The 't Hooft - Polyakov monopole

We have:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu]$$

$$D_\mu = \partial_\mu + ie[A_\mu, \ ]$$

The Lagrangian is:

$$\mathcal{L} = -\frac{1}{4} \mathrm{Tr}(F_{\mu\nu})^2 + \frac{1}{2} \mathrm{Tr}(D_\mu \phi)^2 - V(\phi)$$

with $\phi = \phi_a T_a$

and the action: $S = \int d^4x \, \mathcal{L}$

We shall take the potential as:

$$V = \frac{\lambda}{4} (\mathrm{Tr}\,\phi^2 - a^2)^2$$

By varying the gauge field we have

$$\delta\left(-\frac{1}{4}\mathrm{Tr}(F_{\mu\nu})^2\right) = -\frac{1}{2}\mathrm{Tr}(F^{\mu\nu}\delta F_{\mu\nu})$$

$$= -\mathrm{Tr}\left(F^{\mu\nu}\left(\partial_\mu \delta A_\nu + ie[A_\mu, \delta A_\nu]\right)\right)$$

$$= -\mathrm{Tr}\left(\partial_\mu(F^{\mu\nu}\delta A_\nu) - (\partial_\mu F^{\mu\nu})\delta A_\nu + ie[F^{\mu\nu}, A_\mu]\delta A_\nu\right)$$

$$= \mathrm{Tr}\left(\left(\partial_\mu F^{\mu\nu} + ie[A_\mu, F^{\mu\nu}]\right)\delta A_\nu\right) + \text{surface terms}$$

$$= \mathrm{Tr}(D_\mu F^{\mu\nu} \delta A_\nu) + \text{surface terms}$$

In addition:

$$\delta\left(\frac{1}{2}\mathrm{Tr}(D_\mu\phi)^2\right) = \mathrm{Tr}(D^\mu\phi\,\delta D_\mu\phi)$$

$$= \mathrm{Tr}(D^\mu\phi\, ie[\delta A_\mu, \phi])$$

$$= ie\,\mathrm{Tr}([\phi, D^\mu\phi]\,\delta A_\mu)$$

Therefore:

$$\delta S = \int d^4x \left\{ \mathrm{Tr}(D_\mu F^{\mu\nu}\,\delta A_\nu) + ie\,\mathrm{Tr}([\phi, D^\nu\phi]\,\delta A_\nu) \right\}$$

Therefore, the Euler-Lagrange eqs. for the gauge field is:

$$\boxed{D_\mu F^{\mu\nu} = ie[D^\nu\phi, \phi]}$$

Varying w.r.t. the Higgs field one gets

$$\delta\left(\frac{1}{2}\mathrm{Tr}(D_\mu\phi)^2\right) = \mathrm{Tr}(D^\mu\phi\,\delta D_\mu\phi)$$

$$= \mathrm{Tr}\left(D^\mu\phi\left(\partial_\mu\delta\phi + ie[A_\mu, \delta\phi]\right)\right)$$

$$= \mathrm{Tr}\left(\partial_\mu(D^\mu\phi\,\delta\phi) - (\partial_\mu D^\mu\phi)\delta\phi - ie[A_\mu, D^\mu\phi]\delta\phi\right)$$

$$= -\mathrm{Tr}(D_\mu D^\mu\phi\,\delta\phi) + \text{surface terms}$$

and

$$\delta V = \frac{\lambda}{4}\,2\,(\mathrm{Tr}\,\phi^2 - a^2)\,2\,\mathrm{Tr}(\phi\,\delta\phi)$$

$$= \lambda(\mathrm{Tr}\,\phi^2 - a^2)\,\mathrm{Tr}(\phi\,\delta\phi)$$

Therefore

$$\delta S = -\int d^4x \left\{ \mathrm{Tr}\left(D_\mu D^\mu \phi \, \delta\phi \right) + \lambda \left(\mathrm{Tr}\,\phi^2 - a^2 \right) \mathrm{Tr}\left(\phi \, \delta\phi \right) \right)$$

and so the Euler-Lagrange for the Higgs field is:

$$\boxed{D_\mu D^\mu \phi = -\lambda \left(\mathrm{Tr}\,\phi^2 - a^2 \right) \phi}$$

We now use the ansatz:

$$\boxed{\begin{aligned} \phi_a &= \frac{x_a}{e r^2} H(\xi) \\ A_i^a &= -\varepsilon_{aij} \frac{x^j}{e r^2} \left(1 - K(\xi)\right) \\ A_0^a &= 0 \end{aligned}}$$

with $\xi = a e r$

Since $A_0 = 0$ and nothing depends upon x_0, we have

$$F_{0i} = 0 \qquad D_0 \phi = 0$$

In addition: $\left(r = \sqrt{x_i^2} \right)$

$$\frac{\partial r}{\partial x^i} = \frac{1}{2} \frac{1}{r} 2 x_i = \frac{x_i}{r} = \hat{r}_i$$

A note about the trace: By normalizing

$$[T_a, T_b] = i \varepsilon_{abc} T_c$$

we get that $\mathrm{Tr}(T_a T_b) = K \delta_{ab}$ where K depends upon the representation is taken. Therefore, we shall take the bilinear form Tr to be:

$$\mathrm{Tr}(AB) = \frac{1}{K}\,\mathrm{trace}(AB)$$

such that $\mathrm{Tr}(T_a T_b) = \delta_{ab}$.

We then have
$$\boxed{\phi = \frac{x_a T_a}{e r^2} H(\xi)}$$

and
$$\mathrm{Tr}\,\phi^2 = \frac{1}{e^2}\frac{1}{r^2} H^2(\xi)$$

In addition
$$\boxed{A_i = -\varepsilon_{ija}\frac{x^j T_a}{e r^2}\left(1 - K(\xi)\right)}$$

and so:

$$[A_i, \phi] = -\varepsilon_{ija}\frac{x^j}{e r^2}(1-K)\frac{x_b}{e r^2} H\,[T_a, T_b]$$

$$= -i\,\underbrace{\varepsilon_{abc}\,\varepsilon_{ija}}_{\delta_{bi}\delta_{cj} - \delta_{bj}\delta_{ci}}\frac{(1-K)H}{e^2 r^4}\,x^j x_b T_c$$

$$= -i\left(x_i\,\vec{r}.\vec{T} - r^2 T_i\right)\frac{(1-K)H}{e^2 r^4}$$

$$= -i\left(\hat{r}_i\,\hat{r}.\vec{T} - T_i\right)\frac{(1-K)H}{e^2 r^2}$$

Now

$$\partial_i\phi = \frac{T_i H}{e r^2} - 2\frac{\vec{r}.\vec{T} H \hat{r}_i}{e r^3} + \frac{\vec{r}.\vec{T}}{e r^2}\frac{dH}{dr}\hat{r}_i$$

$$= \frac{1}{e}\frac{1}{r^2}\left[H\left(T_i - 2\,\hat{r}.\vec{T}\,\hat{r}_i\right) + \hat{r}.\vec{T}\,r\frac{dH}{dr}\hat{r}_i\right]$$

Then:

$$D_i \phi = \frac{1}{er^2}\left[\hat{r}_i \hat{r}\cdot\vec{T}\left(r\frac{dH}{dr} - 2H \right) + HT_i \right.$$

$$\left. + H(1-K)\left(\hat{r}_i \hat{r}\cdot\vec{T} - T_i\right)\right]$$

$$= \frac{1}{er^2}\left[\hat{r}_i \hat{r}\cdot\vec{T}\left(r\frac{dH}{dr} - 2H + H(1-K)\right) + HKT_i \right]$$

$$= \frac{1}{er^2}\left[\hat{r}_i \hat{r}_j\left(r\frac{dH}{dr} - 2H + H(1-K)\right) + KH\delta_{ij} \right] T_j$$

We also have:

$$\frac{\partial \hat{r}_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{x_j}{r}\right) = \frac{\delta_{ij}}{r} - \frac{x_j}{r^2}\hat{r}_i = \frac{1}{r}\left(\delta_{ij} - \hat{r}_i\hat{r}_j\right)$$

$$\Rightarrow \partial_i \hat{r}_i = \frac{2}{r}$$

So:

$$\partial_i(D_i\phi) = -\frac{2\hat{r}_i}{er^3}\left[\hat{r}_i \hat{r}_j\left(r\frac{dH}{dr} - 2H + H(1-K)\right) + KH\delta_{ij} \right] T_j$$

$$+ \frac{1}{er^2}\left[\left[\frac{2}{r}\hat{r}_j + \overbrace{\frac{\hat{r}_i}{r}\left(\delta_{ij} - \hat{r}_i\hat{r}_j\right)}^{0}\right]\left(r\frac{dH}{dr} - 2H + H(1-K)\right) \right.$$

$$+ \hat{r}_i\hat{r}_j\left(\cancel{\frac{dH}{dr}}^{①} + r\frac{d^2H}{dr^2} - \cancel{2\frac{dH}{dr}}^{①} + \frac{dH}{dr}(\cancel{1}^{①}-K) - H\frac{dK}{dr}\right)\hat{r}_i$$

$$\left. + \hat{r}_i\delta_{ij}\left(K\frac{dH}{dr} + H\frac{dK}{dr}\right)\right] T_j$$

$$\partial_i(D_i\phi) = \frac{\hat{r}\cdot T}{e r^3}\Big\{ -2\Big[r\frac{dH}{dr} \cancel{- 2H + H(1-K)} \Big] - 2KH$$

$$+ 2\Big[r\frac{dH}{dr} \cancel{- 2H + H(1-K)}\Big] + r^2\frac{d^2H}{dr^2} \cancel{- rK\frac{dH}{dr}} \cancel{- rH\frac{dK}{dr}}$$

$$+ r\Big(\cancel{K\frac{dH}{dr}} + H\frac{dK}{dr}\Big)\Big\}$$

$$= \frac{\hat{r}\cdot\vec{T}}{e r^3}\Big[r^2\frac{d^2H}{dr^2} - 2KH\Big]$$

Now:

$$[A_i, D_i\phi] = -\varepsilon_{ija}\frac{x^j}{e r^2}(1-K)\frac{1}{e r^2}\Big[\hat{r}_i\hat{r}_k\Big(r\frac{dH}{dr} - 2H + H(1-K)\Big)$$

$$+ KH\delta_{ik}\Big]\underbrace{[T_a, T_k]}_{i\varepsilon_{akl}T_l}$$

$$= \frac{-i}{e^2 r^3}\big(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\big)\hat{r}_j(1-K)\Big\{\hat{r}_i\hat{r}_k\Big(r\frac{dH}{dr} - 2H + H(1-K)\Big)$$

$$+ KH\delta_{ik}\Big\}T_l$$

$$= \frac{-i}{e^2 r^3}(1-K)T_l\Big\{\cancel{(\hat{r}_l - \hat{r}_l)\Big(r\frac{dH}{dr} - 2H + H(1-K)\Big)}$$

$$+ KH(3-1)\hat{r}_l\Big\}$$

$$= \frac{-i}{e^2 r^3}\hat{r}\cdot\vec{T}\, 2KH(1-K)$$

Therefore:

$$D_i D_i \phi = \frac{\hat{r}.\vec{T}}{e r^3}\left[r^2 \frac{d^2 H}{dr^2} - 2\cancel{K}H + 2KH(\cancel{1} - K)\right]$$

$$= \frac{\hat{r}.\vec{T}}{e r^3}\left[r^2 \frac{d^2 H}{dr^2} - 2HK^2\right]$$

In addition:

$$\lambda\left(\mathrm{Tr}\,\phi^2 - a^2\right)\phi = \lambda\left(\frac{H^2}{e^2 r^2} - a^2\right)\frac{\hat{r}.\vec{T}}{e r} H$$

$$= \frac{\lambda}{e^2}\frac{\hat{r}.\vec{T}}{e r^3}\left(H^2 - a^2 e^2 r^2\right) H$$

The static eq. for the Higgs become:

$$D_i D_i \phi - \lambda\left(\mathrm{Tr}\,\phi^2 - a^2\right)\phi = \frac{\hat{r}.\vec{T}}{e r^3}\left[r^2 \frac{d^2 H}{dr^2} - 2HK^2 - \frac{\lambda}{e^2} H\left(H^2 - a^2 e^2 r^2\right)\right]$$

Therefore we need

$$\boxed{\xi^2 H'' - 2HK^2 - \frac{\lambda}{e^2} H\left(H^2 - \xi^2\right) = 0}$$

with primes being derivatives w.r.t. $\xi = a e r$

We now have:

$$\partial_i A_j = -\partial_i\left(\varepsilon_{jka}\frac{x^k T_a}{e r^2}(1-K)\right)$$

$$= -\varepsilon_{jka}\frac{T_a}{e}\left(\frac{\delta_{ik}}{r^2}(1-K) + \frac{x^k}{r^2}\frac{dK}{dr}\hat{r}_i - \frac{2x^k(1-K)}{r^3}\hat{r}_i\right)$$

$$= -\varepsilon_{jka}\frac{T_a}{e r^2}\left[(1-K)\delta_{ik} - \hat{r}_i\hat{r}_k\left(r\frac{dK}{dr} + 2(1-K)\right)\right]$$

and

$$\partial_i A_j - \partial_j A_i = -\frac{T_a}{e r^2}\Big[(1-K)\left(\varepsilon_{jia} - \varepsilon_{ija}\right)$$
$$- \left(\varepsilon_{jka}\hat{r}_k\hat{r}_i - \varepsilon_{ika}\hat{r}_k\hat{r}_j\right)\left(r\frac{dK}{dr} + 2(1-K)\right)\Big]$$

$$= \frac{T_a}{e r^2}\left[2\varepsilon_{ija}(1-K) + \left(\varepsilon_{ika}\hat{r}_j - \varepsilon_{jka}\hat{r}_i\right)\hat{r}_k\left(r\frac{dK}{dr} + 2(1-K)\right)\right]$$

Now:

$$[A_i, A_j] = \varepsilon_{ika}\varepsilon_{jlb}\frac{x^k x^l}{e^2 r^4}(1-K)^2\,[T_a, T_b]$$

$$= i\underbrace{\varepsilon_{abc}\varepsilon_{ika}}_{\delta_{bi}\delta_{ck} - \delta_{bk}\delta_{ci}}\varepsilon_{jlb}\frac{\hat{r}_k\hat{r}_l}{e^2 r^2}(1-K)^2 T_c$$

$$= \frac{i(1-K)^2}{e^2 r^2}\left(\varepsilon_{jli}\,\hat{r}\cdot\vec{T}\,\hat{r}_l - \cancel{\varepsilon_{jlk}\hat{r}_k\hat{r}_l T_i}\right) \quad \nearrow 0$$

$$= \frac{i}{e^2 r^2}(1-K)^2\,\varepsilon_{ijl}\,\hat{r}_l\,\hat{r}\cdot\vec{T}$$

Therefore:

$$F_{ij} = \partial_i A_j - \partial_j A_i + ie[A_i, A_j]$$

$$= \frac{T_a}{e r^2} \Big\{ 2\varepsilon_{ija}(1-K) - \left(\varepsilon_{ika}\hat{r}_j\hat{r}_k - \varepsilon_{jka}\hat{r}_i\hat{r}_k\right)\left(r\frac{dK}{dr} + 2(1-K)\right)$$
$$- (1-K)^2 \varepsilon_{ijl}\hat{r}_l\hat{r}_a \Big\}$$

Note that

$$\varepsilon_{jka}\hat{r}_k T_a \hat{r}_i - \varepsilon_{ika}\hat{r}_k T_a \hat{r}_j =$$

$$= (\hat{r}\times\vec{T})_j\hat{r}_i - (\hat{r}\times\vec{T})_i\hat{r}_j$$

But $\quad \varepsilon_{ijk}(\vec{A}\times\vec{B})_k = \varepsilon_{ijk}\varepsilon_{klm}A_l B_m$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_l B_m$$

$$= A_i B_j - B_i A_j$$

Thus,

$$\hat{r}_i(\hat{r}\times\vec{T})_j - \hat{r}_j(\hat{r}\times\vec{T})_i = \varepsilon_{ijk}(\hat{r}\times(\hat{r}\times\vec{T}))_k$$

But $\quad (\hat{r}\times(\hat{r}\times\vec{T}))_k = \varepsilon_{kij}\hat{r}_i(\hat{r}\times\vec{T})_j$

$$= \varepsilon_{kij}\varepsilon_{jlm}\hat{r}_i\hat{r}_l T_m$$

$$= (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})\hat{r}_i\hat{r}_l T_m$$

$$= \hat{r}_k(\hat{r}\cdot\vec{T}) - T_k$$

So, we have:

$$F_{ij} = \frac{1}{er^2}\left\{ \underline{2(1-K)\,\varepsilon_{ija}T_a} - (1-K)^2\,\underline{\varepsilon_{ije}\hat{r}_e}(\hat{r}\cdot\vec{T})\right.$$

$$\left. + \varepsilon_{ijk}\left(\hat{r}_k(\hat{r}\cdot\vec{T}) - \underline{T_k}\right)\left(r\frac{dK}{dr} + 2(1-K)\right)\right\}$$

$$= \frac{1}{er^2}\,\varepsilon_{ijk}\left\{ \left(\cancel{2(1-K)} - r\frac{dK}{dr} - \cancel{2(1-K)}\right)T_k\right.$$

$$\boxed{\cancel{+(\hat{r}\cdot\vec{T})\,\hat{r}_k - (1-K)^2\,\hat{r}_k}}$$

$$\left. + \hat{r}\cdot\vec{T}\,\hat{r}_k\left(r\frac{dK}{dr} + \underbrace{2(1-K) - (1-K)^2}_{\substack{= \\ 2 - \cancel{2K} - 1 + \cancel{2K} - K^2 \\ 1-K^2}}\right)\right\}$$

So:

$$\boxed{F_{ij} = \frac{1}{er^2}\,\varepsilon_{ijk}\left\{\hat{r}_k\,\hat{r}\cdot\vec{T}\left(r\frac{dK}{dr} + 1 - K^2\right) - T_k\,r\frac{dK}{dr}\right\}}$$

Now:

$$\partial_i F_{ij} = +\frac{2}{er^3}\,\hat{r}_i\,\varepsilon_{ijk}T_k\,r\frac{dK}{dr}$$

$$+ \frac{1}{er^2}\,\varepsilon_{ijk}\left\{\left[\left(\frac{\delta_{ik} - \cancel{\hat{r}_i\hat{r}_k}^{\,\to 0}}{r}\right)\hat{r}\cdot\vec{T} + \hat{r}_k\underbrace{\left(\frac{\delta_{il} - \hat{r}_i\hat{r}_l}{r}\right)T_l}_{\to 0}\right]\ast\right.$$

$$\ast\left(r\frac{dK}{dr} + 1 - K^2\right) + \cancel{\hat{r}_k\,\hat{r}\cdot\vec{T}\left(\frac{dK}{dr} + r\frac{d^2K}{dr^2} - 2K\frac{dK}{dr}\right)\hat{r}_i}$$

$$\left. - T_k\left(\frac{dK}{dr} + r\frac{d^2K}{dr^2}\right)\hat{r}_i\right\}$$

$$\partial_i F_{ij} = \frac{1}{er^3}\, \varepsilon_{ijk} \Big\{ 2r\frac{dK}{dr}\, \hat{r}_i T_k$$

$$+ \big(\hat{r}_k T_i - \cancelto{0}{\hat{r}_k \hat{r}_i\, \hat{r}\cdot T}\big)\left(r\frac{dK}{dr} + 1 - K^2\right)$$

$$- \hat{r}_i T_k \left(r\frac{dK}{dr} + r^2 \frac{d^2K}{dr^2}\right)\Big\}$$

$$= \frac{1}{er^3}\, \varepsilon_{ijk}\, \hat{r}_i T_k \Big\{ 2r\cancelto{0}{\frac{dK}{dr}} - r\cancelto{0}{\frac{dK}{dr}} - 1 + K^2 - r\cancelto{0}{\frac{dK}{dr}} - r^2\frac{d^2K}{dr^2}\Big\}$$

$$= -\frac{1}{er^3}\, \varepsilon_{ijk}\, \hat{r}_i T_k \left[r^2 \frac{d^2K}{dr^2} + 1 - K^2 \right]$$

Now:

$$[A_i, F_{ij}] = -\varepsilon_{ilm} \frac{\hat{r}_l}{er}(1-K)\, \frac{\varepsilon_{ijn}}{er^2}\, \times$$

$$\times \Big\{ \hat{r}_n \hat{r}_p \left(r\frac{dK}{dr} + 1 - K^2\right) [T_m, T_p] - r\frac{dK}{dr}\, [T_m, T_n] \Big\}$$

$$= -i \frac{(1-K)}{e^2 r^3}\, \hat{r}_l \left(\delta_{lj}\delta_{mn} - \delta_{ln}\delta_{mj}\right) \times$$

$$\times \Big\{ \hat{r}_n \hat{r}_p \left(r\frac{dK}{dr} + 1 - K^2\right) \varepsilon_{mpq} - r\frac{dK}{dr}\, \varepsilon_{mnq} \Big\} T_q$$

$$= +i \frac{(1-K)}{e^2 r^3} \Big\{ \left(r\cancelto{0}{\frac{dK}{dr}} + 1 - K^2 \right) \hat{r}_p\, \varepsilon_{jpq} - r\cancelto{0}{\frac{dK}{dr}}\, \varepsilon_{jpq}\, \hat{r}_p \Big\} T_q$$

$$= +i \frac{(1-K)(1-K^2)}{e^2 r^3}\, \varepsilon_{jpq}\, \hat{r}_p T_q$$

Therefore:

$$\partial_i F_{ij} + ie[A_i, F_{ij}] = \frac{1}{er^3}\, \varepsilon_{jpq}\, \hat{r}_p T_q \Big[r^2 \frac{d^2K}{dr^2} + \overset{1}{\cancel{K}} - \overset{1}{\cancel{K^2}}$$

$$- (\overset{1}{\cancel{K}} - K)(1 - \overset{🄋}{\cancel{K^2}}) \Big]$$

$$\boxed{\partial_i F_{ij} + ie[A_i, F_{ij}] = \frac{1}{er^3}\, \varepsilon_{jpq}\, \hat{r}_p T_q \left[r^2 \frac{d^2K}{dr^2} + K(1-K^2) \right]}$$

Now,

$$[D_i\phi, \phi] = \frac{1}{er^2}\left[\hat{r}_i \hat{r}_j \left(r\frac{dH}{dr} - 2H + H(1-K) \right) + KH\delta_{ij} \right] *$$

$$* \frac{\hat{r}_\ell}{er} H \underbrace{[T_j, T_\ell]}_{= i\varepsilon_{j\ell q} T_q}$$

$$= \frac{i}{e^2 r^3} K H^2 \varepsilon_{i\ell q}\, \hat{r}_\ell T_q$$

The eq. for the gauge field is:

$$D_i F_{ij} + ie[D_j\phi, \phi] =$$

$$= \frac{1}{er^3}\, \varepsilon_{jpq}\, \hat{r}_p T_q \left[r^2 \frac{d^2K}{dr^2} + K(1-K^2) - KH^2 \right]$$

Therefore we need:

$$\boxed{r^2 \frac{d^2K}{dr^2} + K(1-K^2) - KH^2 = 0}$$

Summarizing, the eq. we have to solve are:

$$r^2 \frac{d^2K}{dr^2} + K(1-K^2) - KH^2 = 0$$

$$r^2 \frac{d^2H}{dr^2} - 2HK^2 - \frac{\lambda}{\epsilon^2} H(H^2 - a^2\epsilon^2 r^2) = 0$$

Defining $\xi = a\epsilon r$ we get

$$\xi^2 K'' - K(K^2-1) - KH^2 = 0$$

$$\xi^2 H'' - 2HK^2 - \frac{\lambda}{\epsilon^2} H(H^2 - \xi^2) = 0$$

with primes being derivatives w.r.t. ξ.

Define $K' \equiv P_K$ $H' = P_H$ and we get

$$K' = P_K$$

$$H' = P_H$$

$$P_K' = \left(K(K^2-1) + KH^2\right)/\xi^2$$

$$P_H' = \left[2HK^2 + \frac{\lambda}{\epsilon^2} H(H^2-\xi^2)\right]/\xi^2$$

Let's define:

$$h = \frac{H}{\xi} \to H = \xi h$$

and so:

$$h' = \frac{H'}{\xi} - \frac{H}{\xi^2}$$

$$h'' = \frac{H''}{\xi} - 2\frac{H'}{\xi^2} + 2\frac{H}{\xi^3}$$

Then:

$$h'' + \frac{2}{\xi} h' = \frac{H''}{\xi} \to \xi^2 H'' = \xi^3 h'' + 2\xi^2 h'$$

Then the eqs become:

$$\xi^2 K'' - K(K^2-1) - K\xi^2 h^2 = 0$$

$$\xi^3 h'' + 2\xi^2 h' - 2\xi h K^2 - \frac{\lambda}{\epsilon^2}\xi h(\xi^2 h^2 - \xi^2) = 0$$

or:

$$\boxed{\begin{aligned} K'' &= K h^2 + K(K^2-1)/\xi^2 \\ h'' &= -\frac{2}{\xi} h' + \frac{2hK^2}{\xi^2} + \frac{\lambda}{\epsilon^2} h(h^2-1) \end{aligned}} \quad (*)$$

Denote $P_K = K'$ and $P_h = h'$

Then we get:

$$\boxed{\begin{aligned} K' &= P_K \\ h' &= P_h \\ P_K' &= K h^2 + \frac{K(K^2-1)}{\xi^2} \\ P_h' &= -2\frac{P_h}{\xi} + \frac{2hK^2}{\xi^2} + \frac{\lambda}{e} h(h^2-1) \end{aligned}}$$

The boundary conditions are:

$K \sim 1 - O(\xi) \qquad H \leq O(\xi) \quad$ as $\xi \to 0$

$K \to 0 \qquad H \sim \xi \quad$ as $\xi \to \infty$

Or:

$h \sim O(1)$ as $\xi \to 0$

$h \to 1$ as $\xi \to \infty$

Let's analyze the behavior of the solution close to the origin $\xi = 0$. We write

$$K = k_0 + k_1 \xi^r + k_2 \xi^{r+1} \ldots$$

$$h = \alpha_1 \xi^s + \alpha_2 \xi^{s+1} \ldots$$

and so,

$$K' = r k_1 \xi^{r-1} + (r+1) k_2 \xi^r \ldots$$

$$K'' = r(r-1) k_1 \xi^{r-2} + r(r+1) k_2 \xi^{r-1} \ldots$$

$$h' = s \alpha_1 \xi^{s-1} + (s+1) \alpha_2 \xi^s + \ldots$$

$$h'' = s(s-1) \alpha_1 \xi^{s-2} + s(s+1) \alpha_2 \xi^{s-1} + \ldots$$

From eqs. (*) on page (14) we get

$$(r-1) r k_1 \xi^{r-2} + r(r+1) k_2 \xi^{r-1} =$$

$$(k_0 + k_1 \xi^r + \ldots)(\alpha_1 \xi^\alpha + \alpha_2 \xi^{\alpha+1} \ldots)^2$$

$$+ (k_0 + k_1 \xi^r)\left((k_0 + k_1 \xi^r)^2 - 1\right)/\xi^2$$

$$= (k_0 + k_1 \xi^r + \ldots)(\alpha_1^2 \xi^{2s} + 2\alpha_1 \alpha_2 \xi^{2s+1} + \ldots)$$

$$+ (k_0 + k_1 \xi^r + \ldots)(k_0^2 - 1 + 2 k_0 k_1 \xi^r + k_1^2 \xi^{2r} + \ldots)/\xi^2$$

$$= k_0 \alpha_1^2 \xi^{2s} + k_0 2 \alpha_1 \alpha_2 \xi^{2s+1} + \alpha_1^2 k_1 \xi^{2s+r} + \ldots$$

$$+ \left[k_0(k_0^2 - 1) + \left[(k_0^2 - 1) k_1 + 2 k_0^2 k_1\right] \xi^r + \ldots\right]/\xi^2$$

We have to take:

$$k_0 = 1$$

and so

$$2k_1 = r(r-1)k_1 \quad \rightarrow \quad r = 2$$

The second eq. in (*) on page 14 gives:

$$s(s-1)\alpha_1\xi^{s-2} + s(s+1)\alpha_2\xi^{s-1} =$$

$$-\frac{2}{\xi}\left(\underline{s\alpha_1\xi^{s-1}} + \cancel{s}(s+1)\alpha_2\xi^{s}\right)$$

$$+\frac{2}{\xi^2}\left(\alpha_1\xi^{s} + \alpha_2\xi^{s+1}\right)\left(1 + k_1\xi^2 + \dots\right)^2$$

$$+\frac{\lambda}{e^2}\left(\alpha_1\xi^{s} + \alpha_2\xi^{s+1}\right)\left(\left(\alpha_1\xi^{s} + \alpha_2\xi^{s+1}\right)^2 - 1\right)$$

$$= -2s\alpha_1\xi^{s-2} - 2(s+1)\alpha_2\xi^{s-1}$$

$$+ 2\alpha_1\xi^{s-2} + 2\alpha_2\xi^{s-1} + \dots$$

Then we need

$$s(s-1)\alpha_1 = 2\alpha_1(1-s) \quad \longrightarrow \quad s = 1$$

So, we have

$$K = 1 + k_1\xi^2 + \dots \qquad K' = 2k_1\xi + \dots$$

$$h = \alpha_1\xi + \dots \qquad h' = \alpha_1 + \dots$$