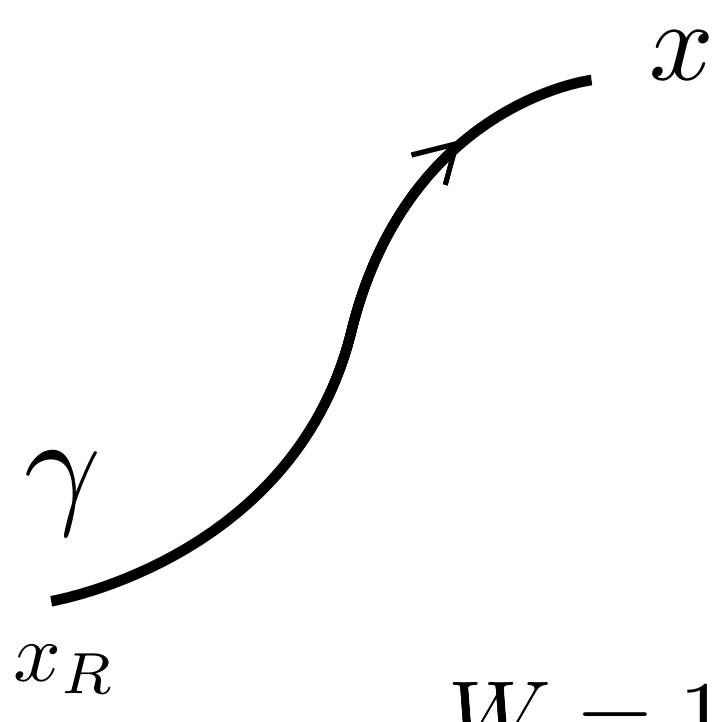
The Wilson Line



$$\frac{d\,W}{d\,\sigma} + i\,e\,A_\mu\,\frac{d\,x^\mu}{d\,\sigma}\,W = 0$$

$$W = 1 - i\,e \int_0^\sigma d\sigma_1\, A_{\mu_1}(\sigma_1)\,\frac{d\,x^{\mu_1}}{d\,\sigma_1} + (i\,e)^2 \int_0^\sigma d\sigma_1\, A_{\mu_1}(\sigma_1)\,\frac{d\,x^{\mu_1}}{d\,\sigma_1} \int_0^{\sigma_1} d\sigma_2\, A_{\mu_2}(\sigma_2)\,\frac{d\,x^{\mu_2}}{d\,\sigma_2} + \ldots$$

Formally

$$W(\gamma) = P_1\, e^{-i\,e\,\int_\gamma A_\mu\,\frac{dx^\mu}{d\sigma}\,d\sigma}\, W_R$$

integration constant

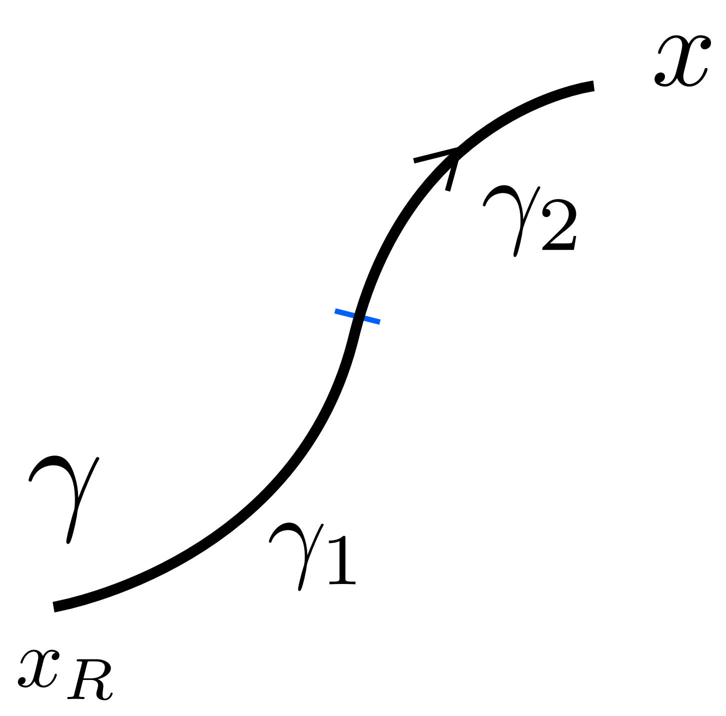
Indeed

$$\frac{d\,W}{d\,\sigma} = -i\,e\,A_{\mu_1}(\sigma)\,\frac{d\,x^{\mu_1}}{d\,\sigma} + (i\,e)^2\,A_{\mu_1}(\sigma)\,\frac{d\,x^{\mu_1}}{d\,\sigma}\int_0^\sigma d\sigma_2\,A_{\mu_2}(\sigma_2)\,\frac{d\,x^{\mu_2}}{d\,\sigma_2} + \ldots$$

$$= -i\,e\,A_\mu(\sigma)\,\frac{d\,x^\mu}{d\,\sigma}\left[1 - i\,e\int_0^\sigma d\sigma_1\,A_{\mu_1}(\sigma_1)\,\frac{d\,x^{\mu_1}}{d\,\sigma_1} + \ldots\right]$$

$$= -i\,e\,A_\mu(\sigma)\,\frac{d\,x^\mu}{d\,\sigma}\,W$$

Sectioning it



$$W(\gamma) = W(\gamma_2)\, W(\gamma_1)$$

Gauge transformation

$$A_\mu \to \bar{A}_\mu = g\, A_\mu\, g^{-1} + \frac{i}{e}\, \partial_\mu g\, g^{-1}$$

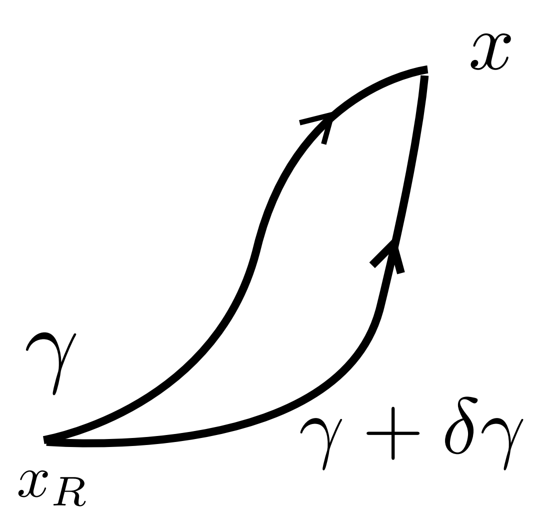
$$\frac{d\,\bar{W}}{d\,\sigma} + i\,e\,\bar{A}_\mu\,\frac{d\,x^\mu}{d\,\sigma}\,\bar{W} = 0 \quad \xrightarrow{\times g^{-1}} \quad g^{-1}\,\frac{d\,\bar{W}}{d\,\sigma} + i\,e\,g^{-1}\,\bar{A}_\mu\,\frac{d\,x^\mu}{d\,\sigma}\,\bar{W} = 0$$

$$g^{-1}\,\frac{d\,\bar{W}}{d\,\sigma} + i\,e\,A_\mu\,\frac{d\,x^\mu}{d\,\sigma}\,g^{-1}\bar{W} - \underbrace{g^{-1}\partial_\mu g\, g^{-1}}_{\downarrow\; -\partial_\mu g^{-1}}\,\bar{W}\,\frac{d\,x^\mu}{d\,\sigma} = 0 \quad \longrightarrow \quad \frac{d\,\left(g^{-1}\,\bar{W}\right)}{d\,\sigma} + i\,e\,A_\mu\,\frac{d\,x^\mu}{d\,\sigma}\,g^{-1}\bar{W} = 0$$

$$\bar{W} = g\, W\, g_R^{-1}$$

$$W(\gamma) \to g\, W(\gamma_1)\, g_2^{-1}\, g_2\, W(\gamma_1)\, g_1^{-1}$$

Varying the curve



$$\frac{dW}{d\sigma} + i\,e\,A_\mu \frac{dx^\mu}{d\sigma} W = 0$$

$$\frac{dW^{-1}}{d\sigma} = -W^{-1}\frac{dW}{d\sigma}W^{-1}$$

$$\frac{d\,\delta W}{d\sigma} + i\,e\,\delta A_\mu \frac{dx^\mu}{d\sigma} W + i\,e\,A_\mu \frac{d\,\delta x^\mu}{d\sigma} W + i\,e\,A_\mu \frac{dx^\mu}{d\sigma}\delta W = 0$$

$$\frac{dW^{-1}}{d\sigma} - i\,e\,W^{-1} A_\mu \frac{dx^\mu}{d\sigma} = 0$$

$\rightarrow W^{-1}$

$\leftarrow \delta W$

$$\frac{d\,(W^{-1}\,\delta W)}{d\sigma} + i\,e\,W^{-1}\,\delta A_\mu\,W \frac{dx^\mu}{d\sigma} + i\,e\,W^{-1}\,A_\mu\,W\frac{d\,\delta x^\mu}{d\sigma} = 0$$

Integrate it

$$(W^{-1}\,\delta W)\,|_0^{\sigma_1} = -i\,e\int_0^{\sigma_1}\left[W^{-1}\frac{\partial A_\mu}{\partial x^\nu}\,W\frac{dx^\mu}{d\sigma}\,\delta x^\nu - \frac{d}{d\sigma}\left(W^{-1}\,A_\mu\,W\right)\,\delta x^\mu + \frac{d}{d\sigma}\left(W^{-1}\,A_\mu\,W\,\delta x^\mu\right)\right]$$

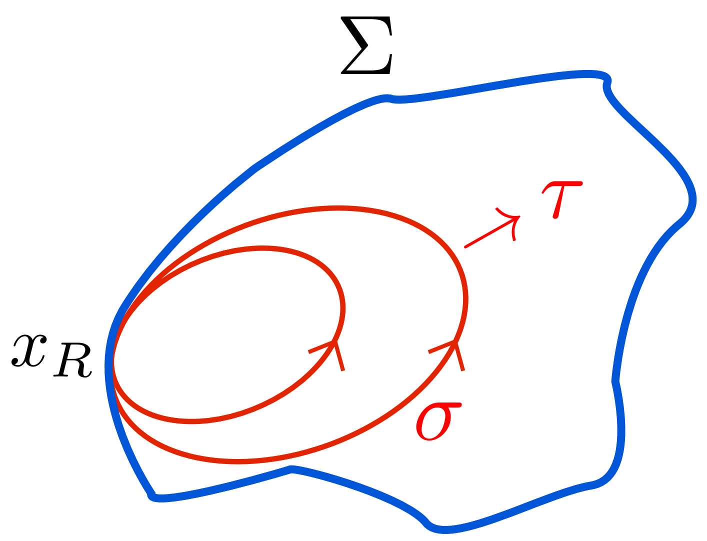
$$(W^{-1}\,\delta\,W)\,|_0^{\sigma_1} = -i\,e\int_0^{\sigma_1}\left[W^{-1}\,\frac{\partial\,A_\mu}{\partial\,x^\nu}\,W\,\frac{d\,x^\mu}{d\,\sigma}\,\delta x^\nu - \frac{d}{d\,\sigma}\,(W^{-1}\,A_\mu\,W)\,\delta\,x^\mu + \frac{d}{d\,\sigma}\,(W^{-1}\,A_\mu\,W\,\delta\,x^\mu)\right]$$

$$-i\,e\,W^{-1}\,A_\nu\,\frac{d\,x^\nu}{d\,\sigma}\,A_\mu\,W - W^{-1}\,\frac{\partial\,A_\mu}{d\,x^\nu}\,\frac{d\,x^\nu}{d\,\sigma}\,W + i\,e\,W^{-1}\,A_\mu A_\nu\,\frac{d\,x^\nu}{d\,\sigma}\,W$$

$$= -i\,e\int_0^{\sigma_1} W^{-1}\left[\frac{\partial\,A_\nu}{\partial\,x^\mu} - \frac{\partial\,A_\mu}{\partial\,x^\nu} + i\,e\,(A_\mu\,A_\nu - A_\nu\,A_\mu)\right]W\,\frac{d\,x^\nu}{d\,\sigma}\,\delta\,x^\mu + \left(W^{-1}\,A_\mu\,W\,\delta\,x^\mu\right)\,|_0^{\sigma_1}$$

Keeping the end points fixed

$$W^{-1}(\gamma)\,\delta W(\gamma) = i\,e\int_0^{2\,\pi} d\sigma\,W^{-1}\,F_{\mu\nu}\,W\,\frac{d\,x^\mu}{d\,\sigma}\,\delta x^\nu$$



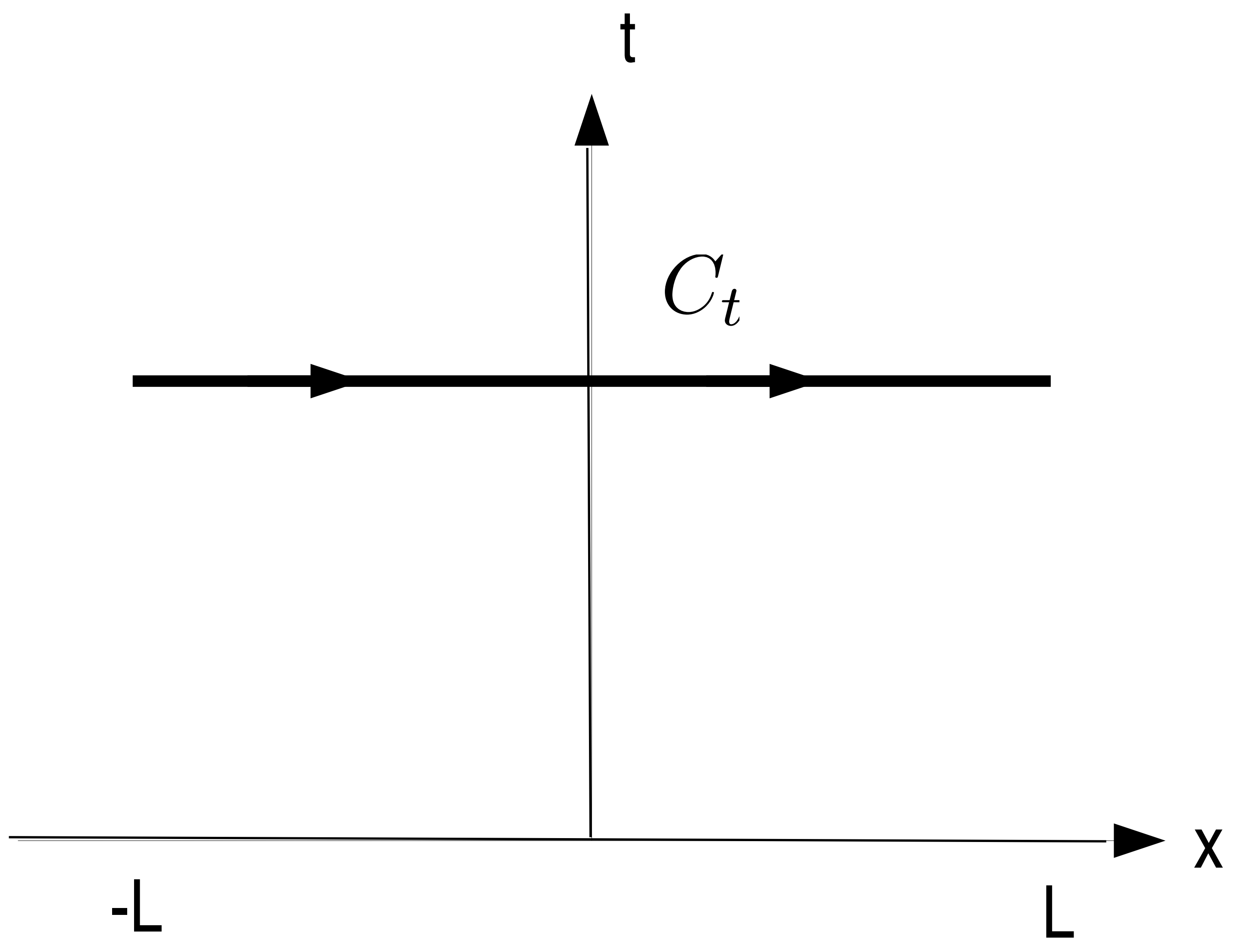
$$W^{-1}(\gamma)\,\delta W(\gamma) = i\,e\int_0^{2\,\pi} d\sigma\, W^{-1}\,F_{\mu\nu}\,W\,\frac{d\,x^\mu}{d\,\sigma}\,\delta x^\nu$$

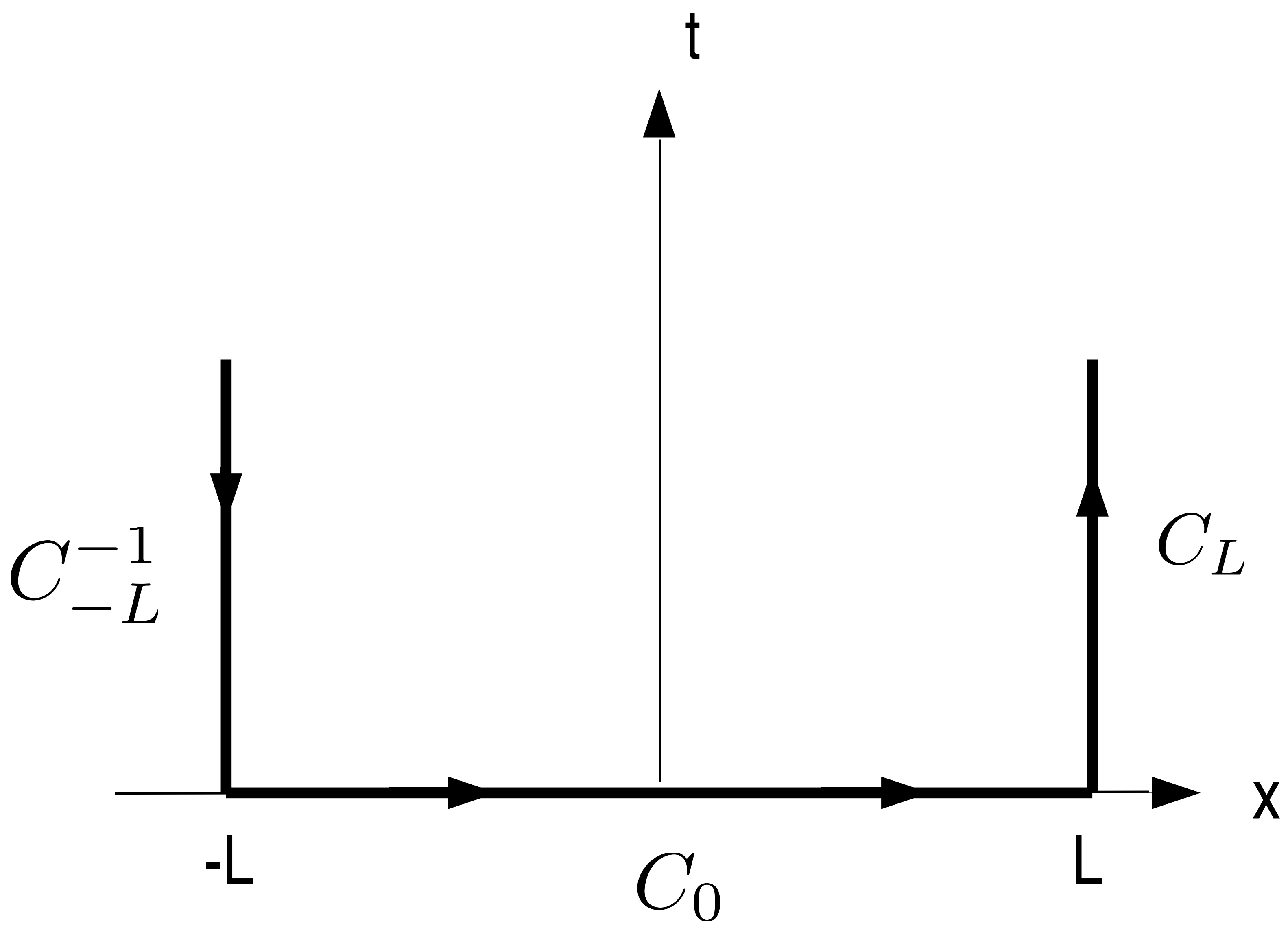
$$\frac{d\,W}{d\,\tau} = i\,e\,W\int_0^{2\,\pi} d\sigma\, W^{-1}\,F_{\mu\nu}\,W\,\frac{d\,x^\mu}{d\,\sigma}\,\frac{d\,x^\nu}{d\,\tau}$$

The non-abelian Stokes Theorem

$$P_1\,e^{-i\,e\,\int_{\partial\Sigma} A_\mu \frac{d\,x^\mu}{d\sigma}\,d\sigma} = P_2\,e^{i\,e\,\int_\Sigma W^{-1}\,F_{\mu\nu}\,W\,\frac{d\,x^\mu}{d\sigma}\,\frac{d\,x^\nu}{d\tau}\,d\sigma\,d\tau}$$

Conservation Laws





$$W(C_t) = W(C_L)\, W(C_0)\, W(C_{-L}^{-1})$$

Boundary cond.

$A_t(-L) = A_t(L)$

$$W(C_{-L}^{-1}) \rightarrow W^{-1}(C_L)$$

Eigenvalues of $W(C_t)$ are conserved

$$\frac{d}{dt}\mathrm{Tr}\left[W(C_t)\right]^n = 0$$

O.Alvarez, LAF, J.S.Guillen
1)[hep-th/9710147], NPB529,689(1998)
2)[arXiv:0901.1654[hep-th]], IJMPA24,1825(2009)
LAF, G.Luchini
3)[arXiv:1205.2088[hep-th]], PRD86,085039(2012)
4)[arXiv:1109.2606hep-th]], NPB858PM(2012)336-365
C.P.Constantinidis, LAF, G.Luchini
5)[arXiv:1508.03049[hep-th]], JHEP12(2015)137
6)[arXiv:1611.07041[hep-th]], JPA 52, 15, 155202 (2019)
7)[arXiv:1710.03359[hep-th]], PRD97(085006)2018