Self-Duality

Topological charge: $Q = \int d^d x \, A_\alpha \, \widetilde{A}_\alpha,$

$$\delta Q = 0 \quad \rightarrow \quad \widetilde{A}_\alpha \frac{\delta A_\alpha}{\delta \varphi_j} - \partial_\mu \left(\widetilde{A}_\alpha \frac{\delta A_\alpha}{\delta \partial_\mu \varphi_j} \right) + A_\alpha \frac{\delta \widetilde{A}_\alpha}{\delta \varphi_j} - \partial_\mu \left(A_\alpha \frac{\delta \widetilde{A}_\alpha}{\delta \partial_\mu \varphi_j} \right) = 0$$

Energy or Euclidean action $S = \frac{1}{2} \int d^n x \left[A_\alpha^2 + \widetilde{A}_\alpha^2 \right]$

$$\delta S = 0 \; \rightarrow \quad A_\alpha \frac{\delta A_\alpha}{\delta \varphi_j} - \partial_\mu \left(A_\alpha \frac{\delta A_\alpha}{\delta \partial_\mu \varphi_j} \right) + \widetilde{A}_\alpha \frac{\delta \widetilde{A}_\alpha}{\delta \varphi_j} - \partial_\mu \left(\widetilde{A}_\alpha \frac{\delta \widetilde{A}_\alpha}{\delta \partial_\mu \varphi_j} \right) = 0$$

Self-duality equation $A_\alpha = \pm \widetilde{A}_\alpha.$

Bound $S = \frac{1}{2} \int d^d x \left[A_\alpha \mp \widetilde{A}_\alpha \right]^2 \pm Q \quad \rightarrow \quad S \geq \mid Q \mid$

1D scalar field models

Static Energy

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} \left(\frac{d\,\varphi}{d\,x} \right)^2 + V\,(\varphi) \right]$$

Topological Charge

$$Q = \int_{-\infty}^{\infty} dx\, G\,(\varphi)\, \frac{d\,\varphi}{d\,x}$$

$$\delta\,Q = \int_{-\infty}^{\infty} dx \left[\delta\varphi \frac{\partial\,G}{\partial\,\varphi} \frac{d\,\varphi}{d\,x} + G \frac{d\,\delta\,\varphi}{d\,x} \right] = \int_{-\infty}^{\infty} dx\,\delta\varphi \left[\frac{\partial\,G}{\partial\,\varphi} \frac{d\,\varphi}{d\,x} - \frac{d\,G}{d\,x} \right] + G\,\delta\,\varphi\,|_{-\infty}^{\infty} = 0$$

$$A_\alpha \equiv \frac{d\,\varphi}{d\,x}, \qquad \widetilde{A}_\alpha \equiv G\,(\varphi) \equiv \sqrt{2\,V\,(\varphi)}$$

Self-dual equation

$$\frac{d\,\varphi}{d\,x} = \pm\sqrt{2\,V\,(\varphi)}$$

2D CP^1 or $O(3)$ σ-model

Topological charge

$$Q = \frac{i}{2}\epsilon_{jk}\int d^2x \frac{\partial_j u \partial_k \bar{u}}{(1+u\bar{u})^2} = \epsilon_{jk}\int d^2x \frac{\partial_j \varphi_1\, \partial_k \varphi_2}{(1+\vec{\varphi}^2)^2}$$

$u = \varphi_1 + i\varphi_2$

Static Energy

$$E = \int d^2x \frac{\partial_j u \partial_j \bar{u}}{(1+u\bar{u})^2} = \int d^2x \frac{(\partial_j \varphi_1)^2 + (\partial_j \varphi_2)^2}{(1+\vec{\varphi}^2)^2}$$

$$A_\alpha = \frac{\partial_j \varphi_1}{1+\vec{\varphi}^2}\ , \quad \widetilde{A}_\alpha = \epsilon_{jk}\frac{\partial_k \varphi_2}{1+\vec{\varphi}^2} \qquad \rightarrow \qquad \partial_j \varphi_1 = \epsilon_{jk}\partial_k \varphi_2,$$

Cauchy-Riemann equations

4D Euclidean Yang-Mills

Pontryagin number $Q_{YM} = \int d^4x \, \mathrm{Tr}\left(F_{\mu\nu}\,\widetilde{F}_{\mu\nu}\right)$

$$\widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\,F^{\rho\sigma}$$

Yang-Mills action

$$S_{YM} = \frac{1}{4}\int d^4x \, \mathrm{Tr}\,(F_{\mu\nu}\,F_{\mu\nu}) = \frac{1}{8}\int d^4x \left[\mathrm{Tr}\,(F_{\mu\nu}\,F_{\mu\nu}) + \mathrm{Tr}\left(\widetilde{F}_{\mu\nu}\,\widetilde{F}_{\mu\nu}\right)\right]$$

$$A_\alpha \equiv F_{\mu\nu} \qquad \widetilde{A}_\alpha \equiv \widetilde{F}_{\mu\nu} \qquad \rightarrow \qquad \underset{\downarrow}{F_{\mu\nu} = \pm\widetilde{F}_{\mu\nu}}$$

Instantons

3D BPS monopoles

Magnetic charge

$$Q = \int d^3x\, \partial_j \mathrm{Tr}\,(B_j\, \phi)$$

$= 0$

$$= \int d^3x\, \left[\partial_j \mathrm{Tr}\,(B_j\, \phi) + i\, e\, \mathrm{Tr}\,([A_j\, ,\, B_j\, \phi])\right]$$

$$= \int d^3x\, \mathrm{Tr}\,[(D_j\, B_j)\, \phi + B_j\, D_j\phi] \quad = \int d^3x\, \mathrm{Tr}\,[B_j\, D_j\phi]$$

$= 0$

Static Energy

$$E = \int d^3x\, \mathrm{Tr}\left[B_j^2 + (D_j\phi)^2\right]$$

$A_\alpha \equiv B_j$ $\quad$ $\tilde{A}_\alpha \equiv D_j\phi$ $\quad \rightarrow \quad$ $B_j = \pm D_j\phi$

BPS equations

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