it is not clear how to generically represent the class of all markings in question. For instance, in case of the philosophers, an initial marking of APRT would have to represent not a *fixed* number of philosophers but *all numbers n* within one and the same marking. (If desired, on the other hand, it is of course perfectly possible to include an initial marking into the definition.)

The general notions of *marking* and dynamic behaviour are defined as follows:

Definition 9.2. A *marking* of APRT is a mapping M which interprets each n-placed $p \in P$ as a set of n-tuples $\{\tau_1, \ldots, \tau_k\}$ where the components $\theta_{i_1}, \ldots, \theta_{i_n}$ of the τ_i, $1 \leqslant i \leqslant n$, are *variable-free terms* of $\mathcal{L}$. Moreover, as a syntactical counterpart to strictness we require all τ_i, τ_j to be mutually distinct in the sense that

$$Ax \nvdash \theta_{i_1} = \theta_{j_1} \wedge \cdots \wedge \theta_{i_n} = \theta_{j_n}. \tag{9.29}$$

We may now define behaviour in analogy with Definition 6.6. First note that in APRTs an assignment α can be conceived as a syntactical replacement with variable-free terms. We define *feasibility* as in Definition 6.6 (i), (ii), except that $\tau_1[\alpha] \neq \tau_2[\alpha]$ in (i) is re-interpreted according to (9.29) above, and (ii) is replaced by $Ax \vdash \varphi_t[\alpha]$.

With these modifications, the notions concerning *enabling*, *firing*, and *reachability* can now be copied literally from Definition 6.6.

A central question in APRTs is, to what extent properties carry over between abstract specifications and concrete net-models. We shall not go very much into detail here, only indicate some immediate possibilities and limitations.

Definition 9.3. A concrete PrT-net PRT $= (N, \mathcal{L}, \lambda, \mathfrak{A}, M_{\text{in}})$ is a *model* of an abstract PrT-net APRT $= (N, \mathcal{L}, \lambda, Ax)$ iff $\mathfrak{A} \models Ax$.

Example 9.4. A natural abstract representation for the dining philosophers is given by the abstract PrT-net $\text{APRT}_{phil} := (N, \mathcal{L}', \lambda, Ax)$, where the net N and the label-function λ are as in Example 6.5. The language $\mathcal{L}'$ is an extension of $\mathcal{L}$, containing an additional constant symbol a and a function symbol f with the intended meaning "right neighbour of philosopher x". Let the set Ax consist of the single equation $r(x) = l(f(x))$.

Consider the following markings of APRT_{phil}:

	M_1	M_2	M_3
think	$\sum_{i=0}^{4} f^i(a)$	$\sum_{i=1}^{4} f^i(a)$	$\sum_{i=0}^{3} f^i(a)$
avail	$\sum_{i=0}^{4} l(f^i(a))$	$\sum_{i=2}^{4} l(f^i(a))$	$\sum_{i=1}^{3} l(f^i(a))$
eat		a	$f^4(a)$

where, $f^i(a)$ is short for the term $\underbrace{f(\cdots f(}_{i\text{-times}}a)\cdots)$.

Let α be the assignment $x \mapsto a$. It is easily verified that $tf[\alpha]$ is enabled in M_1 according to Definition 9.2, and also that $M_1[tf[\alpha]\rangle M_2$.

A typical concrete model of APRT_{phil} is the five philosophers system $\mathrm{PRT5}_{phil}$, where the additional symbols a and f are interpreted as respectively ph_0 and the function $ph_i \mapsto ph_{i\oplus 1}$, with $\oplus$ again denoting addition modulo 5.

Interpreted in $\mathrm{PRT5}_{phil}$, the abstract marking M_1 becomes M_{in}, and M_2 becomes the marking M' in Fig. 8. Hence for these two markings reachability in the abstract and concrete nets coincide.

In contrast, it is a simple matter to check that the marking M_3 in the table above is *not* reachable from M_1 in APRT_{phil}, since a transfer of $f^4(a)$ from *think* to *eat* requires the presence of the term $l(f_5(a))$ on the place *avail*. In $\mathrm{PRT5}_{phil}$, however, the interpretation corresponding to M_3 *is* reachable from M_{in}. The reason is of course that f^5 becomes the identity function in $\mathrm{PRT5}_{phil}$, whereas in APRT_{phil} this is not the case, nor *could be*, because the abstract net must also permit models like $\mathrm{PRT6}_{phil}$ with 6 philosophers, where it is f^6 that is interpreted as identity rather than f^5.

Example 9.4 shows that reachability in actual models does not necessarily imply reachability in the APRT. We mention without formal demonstration that also the converse is not true. From reachability in an APRT we cannot deduce reachability in its models Σ: Suppose there is a marking M of APRT containing two terms θ and η. Then in particular these terms must be different with respect to Σ, in the sense that $Ax \not\vdash \theta = \eta$. However, it is still perfectly possible that the interpretations of θ and η in Σ might evaluate to one and the same element and thus violate the strictness-condition.

After these disappointments one might raise the question, whether abstract PrT-nets have any use at all. The answer is, yes, they do, and one keyword again is *invariants*:

Recall that invariants according to Definition 7.6 refer only to the incidence matrix. Now, by Definition 9.3 an abstract PrT-net shares its incidence matrix with all its realizations. Hence, if **i** is an invariant of an abstract PrT-net, then Proposition 7.7 will automatically hold for all its concrete models. For instance, the invariant **i** from Fig. 7.28 may now be used to show that actually in any system of arbitrary many philosophers, it is not possible that all of them eat at the same time.

9.2 Algebraic nets

Algebraic nets are a variant of abstract nets, which draw essential aspects of their semantics from the general framework of algebraic specifications and abstract data types. The notion is sometimes used quite informally, though there seems to be a general agreement on the following main features: Algebraic nets have no transition-inscriptions, the set of axioms Ax is empty or is concerned only with the specification of equalities between terms, strictness in the sense of (9.29) is not always required, and an initial marking usually forms part of the definition.

In this sense APRT_{phil} in Example 9.4 with a suitable initial marking is an algebraic net.

We refer to [23] for details.

10 Modern trends

Convergence
In the beginning there were essentially two different approaches to high-level nets, based *either* on expressive logic symbolism *or* a precise invariant-calculus. With the advent of unambiguous symbolic invariant-methods, there is a now a trend to define high-level nets mainly via *syntactic inscriptions*. Different models may be related to each another via a common "reference model", which relies on the purely semantic concepts of high-level nets, such as for instance the basic high-level nets proposed in Section 4.

Modeling and analysis requirements
The class of high-level nets is still under development. One reason for the introduction of new net classes is ease of modelling. As a typical example we mention hierarchical construction principles based on coloured nets [15,16].

Often new models are introduced as special-purpose nets, as for instance M-nets which allow composing operations similar to process-algebra [2].

Another impetus for the introduction of new nets is ease of analysis. As an example we mention *well formed coloured nets* where place-domains, firing-modes and arc-inscriptions are constructed from simple basic building blocks. These nets are fine-tuned to allow efficient use of various analysis tools [4,3,11].

Analysis of infinite nets
We have seen that for finite high-level nets with finite token domains, analysis methods easily carry over from low-level nets. Otherwise rather elaborate adaptations are necessary. As an example, we mention a symbolic variant of stubborn sets, which is discussed in [26].

Simulation
In many cases, where analysis-methods are not available, computer-aided simulation appears as an attractive alternative. There are various simulation tools available, among which Design/CPN, cf. [15,16] is probably the most widespread.

Extensions
Similar to the situation in low-level nets, there are many proposals to modify high-level net semantics by means of additional features such as *inhibitor arcs*, *test arcs*, *read only arcs*, *priorities*, etc. Examples can be found for instance in [5,18].

Of special interest in modelling are currently nets with semantics induced by *timing* or *stochastic* considerations. Also these classes of nets transcend high-level nets in the sense of this article. We refer to [1].

11 Assessment and concluding remarks

For application purposes, high-level nets in one form or another constitute the most versatile system model offered by net theory. This is primarily due to the fact that high-level nets permit concise representations of distributed systems while still retaining "locality aspects" on any desired level of detail. High-level nets support a great variety of data structures and permit the use of sophisticated behaviour control languages.

From a formal point of view it is possible to derive high-level nets and their semantic properties from elementary semantics of low-level nets by means of well understood translations. In particular this permits to lift mathematical methods from low- to high-level interpretations, as for instance the invariant-calculus of place-transition nets to matrix-labelled nets and related models such as coloured nets. Other analysis techniques are based on the high-level language proper such as the computation of symbolic invariants by term substitution in PrT-nets, or symbolic methods to simplify reachability analysis.

A recent approach in high-level nets is to proceed to even higher levels of abstraction, where nets are no more conceived as single concrete systems, but rather as a scheme for a whole class of related systems. Again it becomes possible to develop appropriate tools also on this level of discussion. As an example we have seen the generalization of symbolic invariants.

There is of course a lot more to high-level nets than we could cover here. As a starting point for further reading we recommend related papers in this volume and the collection [17], which among others contains various of the original papers mentioned here.

References

1. M. AJMONE MARSAN et al. *Modelling with Generalized Stochastic Petri Nets*. Wiley, 1995.
2. E. BEST, H. FLEISCHHACK, W. FRACZAK, R. HOPKINS, H. KLAUDEL, E. PELZ. A class of composable high level Petri nets. In *Application and Theory of Petri Nets 1995*, vol. 935 of *LNCS*, pp. 103–120. Springer-Verlag, 1995.
3. R. BRGAN, D. POITRENAUD. An efficient algorithm for the computation of stubborn sets of well formed Petri nets. In *Application and Theory of Petri Nets 1995*, vol. 935 of *LNCS*, pp. 121–140. Springer-Verlag, 1995.
4. G. CHIOLA, C. DUTHEILLET, G. FRANCESCHINIS, S. HADDAD. On well formed colored Petri nets and their symbolic reachability graph. In [17], pp. 373–296.
5. S. A. CHRISTENSEN, N. HANSEN. Coloured Petri nets extended with place capacities, test arcs and inhibitor arcs. In *Application and Theory of Petri Nets 1993*, vol. 691 of *LNCS*, pp. 186–205. Springer-Verlag, 1993.
6. C. Chang and H. Keisler. *Model Theory*. North Holland, 1973.
7. P. Cohn. *Universal Algebra*. D. Reidel Publishing Company, 1985.
8. H. GENRICH. Equivalence transformations of PrT-nets. In *Advances in Petri Nets 1989*, vol. 424 of *LNCS*, pp. 179–208. Springer-Verlag, 1990.
9. H. GENRICH, K. LAUTENBACH. System modelling with high-level Petri nets. *Theoretical Comput. Sci.*, 13:109–136, 1981.
10. H. J. GENRICH. Predicate/transition nets. In *Advances in Petri Nets 1986, part I*, vol. 254 of *LNCS*, pp. 207–247. Springer-Verlag, 1987.
11. S. HADDAD, J. ILIÉ, M. TAGHELIT, B. ZOUARI. Symbolic reachability graph and partial symmetries. In *Application and Theory of Petri Nets 1995*, vol. 935 of *LNCS*, pp. 238–257. Springer-Verlag, 1995.

12. P. Huber, A. Jensen, L. Jepsen, K. Jensen. Reachability trees for high-level Petri nets. *Theoretical Comput. Sci.*, 45:261–292, 1986.
13. K. Jensen. Coloured Petri nets and the invariant-method. *Theoretical Comput. Sci.*, 14:317–336, 1981.
14. K. Jensen. Coloured Petri nets: A high level language for system design and analysis. In *Advances in Petri Nets 1990*, vol. 483 of *LNCS*, pp. 342–416. Springer-Verlag, 1990.
15. K. Jensen. *Coloured Petri Nets, vol. 1.* Springer-Verlag, 1992.
16. K. Jensen. *Coloured Petri Nets, vol. 2.* Springer-Verlag, 1995.
17. K. Jensen, G. Rozenberg (eds.) *High-level Petri Nets.* Springer-Verlag, 1991.
18. G. Lakos, S. Christensen. A general systematic approach to arc extensions for coloured Petri nets. In *Application and Theory of Petri Nets 1994*, vol. 815 of *LNCS*, pp. 338–357. Springer-Verlag, 1994.
19. K. Lautenbach, A. Pagnoni. *Invariance and duality in predicate/transition nets and in coloured nets*, Arbeitspapiere der GMD 132, 1985
20. M. Lindqvist. *Parameterized Reachability for Predicate/Transition Nets*, vol. 54 of *Acta Polytechnica Scandinavica, Mathematics and Computer Science.* Helsinki, 1989.
21. M. Lindqvist. Parameterized reachability for predicate/transition nets. In *Advances in Petri Nets*, vol. 674 of *LNCS*, pp. 321–324. Springer-Verlag, 1993.
22. W. Reisig. *Petri-Nets.* Springer-Verlag, 1985.
23. W. Reisig. Petri nets and algebraic specifications. *Theoretical Comput. Sci.*, 80:1–34, 1991.
24. W. Reisig, J. Vautherin. An algebraic approach to high level Petri nets. In *Proc. 8th Workshop on Applications and Theory of Petri Nets*, pp. 51–72, Zaragoza (Spain), 1987.
25. G. Rozenberg, P. S. Thiagarajan. Petri nets: Basic notions, structure, behaviour. In *Current Trends in Concurrency*, vol. 224 of *LNCS*, pp. 585–668. Springer-Verlag, 1986.
26. K. Schmidt. Parameterized reachability trees for algebraic Petri nets. In *Application and Theory of Petri Nets 1995*, vol. 935 of *LNCS*, pp. 392–411. Springer-Verlag, 1995.
27. E. Smith. On net systems generated by process foldings. In *Advances in Petri Nets 1991*, vol. 524 of *LNCS*, pp. 253–276. Springer-Verlag, 1991.
28. E. Smith. A primer on high-level Petri-net theory. In C. Fernandez et. al., editors, *Advanced Course on Petri Nets*, pp. 114–140. Editorial de la Universidad de Santiago de Chile, 1996.
29. E. Smith, W. Reisig. The semantics of a net is a net – an exercise in general net theory. In K. Voss, H. Genrich, and G. Rozenberg, editors, *Concurrency and Nets*, Advances in Petri Nets, pp. 461–480. Springer-Verlag, 1987.
30. V. Sperschneider, G. Antoniou. *Logic: A Foundation for Computer Science.* Addison-Wesley, 1991.
31. A. Valmari. Stubborn sets for coloured Petri nets. In *Proc. 12th Int. Conf. on Appl. and Theory of Petri Nets 1991*, pp. 102–121.