L. A. FERREIRA

Lectures on

Gauge Theories and Lie Algebras

with some applications to

spontaneous symmetry breaking and integrable dynamical systems

Given at the University of Virginia, Fall 1982

by

David Olive

University of Virginia, Charlottesville 22901

and Imperial College, London

Notes Taken By

G. Bhattacharya*

and

N. Turok**

* Present address: Physics Dept., Syracuse University, Syracuse, NY 13210

** Present address: Blackett Lab., Imperial College, London, SW7

Introduction

There seems to be a growing realization that algebraic structures are important in theoretical physics. For example in particle physics modern theories are roughly specified by a choice of simple gauge group. The possible quantum numbers carried by the elementary particles relate to the underlying Lie algebra and its representation theory. (See Slansky's 1981 review.)

A particle theory with an even richer algebraic structure was the dual string model which referred to an infinite spectrum of particle states with different masses and spins. [See the review articles in Jacob 1974.] It is now realized that this is related to the theory of Kac Moody algebras which constitute a class of infinite, but not too infinite, Lie algebras. Theories such as gauge theories, general relativity and supersymmetric theories, which rely on gauge groups of various kinds turned out to be various special cases of this string theory, part of whose algebraic structure referred to the so-called Virasoro algebra which played the role of a gauge algebra. The interrelationships between these theories focussed attention on topological objects in spontaneously broken gauge theories such as vortex lines [Nielsen and Olesen 1973] and monopoles ['t Hooft 1974 and Polyakov 1974] and the question as to whether these could be regarded as quantum objects whose excitations might lie on Regge trajectories.

In particular the presence of magnetic monopoles in theories of the grand unified type was predicted and conjectures were made about the possible form of quantum field theory applicable to these monopoles. [Montonen and Olive 1977]. Definite answers would change our perspectives on these theories. So far there is only one theory in which the quantum field theory of the topological object is known. That is the two dimensional Sine Gordon theory, a theory which exhibits a surprising degree of algebraic structure. Many analytic solutions are known, relating to its complete integrability which implies an infinite number of conservation laws. Finally its quantum equivalence to the fermion massive Thirring model involves an algebraic structure reminiscent of that occurring in the dual string model, the Kac Moody algebras.

Magnetic monopoles occur in four dimensional, spontaneously broken gauge theories and less is known at present. The most promising situation for the conjecture about the monopole quantum field theory concerns a supersymmetric version of the gauge theory with resemblances to the favoured physical theory (Olive 1982b). In such theories the monopole configurations satisfy "self dual" equations and increasingly many analytic solutions are known. [See the volume containing Nahm 1982 and Olive 1982b.] Configurations with special symmetry such as spherical symmetry yield equations with integrability properties rather like Sine Gordon above, and hence interesting algebraic structures.

The suspicion is that the final construction of the monopole quantum field theory will depend on the understanding of a new sort of algebraic structure, richer than those already known in mathematics.

The above discussion explains why I think algebraic structures will play an important role in the future understanding of realistic versions of quantum spontaneous broken gauge theories. [In emphasizing the importance of algebraic or infinitesimal structures I do not wish to imply that global structures will not also play an important role.]

These lectures are aimed to be a first step in explaining and developing the role of algebras by presenting the basic theory of ordinary Lie algebras and explaining how it can be used in the context of gauge theories.

Lectures 1-6 explain the basic formalism of gauge theories leading to the explanation of how topology can be used to classify the instanton solutions. Lectures 7-16 explain the classification of simple Lie algebras and their notation in terms of Dynkin diagrams. Other accounts of this material in the physics literature are due to Racah (1964), Slansky (1981), and Georgi (1982). The source that I have found most useful in the mathematical literature is Humphreys (1972), with Helgason (1978) consulted for the more global aspects. Lectures 17 and 18 explain an application of these results to establish the existence of unexpected conservation laws in certain non linear coupled differential equations, the Toda molecule equations. Lectures 19-25 discuss spontaneous symmetry breaking of gauge invariance and how a complete discussion of possibilities can be given using the Lie algebra theory of lectures 7-16 when the Higgs field lies in the adjoint representation. The Higgs field induces masses both to the gauge particles and to topological objects whose classification is discussed. Although they constitute the "missing link" relating the various topics in these lectures, I do not give a detailed discussion of monopoles since this is readily available in earlier reviews: Goddard and Olive (1978), with a survey update Olive (1982b) and Coleman (1982). These should be referred to for more background information.

The discussion of the relation between symmetry breaking and charge quantization motivates the presentation of the representation theory of Lie algebras in lectures 26 and 28. In lectures 29-31 these results are used to develop solutions to the Toda molecule equations and to show how these equations arise as describing spherically symmetric monopole configurations in spontaneously broken gauge theories. This analysis shows how the apparently disparate ingredients of these lectures are indeed linked together.

The next step would be the analysis of the structure of Euclidean Kac Moody algebras (which underly the Sine Gordon equation for example) but this is beyond the scope of this lecture course which could be regarded as an introduction to their study as reviewed for example by Mac Donald (1981). Some applications along the lines mentioned above can be found in Frenkel (1981).

I would like to express my gratitude to Gautam Bhattacharya and Neil Turok for preparing these notes and also to Paul Fishbane who played an important role.

I am very happy to thank David Brydges, Paul, Michael Fowler, Peter Goddard, Larry Thomas and my other colleagues here for stimulating discussions and suggestions. Finally I wish to express my gratitude to the Center for Advanced Studies and the Departments of Mathematics and Physics for their generous hospitality here at the University of Virginia.

Table of Contents

Lectures

Lecture 1

Let us begin our discussion of gauge theories by reviewing Maxwell's theory of electromagnetism, to establish our notation and conventions. The field strength $F^{\mu\nu}$ viewed as a matrix has components

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} \qquad (1.1)$$

i.e. $F^{oi} = -E_i$ and $F^{ij} = -\epsilon^{ijk}B_k$, $i,j = 1,2,3.$

We use the standard summation convention unless stated otherwise.

$F^{\mu\nu}$ satisfies

$$\partial_\mu F^{\mu\nu} = j^\nu_{e.m.} \qquad (1.2)$$

where $j^\nu_{e.m.} = (\rho, \vec{j})$ is the electromagnetic current four-vector. The duality operation defined by

$$^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \qquad (1.3)$$

has the effect of interchanging $\vec{E}$ with $\vec{B}$ and $\vec{B}$ with $-\vec{E}$.

In the absence of magnetic monopoles the dual tensor satisfies

$$\partial_\mu {}^*F^{\mu\nu} = 0 \qquad (1.4)$$

(1.2) and (1.4) are Maxwell's equations as is easily checked using the formulae above. Another way of writing (1.4) is

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \qquad (1.5)$$

the so-called Bianchi identity. This is the necessary and sufficient condition for us to write, at least locally, that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (1.6)$$

where $A_\mu = (\phi, \vec{A})$ is the four-vector gauge potential, called by mathematicians the 'connection'. We are always free to replace A_μ by

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\chi(x) \tag{1.7}$$

where χ is an arbitrary function of x, which leaves $F_{\mu\nu}$ unaltered. This (unphysical) transformation we call a <u>gauge</u> transformation. Let us see how Maxwell's theory behaves under these transformations.

Consider the motion of a free classical particle. The Hamiltonian is just

$$H = \frac{1}{2m}\vec{p}^{\,2} \tag{1.8}$$

where $\vec p = m\dot{\vec x}$ is the momentum, m the mass, $\dot{\vec x}$ the velocity. In an electromagnetic field, the minimal coupling principle tells us the correct Lorentz equation of motion is obtained by replacing $p^\mu = (H,\vec p)$ with $p^\mu - qA^\mu$, where q is the electric charge so

$$m\dot{\vec x} = \vec p - q\vec A\ , \tag{1.9}$$

and

$$\frac{1}{2m}(\vec p - q\vec A)^2 = H - q\phi \tag{1.10}$$

Now when the theory is quantized, what is important is the canonical momentum $\vec p$ and coordinates $\vec x$; these obey

$$[p_i, x_j] = -i\hbar\delta_{ij} \tag{1.11}$$

In a relativistic theory we write $p_\mu = i\hbar\frac{\partial}{\partial x^\mu} = i\hbar\partial_\mu$. So in an electromagnetic field, the first quantized equations of motion will be obtained by replacing p_μ with (remember $g_{ii} = -1$ in our metric)

$$i\hbar\partial_\mu - qA_\mu = i\hbar(\partial_\mu + \frac{iq}{\hbar}A_\mu) \equiv i\hbar D_\mu \tag{1.12}$$

where D_μ is called the <u>covariant derivative</u>. The Schroedinger equation for a free electron wavefunction ψ is

$$\frac{\vec p^{\,2}}{2m}\psi = E\psi$$

So the minimal coupling principle tells us that in an electromagnetic field ψ will satisfy

$$\frac{-\hbar^2}{2m}D_\mu^2\psi = \frac{-\hbar^2}{2m}(\partial_\mu + \frac{iq}{\hbar}A_\mu)^2\psi = (E-q\phi)\psi \tag{1.13}$$

We would like this equation to be invariant under the gauge transformation (1.7). This requires that ψ and $D_\mu^2\psi$ transform similarly under (1.7) if we have

$$\psi \to \psi'(x) = e^{-iq\chi(x)/\hbar}\psi(x) \tag{1.14}$$

then the change in $\partial_\mu\psi(x)$ is exactly compensated by the change in A_μ and the Schroedinger equation is indeed invariant.

To summarize, we see our equations preserve their form under

$$A_\mu \to A'_\mu(x) = A_\mu(x) + \partial_\mu\chi(x) \tag{1.15a}$$

if

$$\psi \to \psi'(x) = e^{-iq\chi(x)/\hbar}\psi(x) \tag{1.15b}$$

In fact the phase factor in (1.15b) is an element of a group, the circle group U(1) which is called the <u>gauge</u> group. We see it contains $\hbar$ - and so could not have been seen by Maxwell! It is this feature which we will generalize: any Lie group can be chosen as a gauge group. The Maxwell U(1) theory will be a special case of the general gauge theory. The principle of gauge invariance will also govern the other theories we shall discuss - roughly speaking it dictates that physical processes are independent of the phase of the wavefunction at each point of spacetime independently.

We shall construct the theory from objects that transform like ψ under gauge transformations, for instance

$$D_\mu\psi(x) \to D'_\mu\psi'(x) = e^{-iq\chi(x)/\hbar}D_\mu\psi(x)\ , \tag{1.16}$$

and using $\psi(x) = e^{+iq\chi(x)/\hbar}\psi'(x)$ we see that formally we have

$$D_\mu \to D'_\mu = e^{-iq\chi(x)/\hbar} D_\mu e^{+iq\chi(x)/\hbar} \tag{1.17}$$

The electric charge <u>operator</u> Q is defined as the <u>generator</u> of the gauge group $U(1)_{EM}$ (the Maxwell U(1)) in the sense that

$$e^{-iQ\chi/\hbar}\, \psi\, e^{+iQ\chi/\hbar} = e^{-iq\chi/\hbar}\, \psi \tag{1.18}$$

Hence we see that

$$[Q,\psi] = q\psi \tag{1.19}$$

We also see that

$$\begin{aligned} [m\dot{x}^\mu, m\dot{x}^\nu] &= [i\hbar D^\mu, i\hbar D^\nu] \\ &= -\hbar^2 [D^\mu, D^\nu] \\ &= -iq\hbar F_{\mu\nu} \end{aligned} \tag{1.20}$$

using (1.6) and (1.12). In other words, under a gauge transformation,

$$\begin{aligned} \frac{iq}{\hbar} F_{\mu\nu} \to \frac{iq}{\hbar} F'_{\mu\nu} &= [D'_\mu, D'_\nu] \\ &= e^{-iq\chi/\hbar} [D_\mu, D_\nu]\, e^{+iq\chi/\hbar} \\ &= \frac{iq}{\hbar} e^{-iq\chi/\hbar} F_{\mu\nu}\, e^{+iq\chi/\hbar} \\ &= \frac{iq}{\hbar} F_{\mu\nu} \end{aligned}$$

The last equation follows because in the U(1) case everything commutes, and is another way of seeing why $F_{\mu\nu}$ remained invariant. This result will not hold for larger groups. We shall see how the transformation generalizes for non-Abelian groups in the next lecture.

<u>Lecture 2</u>

How do we generalize the principle of gauge invariance to groups of 'phases' larger than U(1) [Yang and Mills, 1954, Shaw, 1955]? An arbitrary group G (defined as a set of elements and an associative product law so that if $g_1, g_2 \in G$, $g_1 g_2 \in G$, with an identity element e and a unique inverse g^{-1} for every $g \in G$) will not provide a <u>gauge</u> theory – we need a continuous or Lie group. A first guess might be

$$\psi(x) \to \psi'(x) = D(g)\psi(x) \tag{2.1}$$

where ψ is a column vector and D(g) a matrix representation of the element g. For elements close to the identify e, we can write

$$g = e^{i\omega_\alpha T_\alpha} = 1 + i\omega_\alpha T_\alpha + O(\omega^2) \tag{2.2}$$

where T_α are the <u>generators</u> of the group and ω_α its parameters. If the generators are to correspond to observable charges then they must be Hermitian. In fact the crucial relations satisfied by the generators are their commutation relations: (determined by the fact that g is a group element)

$$[T_\alpha, T_\beta] = T_\alpha T_\beta - T_\beta T_\alpha = i\, f^\gamma_{\alpha\beta}\, T_\gamma \quad . \tag{2.3}$$

Hermiticity ensures that the $f^\gamma_{\alpha\beta}$ are real numbers, clearly satisfying $f^\gamma_{\alpha\beta} = -f^\gamma_{\beta\alpha}$. The $f^\gamma_{\alpha\beta}$ are the <u>structure constants</u> of the group. They satisfy a Jacobi identity, derived from the identity

$$[T_\alpha, [T_\beta, T_\gamma]] + \text{cycles} = 0 \quad . \tag{2.4}$$

The simplest example of a nonabelian (non-commutative) Lie group is SO(3). This has three generators T_1, T_2 and T_3 satisfying

$$[T_i, T_j] = i\epsilon_{ijk} T_k \quad , \tag{2.5}$$

the angular momentum algebra. There are many inequivalent irreducible representations of the T_i's satisfying (2.5), for example $T_i = 0$, the scalar representation, $T_i = \frac{1}{2}\sigma_i$, the spin ½ representation and so on. In Yang-Mills theories we shall have a Lie group of symmetries at each space time point.

We shall deal with group elements in a specific representation in each case, denoted

$$D(g) = e^{i\omega_a D(T_a)} \tag{2.6}$$

which of course satisfies (under matrix multiplication)

$$D(g_1)\, D(g_2) = D(g_1 g_2)\ . \tag{2.7}$$

The matter fields form multiplets transforming like a column vector under matrix multiplication:

$$\phi_i(x) \to \phi_i'(x) = D_{ij}(g)\phi_j(x) \qquad i = 1 \ldots \text{dimension } (D)$$

The gauge principle says we should demand invariance under local gauge transformations $g(x)$. We can see how this works by looking at kinetic terms in the Lagrangian, which involve $(\partial^\mu\phi(x))^2$, and

$$\begin{aligned} \partial^\mu\phi(x) \to \partial^\mu\phi'(x) &= D(g(x))\partial^\mu\phi + D(\partial^\mu g(x))\phi(x) \\ &= D(g(x))[\partial^\mu\phi + D(g(x))^{-1}\, D(\partial^\mu g(x))\phi(x)]\ . \end{aligned}$$

But $D(g)^{-1}\, D(\partial^\mu g) = D(g^{-1}\partial^\mu g)$ by (2.7).

So

$$\partial^\mu\phi \to \partial^\mu\phi(x) = D(g(x))[\partial^\mu\phi + D(g^{-1}\partial^\mu g)\phi(x)] \tag{2.8}$$

which does not bode well for constructing invariant quantities. However, $g^{-1}\partial_\mu g$ is a linear combination of the generators T_a, as is seen by considering a small displacement ε^μ :

$$\begin{aligned} \varepsilon^\mu g^{-1}\partial_\mu g &= g^{-1}(x)\ (g(x+\varepsilon) - g(x)) \\ &= g^{-1}(x)g(x+\varepsilon) - e \\ &= e + i\varepsilon^\mu W_\mu^a(x)T_a - e \\ &= i\varepsilon^\mu W_\mu^a(x)T_a\ , \end{aligned} \tag{2.9}$$

because $g(x+\varepsilon)$ is a smooth function and close to $g(x)$, and so can be written as $g(x)e^{i\varepsilon^\mu W_\mu^a T_a}$, with $W_\mu^a(x)$, some coefficient functions. We can therefore compensate for the inhomogeneous term is (2.8) by introducing a <u>gauge potential</u> at each point;

$$W^\mu(x) = \sum_{a=1}^{\dim G} W_a^\mu(x)T^a \tag{2.10}$$

and defining a covariant derivative

$$D^\mu\phi = \partial^\mu\phi + ie\,D(W^\mu)\phi \tag{2.11}$$

by analogy with the U(1) theory in the previous lecture. e is the "gauge coupling constant" and will be related to q in lecture 22. How does this transform?

$$\begin{aligned} D_\mu\phi \to (D_\mu\phi)' &= \partial_\nu\phi' + ieD(W_\mu')\phi' \\ &= D(g)\ (\partial_\mu\phi + D(g^{-1}\partial_\mu g)\phi) + ie\ D(g^{-1}W_\mu' g)\phi \end{aligned}$$

using (2.8). We want it to transform simply, like ϕ, so we set

$$(D_\mu\phi)' = D(g)D_\mu\phi = D(g)(\partial_\mu + ie\ D(W_\mu))\phi\ .$$

This must be true for all ϕ, so we have

$$g^{-1}\partial_\mu g + ie\,g^{-1}W_\mu' g = +\,ieW_\mu$$

(in any representation D, so we drop the D).

or

$$\begin{aligned} W_\mu' &= gW_\mu g^{-1} + \frac{i}{e}\,(\partial_\mu g)g^{-1} \\ &= -\frac{i}{e}\,g(D_\mu g^{-1}) \end{aligned} \tag{2.12}$$

With this transformation rule for W_μ, we also find that

$$D^\mu \to D^{\mu\prime} = g\,D^\mu g^{-1}\ , \tag{2.13}$$

showing the analogy with (1.17). Continuing in the same way we find

$$\begin{aligned} [D_\mu, D_\nu] &= [\partial_\mu + ieD(W_\mu),\ \partial_\nu + ieD(W_\nu)] \\ &= ieD(\partial_\mu W_\nu - \partial_\nu W_\mu + ie[W_\mu, W_\nu]) \\ &\equiv ieD(F_{\mu\nu}). \end{aligned} \tag{2.14}$$

We call $F_{\mu\nu}$ the field strength. Notice the new term $[W_\mu,W_\nu]$, making $F_{\mu\nu}$ a nonlinear function of W_μ. This will enrich the theory and lead to many interesting solutions. Under a gauge transformation, using (2.13) we see that

$$F_{\mu\nu} \to F'_{\mu\nu} = g\, F_{\mu\nu} g^{-1} \tag{2.15}$$

and notice that only in an abelian group is $F_{\mu\nu}$ invariant. An obvious candidate for the Lagrangian of the theory is (the trace runs over the group indices)

$$\mathrm{Tr}(D(F_{\mu\nu}F^{\mu\nu})) \quad , \tag{2.16}$$

since it is both Lorentz and gauge invariant.

Lecture 3

One particular representation will play a key role in this course, the adjoint representation. It is defined by

$$g\, T_a\, g^{-1} = T_b d_{ba}(g) \qquad a = 1 \ldots \dim G \tag{3.1}$$

It is easy to check that the matrices $d_{ba}(g)$ satisfy under matrix multiplication

$$d(g_1)d(g_2) = d(g_1 g_2) \tag{3.2}$$

and thus form a representation:

$$\begin{aligned} g_1 g_2\, T_a (g_1 g_2)^{-1} &= T_b d_{ba}(g_1 g_2) \\ &= g_1 (g_2 T_a g_2^{-1}) g_1^{-1} \\ &= g_1 (T_b d_{ba}(g_2)) g_1^{-1} \\ &= g_1 T_b g_1^{-1}\, d_{ba}(g_2) \\ &= T_c d_{cb}(g_1)\, d_{ba}(g_2) \end{aligned} \tag{3.3}$$

and we can equate the coefficients of T_a $a = 1 \ldots$ dimension (G) since they are linearly independent. In fact $d_{ab}(g)$ always has real entries.

In infinitesimal form, let $g = 1 + i\omega^a T_a$, where the ω^a are small. Then (3.1) becomes

$$(1 + i\omega^a T_a)\phi_b T_b (1 - i\omega^a T_a) = T_b d_{ba}(1 + i\omega)\phi_a \tag{3.4}$$

where ϕ_a are arbitrary coefficients of the T_a – it is seen that they transform like a column vector in the adjoint representation. We may re-express (3.4) using $d_{ab}(1) = \delta_{ab}$ as

$$\phi\cdot T + i[\omega, \phi\cdot T] = \phi\cdot T + iT_b d_{ba}(\omega)\phi_a$$

or

$$[\omega, \phi\cdot T] = T\cdot(d(\omega)\phi) \tag{3.5}$$

where $\omega = \omega^a T_a$ and $\phi\cdot T = \phi^a T_a \equiv \Phi$. Then the covariant derivative of the ϕ^a field is

$$(D_\mu\phi)_a = \partial_\mu\phi_a + i\, e\, d_{ab}(W^\mu)\phi_b \tag{3.6}$$

while

$$\begin{aligned} D_\mu\Phi &= T_a D_\mu\phi_a = \partial_\mu\Phi + i\, e\, T_a d_{ab}(W^\mu)\phi_b \\ &= \partial_\mu\Phi + i\, e\, [W^\mu,\Phi] \end{aligned} \tag{3.7}$$

which is a useful form for the covariant derivative in the adjoint representation.

In the general gauge theory, there is also a Bianchi identity, like (1.4) and (1.5) in Maxwell's theory; the Jacobi identity (easily checked to be true for any operator) tells us that

$$[D_\lambda, [D_\mu, D_\nu]] + [D_\mu, [D_\nu, D_\lambda]] + [D_\nu, [D_\lambda, D_\mu]] = 0 \,. \tag{3.8}$$

Using $[D_\mu, D_\nu] = i\, e F_{\mu\nu}$, and

$$\begin{aligned} [D_\lambda, F_{\mu\nu}] &= (\partial_\lambda + i\, e W_\lambda)F_{\mu\nu} - F_{\mu\nu}(\partial_\lambda + i\, e W_\lambda) \\ &= (\partial_\lambda F_{\mu\nu}) + i\, e\, [W_\lambda, F_{\mu\nu}] \\ &= D_\lambda F_{\mu\nu} \end{aligned} \tag{3.9}$$

(the last line is true only for $F_{\mu\nu}$ in the adjoint representation), we obtain the Bianchi identity

$$D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} = 0 \tag{3.10}$$

So if we are in four dimensions, and recall the dual of $F_{\lambda\mu}$ is

$$^*F_{\lambda\mu} = \frac{1}{2}\, \epsilon_{\lambda\mu\nu\rho}\, F^{\nu\rho} \,.$$

The Bianchi identity reads

$$D_\lambda {}^*F^{\lambda\mu} = 0 \tag{3.11}$$

Notice that this is an identity and a consequence of our definition of $F_{\mu\nu}$, not a dynamical equation. The equation of motion of the theory we shall derive is in fact also the analogue of Maxwell's equation (1.2), that is

$$D_\lambda F^{\lambda\mu} = J^\mu \tag{3.12}$$

We proceed to construct the Lagrangian for our gauge fields. Primary requirements are that it be Lorentz and gauge invariant. As we saw in (2.16), $Tr(D(F_{\mu\nu}F^{\mu\nu}))$ is a good candidate. However we must also have a positive definite Hamiltonian (or our theory will not have a stable vacuum). This is a considerable restriction: we have from its definition (2.14) that

$$F_{\mu\nu} = T_a F^a_{\mu\nu} \tag{3.13}$$

So

$$Tr F_{\mu\nu}F^{\mu\nu} = F^a_{\mu\nu}F^{b\mu\nu}\, Tr(T_a T_b) \tag{3.14}$$

and it is easy to show that for a positive definite Hamiltonian one must have $Tr(T_a T_b)$ a positive definite matrix. In fact this matrix is called the <u>Killing metric</u> g_{ab} and it is positive definite if and only if the Lie group is <u>compact</u> and semisimple. Saying a group is <u>semisimple</u> means it has no abelian invariant ideals; that is, there exists no abelian subgroup H such that $g h g^{-1} \in H$ for all $g \in G$, $h \in H$. Moreover, for our purposes we can use positive definiteness of the Killing metric as a definition of compactness. By the cyclic property of the trace,

$$g_{ab} \equiv Tr(T_a T_b) \tag{3.15}$$

is symmetric. <u>If it is positive definite, then there must exist a basis T_a for which it is proportional to the unit matrix i.e.,</u>

$$g_{ab} = K^2\delta_{ab} \,. \tag{3.16}$$

In this basis, the structure constants $f_{ab}^{\ c}$ of the Lie algebra, defined by

$$[T_a, T_b] = i f_{ab}^{\ c} T_c$$

satisfy $f_{ab}^{\ c} = f_{abc}$ and are in fact, as will be shown later, totally antisymmetric in all three indices. We emphasize this will be true only for compact Lie algebras.

Our Lagrangian (3.14) we choose as

$$\mathcal{L} = -\frac{1}{4K^2} \mathrm{Tr}(F^2) = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

$$= -\frac{1}{4} (2 F^a_{oi} F^{aoi} + F^a_{ij} F^{aij})$$

$$= +\frac{1}{2} (E^a E^a - B^a B^a) \qquad (3.17)$$

The corresponding Hamiltonian density is

$$H = \frac{1}{2} (E^a E^a + B^a B^a) \qquad (3.18)$$

Lecture 4

Let us look at a particular example next - an SU(2) gauge theory. Here the algebra is

$$[T_a, T_b] = i \epsilon_{abc} T_c \qquad (4.1)$$

Recall that the covariant derivative of a field ϕ was defined as

$$D_\mu \phi = \partial_\mu \phi + i e D(W_\mu) \phi \qquad (2.11)$$

$$= \partial_\mu \phi + i e W^a_\mu D(T_a) \phi$$

where $D(T_a)$ is the matrix representative of T_a in the representation ϕ transforms under. For instance, if ϕ is an isodoublet, we use the spin $\frac{1}{2}$ representation, for which

$$D(T_a) = \tfrac{1}{2} \sigma_a \qquad (4.2)$$

However if ϕ were an isovector, a three-component vector, it would actually be in the adjoint representation and the theory of the previous lecture would apply. We have

$$d(T_a)_{bc} = i \epsilon_{bac} \qquad (4.3)$$

which follows when we recall that the action of a generator ω on a field ϕ in the adjoint representation was defined by

$$[\omega, T \cdot \phi] = T_a d(\omega)_{ab} \phi_b \,. \qquad (3.5)$$

So choosing $\omega = T_a$ and $\phi_c = \delta_{bc}$, (3.5) reads

$$[T_a, T_b] = T_c d(T_a)_{cb}$$

which by (2.3)

$$= i f_{ab}^{\ c} T_c \qquad (4.4)$$

and we see that

$$d(T_a)_{cb} = i f_{ab}^{\ c} \qquad (4.5)$$

The covariant derivative of a field ϕ in the adjoint representation of SU(2) is thus

$$(D_\mu\phi)_b = \partial_\mu\phi_b - e\, W^a_\mu\, \epsilon_{bac}\, \phi_c \tag{4.6}$$

which can be written

$$\overrightarrow{D_\mu\phi} = \overrightarrow{\partial_\mu\phi} - \overrightarrow{e\, W_\mu} \times \vec{\phi} \tag{4.7}$$

Returning to more general treatment, the Lagrangian (3.17) is used to define the gauge theory via the action $S = \int d^4 x \mathcal{L}_G(x)$. The equations of motion are the conditions for S to be stationary, i.e., $\delta S = 0$. In this case,

$$\delta S = -\frac{1}{2K^2}\int d^4 x\ Tr\ (F_{\mu\nu}\delta F^{\mu\nu}) \tag{4.8}$$

and using (2.14)

$$\begin{aligned}\delta F_{\mu\nu} &= \delta(\partial_\mu W_\nu - \partial_\nu W_\mu + ie[W_\mu,W_\nu]) \\ &= \partial_\mu\delta W_\nu - \partial_\nu\delta W_\mu + ie[\delta W_\mu,W_\nu] + ie[W_\mu,\delta W_\nu] \\ &= D_\mu\delta W_\nu - D_\nu\delta W_\mu \end{aligned}\tag{4.9}$$

since W_ν is in the adjoint representation.

Using (4.8) and (4.9) we find

$$\begin{aligned}\delta S &= -\frac{1}{2K^2}\int d^4 x\ Tr\ (F_{\mu\nu}(D^\mu\delta W^\nu - D^\nu\delta W^\mu)) \\ &= -\frac{1}{K^2}\int d^4 x\ Tr(F_{\mu\nu}D^\mu\delta W^\nu)\end{aligned}\tag{4.10}$$

using the antisymmetry of $F_{\mu\nu}$. We integrate by parts, using $D_\mu Tr = \partial_\mu Tr$ since any trace is a group scalar, to get

$$\delta S = -\frac{1}{K^2}\left(\int d^4 x\ (\partial_\mu Tr(F_{\mu\nu}\delta W^\nu) - Tr((D^\mu F_{\mu\nu})\delta W^\nu)\right) \tag{4.11}$$

The first term is just a surface term (Gauss's theorem) and is zero for suitable δW^ν. The second term tells us that, if δS is to vanish for δW^ν an arbitrary function multiplying each group generator, we must have

$$D_\mu F^{\mu\nu} = 0 \tag{4.12}$$

These are the equations of motion, true in any number of space-time dimensions. However the Bianchi identities in the form

$$D_\mu {}^*F^{\mu\nu} = 0 \tag{3.11}$$

only hold in four dimensions. The similarity between (4.12) and (3.11) distinguishes four dimensions. Notice that the equations of motion (4.11) are highly nonlinear - involving quadratic and cubic terms. This gives them a rich structure of classical solutions.

Interaction terms between the gauge fields and other matter fields may be introduced to the Lagrangian via the minimal coupling principle - for fermion fields ψ for example we get a new term

$$\mathcal{L}_F = \frac{1}{2}\ (\bar{\psi}\gamma^\mu(D_\mu\psi) - (D^\mu\bar{\psi})\gamma_\mu\psi) \tag{4.13}$$

The fermion fields satisfy the equations of motion

$$\gamma^\mu D_\mu\psi = 0 \tag{4.14}$$

while the gauge fields satisfy equations with new terms determined by the variation of $\mathcal{L}_F$ with respect to W_μ,

$$\begin{aligned}\delta\mathcal{L}_F &= +\frac{1}{2}\, e\, \bar{\psi}\, \gamma^\mu D(\delta W_\mu)\psi + \text{hermitian conjugate} \\ &= e\, \bar{\psi}\, \gamma^\mu D(T_a)\psi\delta W^a_\mu\end{aligned}\tag{4.15}$$

using the hermiticity of T_a. (4.11) becomes instead

$$(D_\mu F^{\mu\nu})_a = -e\, \bar{\psi}\, \gamma^\mu D(T_a)\psi \tag{4.16}$$

The right hand side is called a <u>current</u> - it acts as a source for the gauge fields. If several fermion fields ψ_i are involved, all in representations of the same, simple gauge group, the coupling constant e is the same for each (coming from (4.13)) and we get

$$D_\mu F^{\mu\nu} = -\, e \sum_i \bar{\psi}\, \gamma^\nu\, D^i(T_a)\psi_i$$

where D^i is the representation that ψ_i transforms under.

Since we are dealing primarily with the properties of gauge theories here, for the moment we'll forget about the fermions (and most of the observed physics!). We want to look for solutions to the equations (3.11) and (4.11). These depend strongly on the nature of the space-time, as we shall see.

In four dimensions, we can define the dual

$$ {}^*F_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}\,F^{\rho\sigma} \qquad (1.3) $$

If we choose a Euclidean metric, $g_{\mu\nu} = \delta_{\mu\nu}$, then lowering indices alters nothing, and (1.3) reads

$$ {}^*F_{01} = F_{23} \qquad {}^*F_{23} = F_{01} \quad \text{etc.} $$

It's easy to see that

$$ {}^*({}^*F_{\mu\nu}) = F_{\mu\nu} \qquad (4.17) $$

i.e. that $** = 1$, so $*$ has eigenvalues ± 1 and eigenvectors $F^{\pm}_{\mu\nu} = F_{\mu\nu} \pm {}^*F_{\mu\nu}$, satisfying

$$ {}^*F^{\pm}_{\mu\nu} = \pm\, F^{\pm}_{\mu\nu} \; . \qquad (4.18) $$

$F^{+}_{\mu\nu}$ is called self-dual and $F^{-}_{\mu\nu}$ anti-self-dual. For $F_{\mu\nu}$ either self dual or anti-self-dual, the Bianchi identity (3.11) actually implies the equation of motion (4.11)

$$ D^{\mu *}F_{\mu\nu} = 0 \qquad \Rightarrow \qquad D^{\mu}F_{\mu\nu} = 0 $$

which is thus automatically satisfied. The equations (4.18) are <u>first</u>-order but imply second order equations (4.11). This is a typical situation for many non-linear field theories. The most general solution to (4.18) has been constructed by Atiyah, Hitchin, Drinfeld and Manin [1978]. The solutions are called <u>instantons</u>.

Note that in Minkowski space-time there are no configurations satisfying (4.18); the metric is diag (1-1-1-1) and one finds that $** = -1$, thus $*$ has eigenvalues $\pm i$ and only complex eigenvectors. Since the fields are real, there can be no self- or anti-self-dual fields, and hence no classical instantons in Minkowski space-time.

Lecture 5

For todays lecture, we'll remain in four dimensional Euclidean space-time. There the action S is given by

$$S = \frac{1}{4K^2} \int d^4x \; F^a_{\mu\nu} F^a_{\mu\nu} \qquad (5.1)$$

which is clearly greater than or equal to zero. The quantum theory is determined by a path sum (schematically)

$$\int_{\text{all configurations}} \delta W_\mu \, e^{-S} \qquad (5.2)$$

and indeed stationary (minimum) action solutions give the most important contributions. Of course these correspond to classical solutions.

If we examine S in detail, we can rewrite it as [Belavin et al, 1975]

$$S = \frac{1}{8K^2} \int d^4x (F^a_{\mu\nu} \pm {}^*F^a_{\mu\nu})^2 \mp Q \qquad (5.3)$$

where

$$Q = \frac{1}{4K^2} \int d^4x \; F^a_{\mu\nu} \, {}^*F^a_{\mu\nu} \qquad (5.4)$$

and we have used :

$$({}^*F_{\mu\nu})^2 = (F_{\mu\nu})^2. \qquad (5.5)$$

This form of S is possible only in four dimensions. Thus we have

$$S \geq \pm Q \qquad (5.6)$$

and depending on the sign of Q we take the most stringent inequality to get

$$S \geq |Q|, \qquad (5.7)$$

with equality if and only if

$$F^a_{\mu\nu} = \pm {}^*F^a_{\mu\nu} \qquad (5.8)$$

Thus self-dual solutions really are the most important ones. We have come to this conclusion about the classical solutions by looking at quantum theory.

Q is actually a topological quantum number, and has fixed values in each topological sector of the theory, as we shall now see.

Consider the variation of Q,

$$\delta Q = \frac{1}{2K^2} \int d^4x \; \mathrm{Tr}({}^*F_{\mu\nu} \delta F_{\mu\nu})$$
$$= \frac{1}{K^2} \int d^4x \; \mathrm{Tr}\,({}^*F_{\mu\nu} D_\mu \delta W_\nu) \qquad (5.9)$$

s in (4.10), and integrating by parts we find

$$\delta Q = -\frac{1}{K^2} \int d^4x \; \mathrm{Tr}\,((D_\mu {}^*F_{\mu\nu}) \delta W_\nu)$$
$$+ \frac{1}{K^2} \int d^4x \; \partial_\mu ({}^*F_{\mu\nu} \delta W_\nu) \qquad (5.10)$$

he first term is zero by the Bianchi identity (3.11); the second is surface term. So we could have added Q to the action S without effecting he classical equations of motion. However Q is important in the uantum physics.

In fact the density of Q is a total derivative, which may be seen s follows: consider W_μ as a variable number λ multiplied by a fixed ector field $\hat{W}_\mu$ so that

$$\delta W_\mu = \delta(\hat{W}_\mu \lambda) = \hat{W}_\mu \delta\lambda \qquad (5.11)$$

hen the variation of the density of Q under (5.11) is given by (5.10),

$$\delta\left(\frac{1}{4K^2} \; \mathrm{Tr}(F_{\mu\nu} {}^*F_{\mu\nu})\right)$$
$$= \frac{1}{8K^2} \varepsilon_{\sigma\mu\nu\rho} \partial_\sigma \mathrm{Tr}(\lambda(\partial_\mu \hat{W}_\nu - \partial_\nu \hat{W}_\mu - ie\lambda[\hat{W}_\mu, \hat{W}_\nu]) \hat{W}^\rho \delta\lambda) \qquad (5.12)$$

his defines a differential equation in λ which can be integrated to give

$$\frac{1}{4K^2} \mathrm{Tr}(F_{\mu\nu}{}^*F_{\mu\nu})$$

$$= \int_0^1 d\lambda \frac{\delta}{\delta\lambda} \left(\frac{1}{4K^2} \mathrm{Tr}\, F_{\mu\nu}{}^*F_{\mu\nu}\right)$$

$$= \frac{1}{8K^2} \partial_\sigma (2\epsilon_{\sigma\mu\nu\rho} \mathrm{Tr}(\tfrac{1}{2} (\partial_\mu W_\nu) W_\rho - \frac{ie}{3} W_\mu W_\nu W_\rho)$$

= total derivative

$$\equiv \partial_\mu \eta_\mu, \tag{5.13}$$

where η_μ is called the <u>winding current</u>, and may be expressed as

$$\eta_\mu = \frac{1}{4K^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr} \left(\tfrac{1}{2}(\partial_\rho W_\upsilon) W_\nu - \frac{ie}{3} W_\rho W_\sigma W_\nu\right)$$

$$= \frac{1}{16K^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}(F_{\rho\sigma} W_\nu + \frac{2ie}{3} W_\rho W_\sigma W_\nu) \tag{5.14}$$

It was shown that $\mathrm{Tr}(F^*F)$ is important in the quantum theory because the axial current has an anomaly; [Adler 1969, Bell and Jackiw 1969]

$$\partial_\mu j_\mu^J \sim \mathrm{Tr}(F_{\mu\nu}{}^*F_{\mu\nu})$$

and hence the true conserved current is $j_\mu^J - \eta_\mu$. Unfortunately η_μ is not gauge invariant - under a gauge transformation we expect (and find) that

$$\delta\eta_\mu = \partial_\nu t_{\mu\nu}, \tag{5.15}$$

where $t_{\mu\nu} = -t_{\nu\mu}$, because we know that $\delta(\partial_\mu \eta_\mu) = \delta(\mathrm{Tr}(F^*F)) = 0$ and this is guaranteed by (5.15).

We have seen above that Q is a surface integral:

$$Q = \int d^4x \partial_\mu \eta_\mu$$

$$= \int_{S_3} dS^\mu \eta_\mu \tag{5.16}$$

where the integral is over the three-sphere at infinity. We are interested in finite action configurations (remember $S = |Q|$ for self/anti-self-dual configurations) and we expect that a sufficient condition for this would be

$$R^2 F \to 0 \text{ as } R \to \infty$$

here $R^2 = x_\mu x_\mu$ is the radius squared.

However as in the U(1) theory, we can show that if $F_{\mu\nu} = 0$, hen in a simply connected region of space-time the field is a pure gauge,

$$W_\mu = -\frac{1}{ie} g^{-1} \partial_\mu g, \tag{5.17}$$

ith $g(x)$ a continuous function. In fact we can see this by constructing

$$g(x) = T\exp\left[ie \int_{x_0}^x W_\mu dx_\mu\right] \tag{5.18}$$

hich is path-independent if $F_{\mu\nu}$ vanishes and hence well-defined everywhere. his clearly yields (5.17). If $F_{\mu\nu}$ falls off <u>faster</u> than $1/R^2$, this s still true. More precisely, if we choose a gauge where the radial omponent of W vanishes,

$$x \cdot W = 0, \tag{5.19}$$

hen $g = g(\hat{x}^\mu)$ is independent of R and W_μ is given by (5.17). Substituting his into the expression for η_μ, we get

$$\eta_\mu = 0 + \frac{\epsilon^{\mu\nu\rho\sigma}}{24K^2 e^2} \mathrm{Tr}((g^{-1}\partial_\rho g)(g^{-1}\partial_\sigma g)(g^{-1}\partial_\nu g)) \tag{5.20}$$

Since g depends only on angular variables, $g^{-1}\partial_\rho g$ is purely tangential, so η_μ is orthogonal to the sphere at infinity.

Let us see how Q varies under a small gauge transformation,

$$\delta g = gX, \; X \text{ small} \tag{5.21}$$

This yields

$$\delta(g^{-1}\partial_\mu g) = -g^{-1}\delta g\, g^{-1}\partial_\mu g + g^{-1}\partial_\mu \delta g$$

$$= -Xg^{-1}\partial_\mu g + g^{-1}\partial_\mu g X + \partial_\mu X$$

$$= [g^{-1}\partial_\mu g, X] + \partial_\mu X, \tag{5.22}$$

where we have used $\delta(g^{-1}) = -g^{-1}(\delta g)g^{-1}$. Thus, from (5.20),

$$\delta\eta_\mu = \frac{\epsilon_{\mu\nu\rho\sigma}}{8K^2e^2} \mathrm{Tr}(\delta(g^{-1}\partial_\nu g)g^{-1}\partial_\rho g\, g^{-1}\partial_\sigma g)$$

$$= \frac{\epsilon_{\mu\nu\rho\sigma}}{8K^2e^2}\mathrm{Tr}(\partial_\nu X\, g^{-1}\partial_\rho g\, g^{-1}\partial_\sigma g) \qquad (5.23)$$

where the other term vanishes using the cyclic property of the trace and the antisymmetry of $\epsilon_{\mu\nu\rho\sigma}$. In fact,

$$\delta\eta_\mu = \partial_\nu(\frac{\epsilon^{\mu\nu\rho\sigma}}{8K^2e^2}\ \mathrm{Tr}\ X\, g^{-1}\, \partial_\rho g\, g^{-1}\partial_\sigma g), \qquad (5.24)$$

where the extra derivatives vanish by the antisymmetry of $\epsilon_{\mu\nu\rho\sigma}$, so indeed

$$\delta\eta_\mu = \partial_\nu t_{\mu\nu} \qquad (5.25)$$

and $t_{\mu\nu}$ is antisymmetric, as expected by Eq. (5.15). We save evaluation of the variation of Q itself for the next lecture.

Lecture 6

This lecture we'll discuss some of the global aspects of the construction above. For a general reference on this and the previous lecture, see my review [Olive 1979]. The function g(x) defined through $W_\mu = \frac{i}{e}\, g^{-1}\, \partial_\mu g$ is actually a 'map' (in mathematical terminology) from S_3 to the gauge group G. g(x) winds around the group as x varies over the surface of the sphere at infinity.

Consider a finite gauge transformation h of η_μ built up from infinitesimal ones like X in (5.21). From (5.25), we have for an infinitesimal transformation

$$\delta Q = \int dS_\mu \partial_\nu\, t_{\mu\nu} = 0 \qquad (6.1)$$

by Stokes' theorem, so Q is invariant for any gauge transformation h built up in this way, or for $g \to gh$,

$$Q(g) = Q(gh) \qquad (6.2)$$

Q is thus invariant under any smooth gauge deformation of the configuration, or in other words takes the same value over all spheres which are smoothly deformable to each other, analogous in that sense to the charge in Quantum electrodynamics. It is a topologically conserved quantity, the Pontryagin, or Instanton, or Winding number. Note that since the area of the sphere $S_3 \sim R^3$, we thus expect η to be $\sim \frac{1}{R^3}$ at large R.

Let us briefly introduce the concept of homotopy. If h(x) is constructed as above, we clearly have some function $h(x,\lambda)$, $0 \le x \le 1$, with $h(x,0) = 1$ and $h(x,1) = h(x)$, which is the intermediate function as we build up h(x) from infinitesimal gauge transformations. We say h(x) is homotopic to the identity if it can be obtained in this way, i.e., if there exists such an $h(x,\lambda)$. If h(x) is not homotopic to 1, we can easily show that

Theorem 6: $\qquad Q(gh) = Q(g) + Q(h)$;

here g and h are two maps as above. This theorem has an obvious corollary, namely

Corollary 6.1: If h is homotopic to 1, then Q(h) = 0.

To prove Theorem 6, we can, when neither g nor h are homotopic to 1, smooth out g and h on S_3 to the identity except for a small area for g and a different small area for h. Then the result is obvious. Thus Q is an additive charge. Another corollary is

Corollary 6.2: $Q(g^n) = nQ(g)$

so Q is also quantized.

Finally let us construct explicitly an example of a g(x) for which $Q(g) \neq 0$. Take an SU(2) gauge theory, and let

$$g(x) = y_0 + i\,\vec{y}\cdot\vec{\sigma}$$

where $\vec{\sigma}$ are the Pauli matrices and $y_\mu = x_\mu/\sqrt{x^2}$. We find

$$\det g = y_0^2 + y_1^2 + y_2^2 + y_3^2 = 1$$

and g is clearly unitary, so g is in SU(2). This defines a map - in fact SU(2) is topologically just S_3 which is why it was so easy - we are mapping one sphere onto another. Q merely counts how many times g(x) wraps around the group (a second sphere) as x wraps around the sphere at infinity.

Because of the inequality $A \geq |Q|$, the lowest winding number configurations are evidently the most interesting, and as mentioned above the self- or anti-self dual ones are the most important of these. In fact all the self- or anti-self-dual instanton solutions can (in principle) be constructed using the ADHM construction [Atiyah, Drinfeld, Hitchin, Manin, 1978]. They are not however very physical - living in Euclidean space-time only. We'll see later how analogous solutions for magnetic monopoles do exist in Minkowski space-time, but these are slightly different in that one component of $F_{\mu\nu}$, coming from the 'higgs' field, tends to a constant value at infinity instead of falling off like $\frac{1}{R^2}$. The construction of these has been partially achieved by Nahm [Nahm, 1982] and others, but is still not complete. As we shall see, one such

construction relates to the inverse scattering method [lectures 30,31].

Lecture 7

We now turn to the discussion of the properties of Lie groups, and begin with a quick review of some essential features:

a) The generators T_α of the group close under commutation.

$$[T_\alpha, T_\beta] = i f^\gamma_{\alpha\beta} T_\gamma \tag{7.1}$$

$\alpha, \beta, \gamma = 1, 2, \ldots, \dim G$

b) The adjoint representation is defined by the generators themselves, as in the previous lecture, through the relation

$$g T_\alpha g^{-1} = T_\beta\, d_{\beta\alpha}(g) \ . \tag{7.2}$$

The dim G x dim G matrices $d_{\beta\alpha}(g)$ form the adjoint representation, which is real.

Taking g to be an element infinitesimally close to the identity, i.e. with

$$g = 1 + i\epsilon^\gamma T_\gamma \tag{7.3}$$

we observe from (7.2) that

$$d_{\alpha\beta}(T_\gamma) = i f^\alpha_{\gamma\beta} \ . \tag{7.4}$$

c) The Killing form $g_{\alpha\beta}$ is defined in any representation as

$$g_{\alpha\beta} = \mathrm{Tr}(D(T_\alpha T_\beta)) \ . \tag{7.5}$$

In the adjoint representation we have from (7.4)

$$g_{\alpha\beta} = - f^i_{\alpha j}\ f^j_{\beta i} \ . \tag{7.6}$$

It is clearly symmetric. If G is semisimple it can be shown that $g_{\alpha\beta}$ is the same in all representation up to an overall constant. The proof will be given in the next lecture. Hence for semisimple groups the Killing form is given by (7.6) in terms of the structure constants of the group.

d) If we define

$$f_{\alpha\beta\delta} = f^\gamma_{\alpha\beta}\, g_{\gamma\delta} \tag{7.7}$$

then

$$\begin{aligned} f_{\alpha\beta\delta} &= f^\gamma_{\alpha\beta}\, \mathrm{Tr}(d(T_\gamma T_\delta)) \\ &= -i\mathrm{Tr}(d([T_\alpha, T_\beta] T_\delta)) \ . \end{aligned} \tag{7.8}$$

Hence for simple groups $f_{\alpha\beta\delta}$ is totally antisymmetric with respect to the indices α, β and δ .

If the group is <u>compact</u> the matrix $[g]$ is positive definite and consequently by a suitable choice of the basis we can make

$$g_{\alpha\beta} = \delta_{\alpha\beta} \ . \tag{7.9}$$

In such a basis we have

$$[T_\alpha, T_\beta] = i f_{\alpha\beta\gamma} T_\gamma \ . \tag{7.10}$$

Thus the structure constants can be made to be totally antisymmetric for the compact simple Lie groups. For all the simple Lie groups of physical interest they are indeed chosen to be so.

<u>Examples</u> of gauge groups of physical interest:

1) Maxwell's electromagnetic theory: The gauge group is U(1) (see lecture 1) and the generator is Q (electric charge); it is trivially abelian as $[Q,Q] = 0$. The photon belongs to the adjoint representation and thus carries no electric charge. Electrons belong to a representation where $Q \neq 0$ and thus can carry charge. In the U(1) theory, clearly $\mathrm{Tr}(Q^2) = 0$ for Q in the adjoint representation (since all structure constants vanish); thus we cannot use the trace to construct invariant quantities from the gauge fields. However, for other representations $\mathrm{Tr}(Q^2)$ need not be equal to zero. On the other hand for semisimple groups we shall see that the trace is essentially independent of representation.

2) Electro-weak theory: The gauge group is SU(2) x U(1) (Weinberg 1967, Salam 1968), and the generators are T_1, T_2, T_3 (weak isospin) and Y (weak hypercharge) for SU(2) and U(1) respectively. The electric charge Q is defined to be $Q = T_3 + Y$. The relevant commutation relations are

$$[T_i, T_j] = i\epsilon_{ijk} T_k ,$$

$$[T_i, Y] = 0 ,$$

$$[Y, Y] = 0 .$$

Since U(1) is an invariant subgroup, the gauge group is not semi-simple.

3) Add Nuclear forces (the Standard Model): The gauge group is SU(3) x SU(2) x U(1), where SU(3) is associated with the strong (colour) interaction and SU(2) x U(1) is from example 2). Notice that SU(3) x SU(2) is semisimple but not simple. There is a theorem that the most general compact Lie group is a direct product of U(1)'s and simple groups with the identification of some finite discrete subgroups of the factors. SU(3) x SU(2) x U(1) is a typical example here, actually being [SU(3) x SU(2) x U(1)]/Z_6. Z_6 being the cyclic group of order 6.

4) Unified theories: For aesthetic purposes it is desirable that the gauge group discussed above be a subgroup of a compact simple group which can be taken as the starting unified theory, subsequently breaking down to SU(3) x SU(2) x U(1). The simplest example is SU(5) (Georgi-Glashow, 1974). Alternatively one can have SO(10), E_6, E_7 or E_8 each of which contains SU(5). Indeed, this sequence is nested, $SU(5) \subset SO(10) \subset E_6 \subset E_7 \subset E_8$, suggesting that E_8 could be a unified theory broken in sequential fashion into the Standard Model.

We see that simple groups furnish both the building blocks of unified theories and the ultimate goal of a fully unified theory.

We now proceed to study their classification and enumeration.

We turn to the more general discussion on the Lie algebra g associated with the Lie group G. An algebra is, roughly speaking, a system defined by the commutation relations. First we define some important concepts.

i) An Abelian algebra is one where all the structure constants $f^{\gamma}_{\alpha\beta} = 0$, i.e., all the elements commute among themselves.

ii) A subalgebra h of g is a subset, $h \subset g$, such that it is closed under commutation, i.e.

$$[h, h] \subset h . \tag{7.11}$$

iii) h is an invariant subalgebra of g when

$$[h, g] \subset h . \tag{7.12}$$

iv) An algebra is semisimple if it has no abelian invariant subalgebra, and it is simple if it has no invariant subalgebra.

Theorem 7.1: (no proof) A Lie algebra is semisimple if the Killing form evaluated in the adjoint representation is non-singular, i.e.

$$\det(Tr(d(T_\alpha T_\beta)) \neq 0 . \tag{7.13}$$

v) A semisimple Lie algebra is compact if the Killing form is positive definite.

Theorem 7.2: A compact semisimple Lie algebra g with a subalgebra h can be written as

$$g = h + P , \tag{7.14a}$$

where

$$[h, P] \subset P . \tag{7.14b}$$

Proof: Define P to be the orthogonal complement of h, i.e.,

$$Tr(h P) = 0 .$$

Then, using the cyclic property of the trace,

$$\mathrm{Tr}(\underline{h}\ [\underline{h},\underline{P}]) = \mathrm{Tr}([\underline{h},\underline{h}]\underline{P})$$
$$= \mathrm{Tr}(\underline{h}\ \underline{P}) = 0 \quad .$$

Hence $\quad [\underline{h},\underline{P}] \subset \underline{P}$. Q.E.D.

Theorem 7.3: A compact semisimple Lie algebra is a direct sum of mutually commuting simple subalgebras.

Proof: If $\underline{g}$ is not simple it has an invariant subalgebra $\underline{h}$ where $[\underline{h},\underline{g}] \subset \underline{h}$. On the other hand from Theorem 7.2 we have $[\underline{h},\underline{P}] \subset \underline{P}$. Hence we must have $[\underline{h},\underline{P}] = 0$.

We now need to show that $\underline{P}$ is a subalgebra. That follows from

$$\mathrm{Tr}([\underline{P},\underline{P}]\ \underline{h}) = \mathrm{Tr}(\underline{P}\ [\underline{P},\underline{h}]) = 0 \quad .$$

Hence $[\underline{P},\underline{P}] \subset \underline{P}$. Q.E.D.

If P is not simple repeat the process.

:

Lecture 8

We continue with some points related to the compact Lie groups, their algebra and their representations. Detailed description of representation theory will occur late in the course. We assume the reader is familiar with elementary features of finite groups, as might be discussed for example in [Hamermesh (1962)].

It was mentioned in the last lecture that for a simple Lie algebra the Killing form $\mathrm{Tr}(D(T_a)D(T_b))$ is the same for all representations except for an overall factor. We prove it here.

Recall that the adjoint representation matrix $d_{ba}(f)$ is defined from the relation

$$f\ T_a\ f^{-1} = T_b\ d_{ba}\ (f) \quad \text{for all } f \in G.$$

If G is simple the matrix d is irreducible, i.e., the vector space of dim G cannot be decomposed into two or more subspaces with the property that under an arbitrary group operation elements of the particular subspace would mix only between themselves.

Consider a Killing form $\bar{g}$ as a matrix

$$\bar{g}_{ab} = \mathrm{Tr}\ (D(T_a)D(T_b)) \ .$$

Then $\bar{g}_{ab} = \mathrm{Tr}\ (D(f\ T_a\ f^{-1}\ f\ T_b\ f^{-1}))$ (f an arbitrary element of G)

$$= \mathrm{Tr}(D(T_{a'}\ d_{a'a}\ (f)\ T_{b'}\ d_{b'b}(f))$$
$$= d^T_{a'a}(f)\ \mathrm{Tr}(D(T_{a'}T_{b'}))d_{b'b}(f).$$

Hence
$$\bar{g} = d^T\ \bar{g}\ d \ , \tag{8.1}$$

i.e., $\bar{g}$ is an invariant tensor.

(8.1) holds for all representations and, in particular, for the adjoint one, where the Killing form is denoted by g_{ab}.

Now if the algebra is semisimple, i.e., if $\det g \neq 0$ and g^{-1} exists, then

$$g^{-1}\bar{g} = (d^T g\, d)^{-1}\, d^T\, \bar{g}\, d = d^{-1}\, g^{-1}\, \bar{g}\, d \quad .$$

Hence

$$d(f)\, g^{-1}\, \bar{g} = g^{-1}\, \bar{g}\, d(f) \qquad \text{for all } f \in G \, . \tag{8.2}$$

For simple algebras, $d(f)$ is irreducible, and we can use Schur's lemma which says that any quantity that commutes with all the elements of a group in an irreducible representation must be a constant multiple of the unit matrix. Thus

$$g^{-1}\, \bar{g} = \lambda\, \mathbb{1} \ ,$$

$$\text{i.e., } \bar{g} = \lambda\, g \, . \tag{8.3}$$

This establishes that for a simple group all matrices formed from the trace of two generators are proportional, the proportionality depending only on the representation.

Equipped with the basic ingredients of Lie groups and their algebra we are now in a position to start the theory of their classification and representations. The basic aspects of representation theory can be recaptured from the well-known angular momentum algebra, i.e., the SO(3) Lie algebra

$$[T_i, T_j] = i\, \varepsilon_{ijk}\, T_k \ . \tag{8.4}$$

Since the generators are hermitian we can always choose one of them, traditionally T_3, to be diagonal. Writing

$$T^{\pm} = T_1 \pm i\, T_2 \quad , \tag{8.5}$$

we have

$$[T_3, T^{\pm}] = \pm\, T^{\pm} \tag{8.6a}$$

$$[T^{+}, T^{-}] = 2T_3 \ . \tag{8.6b}$$

(8.6) typically characterizes the SO(3) algebra to be of rank 1 (since none of the generators commute with any other), dimension 3 (the number of linearly independent generators, i.e., T_3, T_+ and T_-) and having roots ± 1 (if (8.6a) is considered as eigenvalue equations).

For the representation of SO(3) one has to further consider the operator $T_1^2 + T_2^2 + T_3^2$ which commute with all the generators. Consequently the basis of the vector space where the matrices representing the group elements act can be chosen to be the simultaneous eigenstates of this operator along with T_3. Thus the basis $|j, m\rangle$ is labelled by spin j (the eigenvalue of $T_1^2 + T_2^2 + T_3^2$ is $j(j+1)$, j being half-integer ≥ 0) and weight m (the eigenvalue of T_3 that can take values from j to $-j$ in integral steps, a feature connected with the roots being ± 1). For a general compact simple algebra the representation is a generalization of this concept.

The first step is to find the Cartan subalgebra which is formed out of the maximal set of linearly independent generators that commute among themselves. One starts with an arbitrary one H_1 ($= T_3$ in SO(3) case) and looks for H_2 which is linearly independent and commutes with H_1. Then look for another, H_3, linearly independent of H_1 and H_2 and commuting with both of them. The process is continued till one exhausts the set. In this way one finds r commuting generators: r is the rank of G and is the dimension of the Cartan subalgebra. By choosing suitable linear combinations one can make them orthonormal, i.e.,

$$\mathrm{Tr}(H_i H_j) = \delta_{ij}, \quad i,j = 1, \ldots\ldots\, r. \tag{8.7}$$

[Note that $f^{-1}H_1 f$, $f^{-1}H_2 f$, ... $f^{-1}H_r f$, where f is an arbitrary element of G, also form an orthonormal basis of the Cartan subalgebra and hence this conjugated set is an equally good choice. In fact, it can be shown that all possible choices of the Cartan subalgebra are related to each other by such conjugation.] By taking the H_i's to be hermitian one can always simultaneously diagonalize them.

The next stage is the construction of the step operators (generalization of $T_1 \pm i T_2$ in the SO(3) case) out of the remaining dim G − rank G generators T_m. One can always make them orthonormal, i.e.,

$$\mathrm{Tr}\,(T_i T_j) = \delta_{ij}$$
$$\mathrm{Tr}\,(H_i T_m) = 0 \tag{8.8}$$

The last equation follows from Theorem (7.2).

Notice that by construction the T_m's form an orthogonal complement to the Cartan subalgebra and hence also according to Theorem (7.2)

$$[H_i, T_m] = i\, f_{imn}\, T_n\ , \quad m, n = 1, \ldots \dim G - \mathrm{rank}\, G\ , \tag{8.9}$$

where the f_{imn}, according to (7.8), are real and antisymmetric in m and n. This means that if we write

$$[H_i, T_m] = (h^i)_{mn}\, T_n\ , \tag{8.10}$$

the matrices h^i (i = 1, ... r) of dimension (dim G − rank G) x (dim G − rank G) are hermitian. Consequently the h^i's can be diagonalized by unitary transformations without affecting the H_i's but changing the T_m's to a new basis, E_α's. The final algebraic form after such unitary transformations is

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{8.11}$$

The dim G − rank G vectors α are the roots, and have r = rank G components. The i^{th} component α_i is the eigenvalue of H^i. The dim G − rank G number of generators E_α are called the step operators. They are complex linear combinations of the original hermitian T_m's.

Lecture 9

In the last lecture we observed that for any given compact and simple Lie algebra one could, by a suitable choice of basis, split the generators into two sets, namely, r mutually commuting H_i's (forming the Cartan Subalgebra) and dim G-r E_α's (step operators). The root-system of the algebra is defined by Eq. (8.11), with dim G-r roots α_i which are the eigenvalues of the (dim G-r) x (dim G-r) hermitian matrices $(h_i)_{mn} \equiv i\, f_{imn}$, according to Eq. (8.9).

We now notice that the matrices h_i, apart from being hermitian, have their off-diagonal elements purely imaginary, i.e.,

$$h_i^\dagger = h_i \quad \text{and } h_i^* = -h_i\ . \tag{9.1}$$

As a consequence, if V is an eigenstate of h_i with the eigenvalue α_i which is real, i.e.,

$$h_i V = \alpha_i V\ , \tag{9.2}$$

then $$-h_i V^* = h_i^* V^* = \alpha_i V^*\ . \tag{9.3}$$

Thus $-\alpha_i$ is also an eigenvalue, and we find that roots always come in pairs $\pm\alpha$.

To visualize the root system let us consider two simple examples.

Example 1: SU(2): This has dimension 3, rank 1 and the roots are ± 1. These can be easily seen from the algebra of Pauli matrices with $H = \sigma_3$, $E_\pm = \sigma_1 \pm i\sigma_2$.

Example 2: SU(3): By definition this is the group of all 3 x 3 unitary matrices with determinant 1. Writing an arbitrary element of SU(3) as $\exp\,[i\vec{\theta}\cdot\vec{T}]$, we find that unitarity would imply the generators T_i's to be hermitian and the determinant 1 would imply $\mathrm{Tr}T_i = 0$, (since $\det(\exp A) = \exp(\mathrm{Tr}A)$).

The number of linearly independent matrices to express any arbitrary traceless hermitian 3 x 3 matrix is 9 − 1 = 8. Hence the dimension of the algebra is 8.

iii) We further notice from (9.4) that

$$[H_i, [E_\alpha, E_\beta]] = (\alpha_i + \beta_i)\, [E_\alpha, E_\beta] \ . \tag{9.8}$$

By definition $[E_\alpha, E_\beta]$ is an element of the Lie algebra. Therefore either $\alpha_i + \beta_i$ is a root associated with the step operator $E_{\alpha+\beta} = c[E_\alpha, E_\beta]$ (c a constant) or $\alpha_i + \beta_i$ is not a root and $[E_\alpha, E_\beta] = 0$.

The basis in which $Tr(H_i H_j) = \delta_{ij}$, $Tr(H_i E_\alpha) = 0$, and $Tr(E_\alpha E_\beta) = \delta_{\alpha+\beta,0}$ is called the <u>Cartan-Weyl basis</u>.

Having explored all the features obtainable from the basic definition of the Cartan subalgebra and the step operators let us focus our attention on any one root α. We shall show that there is an SU(2) subalgebra associated with each root α.

Since α a root implies $-\alpha$ also a root, corresponding to a step operator E_α there will be a step operator $E_{-\alpha}$. Now

$$[H_i, [E_\alpha, E_{-\alpha}]] = (\alpha_i - \alpha_i)\, [E_\alpha, E_{-\alpha}] = 0 \ . \tag{9.9}$$

Hence $[E_\alpha, E_{-\alpha}]$ must belong to the Cartan subalgebra and therefore we can write

$$[E_\alpha, E_{-\alpha}] = x_i H_i \ .$$

Since we have chosen $Tr(H_i H_j) = \delta_{ij}$, we find

$$\begin{aligned} x_i &= Tr(H_i x_j H_j) = Tr(H_i [E_\alpha, E_{-\alpha}]) \\ &= Tr([H_i, E_\alpha]\, E_{-\alpha}) = \alpha_i\, Tr(E_\alpha E_{-\alpha}) \\ &= \alpha_i \end{aligned}$$

since one can rescale so that $Tr(E_\alpha E_{-\alpha}) = 1$, as in the Cartan-Weyl basis.

Thus we obtain

$$[E_\alpha, E_{-\alpha}] = \alpha\cdot H \ . \tag{9.10}$$

We now notice that E_α, $E_{-\alpha}$ and $\alpha\cdot H$ close among themselves, since

$$[\alpha\cdot H, E_\alpha] = \alpha^2 E_\alpha \tag{9.11}$$

$$[\alpha\cdot H, E_{-\alpha}] = -\alpha^2 E_{-\alpha} \ . \tag{9.12}$$

Clearly we have formed an SU(2) subalgebra. Such SU(2) subalgebras in compact simple Lie algebras are the generalizations of the I-spin, U-spin and V-spin concepts in SU(3). Fitting such subalgebras together into the large group G is non-trivial.

Using the SU(2) subalgebras we will now obtain a powerful result which will be extensively used during this course.

Define $$H_\alpha = 2\,\frac{\alpha\cdot H}{\alpha^2} \ . \tag{9.13}$$

Then $$[H_\alpha, E_{\pm\alpha}] = \pm 2E_{\pm\alpha} \ . \tag{9.14}$$

It is clear from (9.14) and (9.10) that H_α is like $2T_3$ in the SU(2) subalgebra formed by H_α, $E_{\pm\alpha}$. We already know T_3 has ½-integer eigenvalues, and as a consequence <u>H_α must have integer eigenvalues.</u>

Now,

$$[H_\alpha, E_\beta] = 2\,\frac{\beta\cdot\alpha}{\alpha^2}\, E_\beta \ . \tag{9.15}$$

Let $|m\rangle$ be an eigenstate of H_α with an integer eigenvalue m. Then from (9.15) we will have

$$H_\alpha (E_\beta |m\rangle) = \left(m + \frac{2\beta\cdot\alpha}{\alpha^2}\right) E_\beta |m\rangle \ .$$

So $E_\beta|m\rangle$ is an eigenstate of H_α with the eigenvalue $m + \frac{2\beta\cdot\alpha}{\alpha^2}$ and that must be an integer. Therefore we have the central result

$$\frac{2\beta\cdot\alpha}{\alpha^2} = \text{integer for any two roots } \alpha \text{ and } \beta. \tag{9.16}$$

In fact the restriction can be shown to be stronger than (9.16). This follows from the Schwartz inequality

$$\alpha\cdot\beta = ||\alpha||\ ||\beta||\ \cos\theta \le ||\alpha||\ ||\beta|| \ .$$

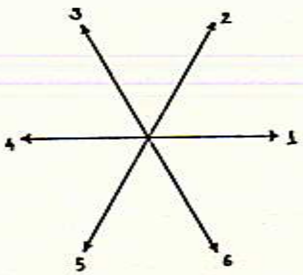
To express the Cartan subalgebra, i.e., to write an arbitrary diagonal 3 x 3 real traceless matrix we need two linearly independent diagonal matrices. Hence the rank of SU(3) is 2.

Therefore there are 8 - 2 = 6 roots of SU(3). Usually one takes the generators of the Cartan subalgebra to be T_3 (isospin) and T_8 (hypercharge). The first pair of step operators are $T_1 \pm iT_2$ which have their roots with components (1,0) and (-1,0) from the algebra

$$[T_3, T_1 \pm iT_2] = \pm (T_1 \pm iT_2) \ ,$$

$$[T_8, T_1 \pm iT_2] = 0 \ .$$

The other roots are obtained by taking the step operators for U-spin ($T_6 \pm iT_7$) and V-spin ($T_4 \pm iT_5$). Notice that all the roots are of equal length and the angle between each adjacent pair of roots is $\pi/3$. We can summarize our information in a <u>root diagram</u> for SU(3):



As we see from the preceding example, the root diagrams are highly regular. It is this feature that helps to classify the associated Lie algebras.

Let us now see how much information we can obtain about the roots and the remaining part of the algebra.

i)
$$[H_i, E_\alpha E_\beta]$$
$$= [H_i, E_\alpha]E_\beta + E_\alpha [H_i, E_\beta]$$
$$= (\alpha_i + \beta_i) E_\alpha E_\beta \qquad \text{(from 8.11)} \ . \tag{9.4}$$

Taking trace of both sides we have, since $Tr[A,B] = 0$,

$$(\alpha_i + \beta_i) \, Tr \, (E_\alpha E_\beta) = 0 \ . \tag{9.5}$$

Therefore, $\quad Tr(E_\alpha E_\beta) = 0 \quad$ unless $\quad \alpha_i + \beta_i = 0$. (9.6)

So we observe that two step operators are orthogonal unless they have equal and opposite roots. We now observe that E_α is orthogonal to itself and so if it is orthogonal to all others the determinant of the Killing form would be zero. But this cannot happen if the algebra is simple. Therefore corresponding to every root α there will be another root $-\alpha$ and $Tr(E_\alpha E_{-\alpha}) \neq 0$. This establishes by a different method that roots come in pairs of opposite sign.

ii) Next we notice that because the H's commute

$$[H_i, H_j E_\alpha] = \alpha_i \, H_j E_\alpha \ .$$

Taking the trace of both sides we obtain

$$\alpha_i Tr(H_j E_\alpha) = 0 \ .$$

Since this is true for all α_i, we have

$$Tr(H_j E_\alpha) = 0 \ . \tag{9.7}$$

This tells us that the H's are orthogonal to all the step operators.

Therefore $$4\frac{\beta\cdot\alpha}{\alpha^2}\,\frac{\beta\cdot\alpha}{\beta^2} = mn = 4\cos^2\theta \le 4\,, \tag{9.17}$$

where m,n are integers. Hence the product of any two allowed integer values $\frac{2\alpha\cdot\beta}{\alpha^2}$ and $\frac{2\alpha\cdot\beta}{\beta^2}$ must lie between 0 and 4. The zero value obviously holds only for $\alpha\perp\beta$. All the possible values of $2\frac{\alpha\cdot\beta}{\alpha^2}$ and $2\frac{\alpha\cdot\beta}{\beta^2}$ are tabulated below. For α not parallel to β.

$2\frac{\alpha\cdot\beta}{\alpha^2}$	$\frac{2\alpha\cdot\beta}{\beta^2}$	$4\frac{(\alpha\cdot\beta)^2}{\alpha^2\beta^2}$
0	0	0
±1	±1	1
±1	±2	2
±1	±3	3

(9.18)

and interchanges.

When $\alpha\|\beta$, the possible values are:

$$\frac{2\alpha\cdot\beta}{\alpha^2} = \pm 2 \qquad \frac{2\alpha\cdot\beta}{\beta^2} = \pm 2$$

$$\text{or}\quad \frac{2\alpha\cdot\beta}{\alpha^2} = \pm 1 \qquad \frac{2\alpha\cdot\beta}{\beta^2} = \pm 4 \tag{9.19}$$

$$\text{or}\quad \frac{2\alpha\cdot\beta}{\alpha^2} = \pm 4 \qquad \frac{2\alpha\cdot\beta}{\beta^2} = \pm 1$$

The first set of (9.19) is obvious, i.e., a root is trivially parallel to itself and for every root there is an equal and opposite root — a fact already established earlier. The next two sets imply that either $\alpha = \pm 2\beta$ or $\beta = \pm 2\alpha$. However these two are ruled out as a consequence of (9.8). According to (9.8) $\pm 2\alpha$ could be the allowed roots if only there exist step operators $[E_\alpha, E_\alpha]$ and $[E_{-\alpha}, E_{-\alpha}]$. But they are obviously zero and cannot thus be step operators.

We will now see that (9.18) and the first part of (9.19) put severe restrictions on the lengths of the roots as well as on the angles between them.

That follows from the relation

$$\left(\frac{2\alpha\cdot\beta}{\beta^2}\right)\Big/\left(\frac{2\alpha\cdot\beta}{\alpha^2}\right) = \frac{\alpha^2}{\beta^2}\ .$$

Consequently α^2/β^2 can take values 1, 2, 3, $\frac{1}{2}$ and $\frac{1}{3}$ only unless $\alpha\cdot\beta = 0$. There is only one simple Lie group where a root can be three times longer than another and that is the exceptional group G_2. In most of the simple Lie groups the roots are of equal length (the exceptions being SO(2r + 1), Sp(2r), F_4). The restrictions on the angles arise from the fact that $2\alpha\cdot\beta/\beta^2 = 2\frac{\|\alpha\|}{\|\beta\|}\cos\theta$ = integer.

Another consequence of the SU(2) subalgebra is the <u>Weyl group</u> (or the Kaleidoscope group) symmetry of the root system. This group is formed by certain reflection operators in the root space. We now discuss it in detail.

$$\text{Define}\quad F_{\pm\alpha} = \sqrt{\frac{2}{\alpha^2}}\,E_{\pm\alpha}\ . \tag{9.20}$$

Then $$[F_\alpha, F_{-\alpha}] = H_\alpha$$

and $$[H_\alpha, F_{\pm\alpha}] = \pm 2\,F_{\pm\alpha}\,. \tag{9.21}$$

Notice that this is exactly the algebra of T_+, T_- and $2T_3$ of the angular momentum generators. This suggests we define

$$T_2(\alpha) = \frac{F_\alpha - F_{-\alpha}}{2i} \tag{9.22}$$

and consider a rotation by π about the 2-axis. This would reverse the 3-axis, i.e.,

$$e^{i\pi T_2(\alpha)}\,T_3\,e^{-i\pi T_2(\alpha)} = -T_3\ .$$

or

$$e^{i\pi T_2(\alpha)}\,\alpha\cdot H\,e^{-i\pi T_2(\alpha)} = -\alpha\cdot H\ . \tag{9.23}$$

Consider now a vector $x \perp \alpha$;

$$[x\cdot H,\ E_{\pm\alpha}] = \pm\alpha\cdot x\ E_{\pm\alpha} = 0 \ ,$$

so

$$e^{i\pi T_2(\alpha)}\ x\cdot H\ e^{-i\pi T_2(\alpha)} = x\cdot H \tag{9.24}$$

So we notice that $\alpha\cdot H$, which is a projection of H on α, is reversed under the rotation while $x\cdot H$ remains invariant. Such a transformation is a mirror reflection. In general for any vector y we will have

$$e^{i\pi T_2(\alpha)}\ y\cdot H\ e^{-i\pi T_2(\alpha)} = (y - \frac{2\alpha\cdot y}{\alpha^2}\alpha)\cdot H \ . \tag{9.25}$$

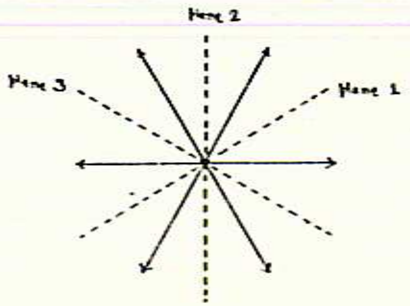
In the root space we can define a reflection operator $\sigma_\alpha(y)$ which takes a vector y to its mirror image, the mirror being perpendicular to the root α:

$$\sigma_\alpha(y) = y - \frac{2\alpha\cdot y}{\alpha^2}\alpha \ .$$

Such reflections in the root space are called <u>Weyl reflections</u>. (9.25) now reads

$$e^{i\pi T_2(\alpha)}\ y\cdot H\ e^{-i\pi T_2(\alpha)} = \sigma_\alpha(y)\cdot H \ .$$

In SU(2) there is only one such mirror plane and under Weyl reflection the two roots just interchange. In SU(3) there are three such reflection planes,



Symmetries of the root diagram under reflection on these planes are clear from the diagram. We will show in the next lecture that root diagrams for all compact and simple Lie algebras are <u>invariant under Weyl reflections</u>.

<u>Lecture 10</u>

In this lecture we shall establish that root diagrams are invariant under Weyl reflection and form a complete description of gauge groups. Recall that

$$S_\alpha(x\cdot H)S_\alpha^{-1} = \sigma_\alpha(x)H \ . \tag{10.1}$$

where $S_\alpha = e^{i\pi T_2(\alpha)}$.

We now see that

$$[x\cdot H,\ E_\beta] = \beta\cdot x\ E_\beta \tag{10.2}$$

and so if we denote the transformed step operator by E'_β, i.e.,

$$E'_\beta = S_\alpha E_\beta S_\alpha^{-1} \tag{10.3}$$

then from (10.3) we have

$$[\sigma_\alpha(x)\cdot H,\ E'_\beta] = \beta\cdot x\ E'_\beta \ .$$

Let us denote $\sigma_\alpha(x)$ by y. Clearly, since $\sigma^2 = 1$ (the reflection group consists of two elements, 1 and σ, with $\sigma^2 = 1$) we have

$$x = \sigma_\alpha(y) \ .$$

Hence $\beta\cdot x = \beta\cdot\sigma_\alpha(y) = \sigma_\alpha(\beta)\cdot y$, as can be easily seen by a graphical representation. Thus

$$[y\cdot H,\ E'_\beta] = \sigma_\alpha(\beta)\cdot y\ E'_\beta$$

and that means

$$[H_i,\ E'_\beta] = \sigma_\alpha(\beta)\ E'_\beta \ . \tag{10.4}$$

(10.4) implies that <u>if β is a root so is $\sigma_\alpha(\beta)$</u> with the associated step operator being $E_{\sigma_\alpha(\beta)}$ which is proportional to

$$e^{i\pi T_2(\alpha)}\ E_\beta e^{-i\pi T_2(\alpha)} \tag{10.5}$$

Thus by Weyl reflections one root can give rise to several roots. However in general one cannot get all the roots from one. The reason is that the Weyl reflected roots have the same length as the original and hence one cannot generate roots which are of different lengths (as in G_2, $SO(2r+1)$, $Sp(2r)$, F_4).

So we observe that the root-diagram is a set of vectors in a Euclidean space of dimension equal to the rank of the algebra and having the following properties:

(1) Number of roots = dim G − rank G .

(2) If α and β are roots, then $\frac{2\alpha\cdot\beta}{\alpha^2}$ is an integer.

(3) If α and β are roots, then $\sigma_\alpha(\beta)$ is a root.

(4) If α and β are roots and $\beta = n\alpha$, then $n = \pm 1$.

The first property reflects the fact that the roots are nondegenerate, i.e., corresponding to each step operator there is a distinct root. This statement, which is not proven here, establishes (1); the other statements have been shown earlier.

To every simple compact Lie algebra we have thus constructed a root diagram with properties (1)-(4). Conversely, it can also be shown that there exists a unique compact simple Lie algebra (up to isomorphisms) for each set of roots that satisfy prop. (1)-(4). In other words, classifying the root systems is equivalent to classifying the Lie algebra. Because of the finite number of possibilities of angles between the roots and ratios of their lengths, the root systems follow a regular pattern and the classification is based on the studies of such polytopes. (This process is equivalent to the classification of crystal structures in any number of dimensions).

As a simple example, we will classify all 2-dimensional root systems and hence all rank 2 simple Lie algebras.

Consider two arbitrary distinct roots α and β. Since $-\alpha$ is also a root we can, if $\alpha\cdot\beta < 0$, interchange α and $-\alpha$ and hence without any loss of generality have

$$\alpha\cdot\beta \geq 0$$
$$\beta^2 \geq \alpha^2 .$$

So now the possibilities are

i) $\alpha\cdot\beta = 0$. No information about the length (the algebra is not simple).

ii) $\frac{2\alpha\cdot\beta}{\beta^2} = 1 = \frac{2\alpha\cdot\beta}{\alpha^2}$

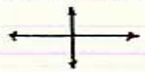
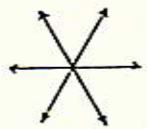
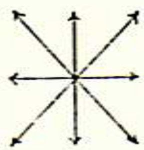
In such a case $||\alpha|| = ||\beta||$ and $\cos\theta = \frac{1}{2}$, i.e. $\theta = \pi/3$

iii) $\frac{2\alpha\cdot\beta}{\beta^2} = 1$ and $\frac{2\alpha\cdot\beta}{\alpha^2} = 2$

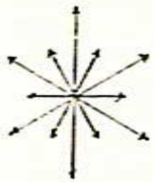
In such a case $||\beta|| = \sqrt{2}\,||\alpha||$ and $\cos\theta = \frac{1}{\sqrt{2}}$, i.e., $\theta = \pi/4$

iv) $\frac{2\alpha\cdot\beta}{\beta^2} = 1$, and $\frac{2\alpha\cdot\beta}{\alpha^2} = 3$

In such a case $||\beta|| = \sqrt{3}\,||\alpha||$ and $\cos\theta = \frac{\sqrt{3}}{2}$, i.e., $\theta = \pi/6$. The possible diagrams are as follows

	root diagrams	group names
1)		$D_2 \sim SO(4)$ (not simple) $\sim SU(2) \times SU(2)$
2)		$A_2 \sim SU(3)$ $\lvert\beta\rvert = \lvert\alpha\rvert$
3)		$B_2 \sim SO(5)$, $C_2 \sim Sp(4)$ $\lvert\beta\rvert = \sqrt{2}\lvert\alpha\rvert$

root diagrams	group names

4) 

$G_2 \quad |\alpha| = \sqrt{3}\,|\beta|$

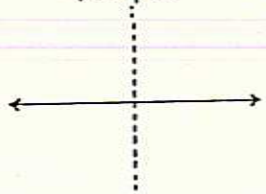
The notations D_2, A_2, B_2, C_2 are familiar to mathematicians, while SO(4), SU(3), SO(5) and Sp(4) are more familiar to physicists.

Lecture 11

The Weyl Group: We now return to the Weyl-reflection symmetries of the root systems of simple and compact Lie algebras. We have proven earlier that reflections of all the roots on the hyperplane perpendicular to any arbitrarily chosen root would also be roots. Now instead of choosing one such reflection hyperplane one can consider all possible hyperplanes and reflections on them in a sequential way. Obviously the root-system remains invariant under all possible sequences of reflections. Therefore all these reflection operators form a group which is called the Weyl group. Since the root-system is finite this group must be finite. Further, according to (10.1), every reflection operation is associated with the gauge transformation and hence so will be any sequence of such operations. Therefore all the elements of the Weyl group can be achieved by gauge-transformations.

We will identify the Weyl groups in two particular root-systems, namely for SU(2) and SU(n).

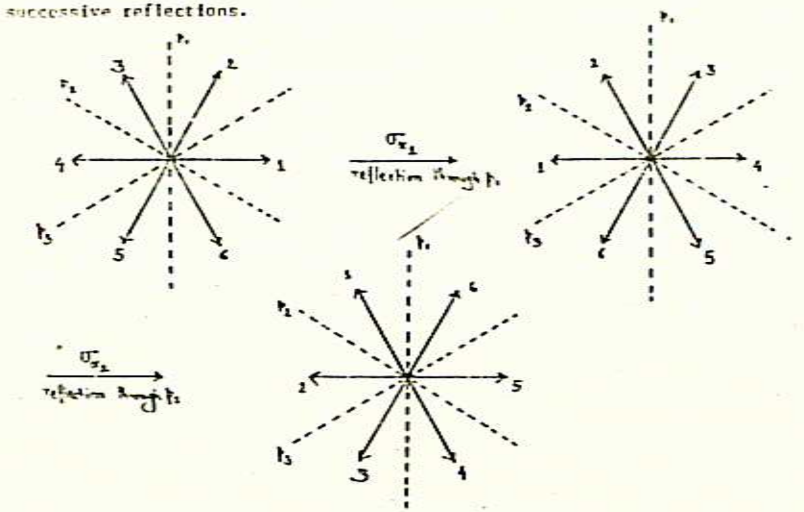
i) SU(2):



The group consists of only two elements since there is only one reflection plane. Thus the group is the cyclic group Z_2 [A cyclic group with n elements is called Z_n; its one dimensional representation is $e^{2\pi i/n}$.]

ii) SU(3):

Label the hyperplanes $p_1, \ldots, p_3$ once and for all and consider the successive reflections.



Thus $\sigma_{\alpha_2}\sigma_{\alpha_1}$ is a rotation of the root-system by $2\pi/3$. Another $\sigma_{\alpha_2}\sigma_{\alpha_1}$ would rotate it by a further $2\pi/3$. Let us look at the equal and opposite roots labelled by 1, 4. Apart from the rotation of $2\pi/3$ and $4\pi/3$ by the operation $\sigma_{\alpha_2}\sigma_{\alpha_1}$ and $(\sigma_{\alpha_2}\sigma_{\alpha_1})^2$ respectively, we can also have its 2π rotation by reflecting them on the hyperplanes perpendicular to itself. Thus we can take any root into any arbitrary position keeping its original opposite root always opposite. The Weyl group is therefore a permutation of 3 objects (the root pairs 1, 4; 2, 5; and 3, 6) among themselves, i.e., S_3. It has six elements.

In general the Weyl group of SU(n) is S_n, with n! elements. It is to be noted that the Weyl group does not exhaust all the symmetries. This is because its elements are connected with some special gauge transformations which are realizable from the identity element in a continuous sense.

Thus the inversion symmetry $\alpha \to -\alpha$ which is clearly a symmetry of the root system is not realizable from the Weyl group of SU(N) unless N = 2. In the context of SU(3) symmetry in elementary particle physics, charge conjugation is an example which is not realizable through SU(3) gauge transformations. In SO(8) there are other such symmetries that cannot be obtained from the Weyl group.

<u>Weyl chambers</u>

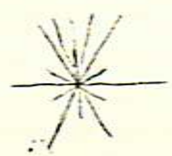
Let us denote by $\Phi(g)$ the set of roots of a compact and simple Lie algebra g, rank r. The mirror hyperplanes responsible for the Weyl reflections are defined by the equations $\alpha \cdot x = 0$ for all $\alpha \in \Phi(g)$. These hyperplanes partition the r-dimensional vector space into disjoint cones each of which is called a <u>Weyl chamber</u>. The action of the Weyl group takes one Weyl chamber to another and hence the group acts on the Weyl chambers. The number of elements of the Weyl group is the number of Weyl chambers. Generally speaking, as we shall see, there are more roots than Weyl chambers.

Let us choose a particular Weyl chamber and a vector x lying <u>within</u> it. We first notice that for all $\alpha \in \Phi(g)$ $\alpha \cdot x \neq 0$, since $\alpha \cdot x = 0$ would mean x is lying on one of the hyperplanes and hence not <u>within</u> the chamber. Moreover, as x moves within the chamber $\alpha \cdot x$ keeps the same sign for all $\alpha \in \Phi(g)$, since a sign-change would mean that for some x, $\alpha \cdot x$ has to become zero. One now defines (with respect to a specific chamber)

i) α is a <u>positive root</u> if $\alpha \cdot x > 0$ (11.1)

ii) α is a <u>negative root</u> if $\alpha \cdot x < 0$ (11.2)

Clearly α and $-\alpha$ have opposite signs.



Lemma: Let α and β be two distinct roots. Then if $\alpha\cdot\beta > 0$, $\alpha-\beta$ is also a root and if $\alpha\cdot\beta < 0$, $\alpha+\beta$ is also a root.

Proof: Since $\alpha\cdot\beta > 0$ implies $(-\alpha)\cdot\beta < 0$, we can, without any loss of generality, take $\alpha\cdot\beta < 0$. We notice from (9.18) that either $2\,\frac{\alpha\cdot\beta}{\alpha^2}$ or $2\,\frac{\alpha\cdot\beta}{\beta^2}$ would be equal to -1 and without any loss of generality we can take

$$\frac{2\alpha\cdot\beta}{\alpha^2} = -1 \ .$$

By Weyl reflection $\sigma_\alpha(\beta) = \beta - \frac{2\alpha\cdot\beta}{\alpha^2}\,\alpha = \beta+\alpha$, and that must be a root.

We will require this lemma to prove the following important theorem.

Theorem II.1: There exists rank-g number of simple roots $\alpha_1, \ldots \alpha_r$, such that any positive root α can be written as:

$$\alpha = \sum_{i=1}^{r} n_i\alpha_i \tag{11.3}$$

with n_i's being integers ≥ 0 ;

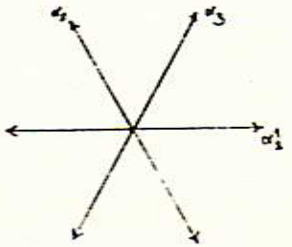
and any negative root β can be written as

$$\beta = \sum_{i=1}^{r} m_i\alpha_i \tag{11.4}$$

with m_i's being integers ≤ 0.

Moreover, $\alpha_i\cdot\alpha_j \leq 0$, i.e. the simple roots make obtuse angles with each other.

[A positive root α is called simple if it cannot be written as a sum of two positive roots. As an example, consider SU(3), whose root diagram is the familiar



Clearly α_1 and α_2 can be taken as simple, whereas α_3 cannot be. Note also

that α_1 and α_3 make an obtuse angle, as they must.]

Proof: The proof is given in three steps:

Step 1

Suppose there exists a set of roots $\{\beta_i\}$ not expressible in the form (11.3). Since the α_i's trivially satisfy (11.3), $\{\beta_i\}$ cannot include them and hence by definition the latter must be all written as the sum of two positive roots. Let us choose β to be the one of $\{\beta_i\}$ such that its scalar product with x, an arbitrarily chosen vector within the Weyl chamber, has the least value. Now, since $\beta = \gamma_1 + \gamma_2$, γ_1 and γ_2 positive, we will have $\gamma_1 = \beta - \gamma_2$. Hence, if γ_2 satisfies (11.3), γ_1 will not and vice-versa. Let us take γ_1 to belong to $\{\beta_i\}$. We also notice that $\beta\cdot x = \gamma_1\cdot x + \gamma_2\cdot x$, implying $\beta\cdot x$ is larger than $\gamma_1\cdot x$ (as well as $\gamma_2\cdot x$). This is clearly contradictory to the starting point that $\beta\cdot x$ has the lowest value. Hence the set $\{\beta_i\}$ must be null.

(11.4) can be proved in a similar way.

Step 2: Let α and β be two distinct simple roots. If $\alpha\cdot\beta > 0$, according to the lemma, $\alpha-\beta$ would be a root which, by definition, has to be either positive or negative. But that violates (11.3) or (11.4). Hence we see that if α and β are simple roots

$$\alpha\cdot\beta \leq 0 \tag{11.5}$$

Step 3: It is clear from (11.3) and the fact that the roots span an r-dimensional space that there are at least r simple roots. That there are precisely r and no more follows from their linear independence, now to be shown. If not,

$$\sum_{i=1}^{r} \mu_i\alpha_i = 0 \quad \text{with not all } \mu_i\text{'s zero.}$$

Of these μ_i's some are positive and some are negative. Call the positive set s_i and the negative set $-t_i$, with s_i's and t_i's ≥ 0. Hence

$$\Sigma\, s_i\alpha_i = \Sigma\, t_j\alpha_j = \epsilon$$

Then $$\epsilon^2 = \Sigma\, s_i t_j(\alpha_i\cdot\alpha_j) \le 0 \quad \text{(according to step 2)}$$

This is clearly impossible unless $\epsilon^2 = 0$, i.e. $\epsilon = 0$, for which all the s_i's and t_i's are zero.

Q.E.D.

A consequence of the lemma is that the commutator of step operators E_α and $E_{-\beta}$, where α and β are two simple roots, has the form

$$[E_\alpha, E_{-\beta}] = \delta_{\alpha,\beta}\, H_\beta \tag{11.6}$$

This follows because if α and β are simple, α-β cannot be a root because it has mixed signs for $\alpha \neq \beta$. This is a new result and plays an important role in some dynamical theories which will be discussed later. [Lecture 16]

Lecture 12

In the last lecture we found that the root system of any compact and simple Lie algebra of rank r can be given a basis in terms of r simple r-dimensional roots α_i (i = 1, r) such that any root α can be written as

$$\alpha = \sum_{i=1}^{r} n_i\alpha_i$$

with the property that either all the n_i's ≥ 0 (for the positive roots) or all the n_i's ≤ 0 (for the negative roots). This fact was established by considering one of the various Weyl chambers of the root systems. These Weyl chambers are connected to each other by Weyl group action. We now state a theorem whose proof may be found in Humphreys [1972], p. 51.

Theorem 12.1: There exists one and only one element of the Weyl group W(g) which takes one Weyl chamber to any other.

This shows that by a gauge transformation one can go from one Weyl chamber to another and consequently all the choices of simple roots are gauge-related.

Example: To illustrate these concepts we consider the SU(3) root system and define a Weyl chamber by the shaded region

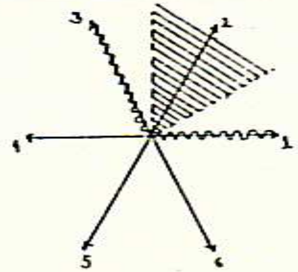


Fig. 12.1

With respect to this Weyl chamber 1, 2 and 3 are the positive roots and the others are negative. Further 1 and 3 are the simple roots.

However by considering other Weyl chambers we find that any of the pairs (1,3), (2,4), (3,5), (4,6), (5,1) and (6,2) could be chosen as the simple roots. All these have one feature in common – the angle between the pair. While the Weyl group operations, i.e., certain gauge transformations, change one set of simple roots to another, they would certainly keep the angle between them invariant. Therefore, to define the root system of a certain algebra the quantities of importance are these angles. This leads us to the Cartan matrix which is an r x r matrix with the elements

$$K_{ij} = \frac{2\alpha^i\cdot\alpha^j}{(\alpha^j)^2} \qquad i,j = 1, \ldots\ldots r. \tag{12.1}$$

Some of the properties of K_{ij} are listed below.

i) $$K_{ij}K_{ji} = \frac{4(\alpha^i\cdot\alpha^j)^2}{(\alpha^i)^2(\alpha^j)^2} = (2\cos\theta)^2 \tag{12.2}$$

(no summation on i or j)

Hence $$\cos\theta = -\frac{1}{2}\sqrt{K_{ij}K_{ji}} \tag{12.3}$$

The negative sign is because of the obtuse angle between the simple roots.

ii) $$\frac{K_{ij}}{K_{ji}} = \frac{(\alpha^i)^2}{(\alpha^j)^2} = \text{square of the ratio of the lengths of } \alpha^i \text{ and } \alpha^j \tag{12.4}$$

These two properties tell us that the Cartan matrix gives us the complete description of the lengths of the simple roots and the angles between them up to a scale and overall rotation.

iii) $K_{ii} = 2$ (obvious from the definition), i.e., all the diagonal entries are 2. Only the off-diagonal elements contain real information.

iv) $K_{ij}K_{ji} = (2\cos\theta)^2 = 0, 1, 2, 3$ (but not 4 since $i \neq j$).

v) Since $\alpha^i\cdot\alpha^j < 0$ (from (9.18)), the off-diagonal elements are negative integers, i.e., 0, -1, -2, or -3 (from (9.18) again).

vi) If $K_{ij} = 0$

then $K_{ji} = 0$, since $\alpha^i \perp \alpha^j$

Properties iii)-vi) give us an idea of the elements of the Cartan matrix. In the following exercise we will evaluate the Cartan matrix for all rank-2 algebras (for rank 1 algebra, i.e., SU(2) or SO(3), the Cartan matrix is trivial – containing a single element 2).

Ex: SU(3): From diagram 12.1, it is clear that the off diagonal elements are both -1 (since the simple roots have the same length and the angle between them is $\frac{2\pi}{3}$).

$$K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Ex: SO(4): The simple roots are perpendicular to each other and hence

$$K = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Since the ratio of the lengths are not determined, SO(4) is not a simple algebra.

Ex: SO(5): The root diagram is

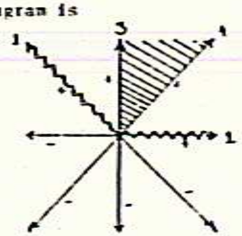


Fig. 12.2

With respect to the shaded Weyl chamber the simple roots are denoted by the wavy lines 1 and 2 while the other two positive roots are 3 and 4. (It is easy to check that

$$\alpha^4 = 2\alpha^1 + \alpha^2$$

and $$\alpha^3 = \alpha^1 + \alpha^2.)$$

The Dynkin diagrams for all rank 1 and rank 2 compact algebras can be easily obtained from their Cartan matrices, which were evaluated in the previous section. Alternatively, the simple roots of the corresponding diagrams of the root-systems can be directly deduced through the rules i), ii) and iii):

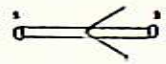
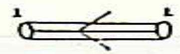
	Algebra	Rank	Dynkin diagram
a)	SU(2)	1	
b)	SU(3)	2	 (simple roots have the same lengths)
c)	SO(4)	2	 (n = 0, the diagram is disconnected implying SO(4) = SU(2) x SU(2))
d)	SO(5)	2	 ($\alpha_2^2 = 2\,\alpha_1^2$)
e)	G_2	2	 ($\alpha_2^2 = 3\,\alpha_1^2$)

Fig. 12.5

Apart from rules (i)-(iii), there are further restrictions on the construction of Dynkin diagrams. In other words, one does not have Lie algebras for every topologically possible diagram obtainable from the three rules. This will be discussed later in Lectures 14 and 15.

We will now show how to construct the Dynkin diagrams for the well-known simple Lie algebras (i.e., those for which the corresponding groups are

defined through matrices), SU(n), SO(n) and Sp(2n). The latter two will be worked out in the next lecture. These are called the classical groups.

SU(n): The group SU(n) is realized through n x n unitary matrices of determinant 1. It has dimension n^2-1, rank n-1 and $n^2-1 - (n-1) = n(n-1)$ roots. We will first derive the root-system of the SU(n) algebra.

Let A be an element of the Cartan subalgebra. It can be represented by a diagonal matrix -

$$A = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \qquad \text{with } \Sigma\, a_i = 0 \qquad (12.9)$$

$$\text{i.e. } (A)_{ij} = a_i\,\delta_{ij}$$

The constraint arises since the elements of the Lie algebra have to be traceless.

A convenient way to define the step operators is to introduce n^2-n number of nxn-matrices $E_{(ij)}$ ($i \neq j$) each of which has only one nonzero entry defined through

$$(E_{(ij)})_{k\ell} = \delta_{ik}\,\delta_{j\ell} \qquad \begin{matrix} i,j = 1, \ldots\ldots\ldots n \\ k,\ell = 1, \ldots\ldots\ldots n \end{matrix} \qquad (12.10)$$

As an example $\qquad E_{12} = \begin{pmatrix} 0\,1\,0 & \ldots\ldots\ldots 0 \\ 0\,0\,0 & \ldots\ldots\ldots 0 \\ 0\,0\,0 & \ldots\ldots\ldots 0 \\ 0\,0\,0 & \ldots\ldots\ldots 0 \end{pmatrix}$

$$\begin{aligned}
\text{Now} \quad [A, E_{(ij)}]_{km} &= \sum_\ell A_{k\ell}\,(E_{ij})_{\ell m} - \sum_\ell (E_{ij})_{k\ell}A_{\ell m} \\
&= \sum_\ell a_k\,\delta_{k\ell}\,\delta_{i\ell}\,\delta_{jm} - \sum_\ell a_m\,\delta_{\ell m}\,\delta_{ik}\,\delta_{j\ell} \\
&= a_k\,\delta_{ik}\,\delta_{jm} - a_m\,\delta_{jm}\,\delta_{ik} \\
&= (a_k - a_m)\,(E_{(ij)})_{km} \\
\text{i.e. } [A, E_{(ij)}] &= (a_k - a_m)\,E_{(ij)}
\end{aligned} \qquad (12.11)$$

Now α^2 is $\sqrt{2}$ times longer than α^1 and the angle between them is $\frac{3\pi}{4}$; consequently

$$K = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad .$$

Ex: G_2: The root diagram is

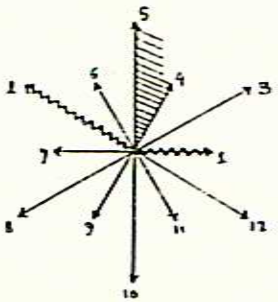


Fig. 12.3

With respect to the shaded Weyl chamber the simple roots are 1 and 2, the latter being $\sqrt{3}$ times longer than the former. The angle between them is $\frac{5\pi}{6}$. It is easy to see that $\alpha_6 = \alpha_2 + \alpha_1$, $\alpha_4 = \alpha_2 + 2\alpha_1$, $\alpha_3 = \alpha_2 + 3\alpha_1$, and $\alpha_5 = 2\alpha_2 + 3\alpha_1$. The Cartan matrix is

$$K = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Notice that $K_{ij} = K_{ji}$ only when the roots are of equal length (see property (ii)). So the Cartan matrix is symmetric only when the simple roots have equal length. It turns out that in such a case all the roots of the algebra will have equal lengths.

Dynkin diagrams

It is clear from the preceding discussion that only the off-diagonal elements of the Cartan matrices are of importance, the diagonal elements being always 2. A way of coding the off-diagonal elements is to represent them

through Dynkin diagrams. The coding is done with the following rules:

i) Draw r points corresponding to α^1 .... α^r, r being the rank of the algebra.

ii) Join i to j by $K_{ij} K_{ji}$ lines (the number of such lines can be 0, 1, 2 or 3 only).

iii) If the number of lines between i and j exceeds one put an arrow on the lines directed towards the one whose corresponding simple root has shorter length than the other.

The third rule needs some clarification. Notice that if $K_{ij}K_{ji} \neq 0$, one of K_{ij} or K_{ji} must be equal to -1 (see (9.18)). Let us take $K_{ij} = -1$.

Then $$K_{ji} = -K_{ij}K_{ji} = -n, \tag{12.5}$$

where n is the number of lines joining i and j.

Also $$\frac{(\alpha^i)^2}{(\alpha^j)^2} = \frac{K_{ij}}{K_{ji}} = \frac{1}{K_{ij}K_{ji}} \quad . \tag{12.6}$$

and hence $$(\alpha^j)^2 \geq (\alpha^i)^2 \tag{12.7}$$

When $n = 1$, $K_{ij} = K_{ji} = -1$ and $(\alpha^i)^2 = (\alpha^j)^2$. For $n > 1$, the two roots are necessarily unequal. So we notice that for $n > 1$,

$$\text{either} \quad K_{ij} = -1 \quad \text{and } K_{ji} = -n \; ; \; (\alpha^j)^2 = n(\alpha^i)^2 \tag{12.8}$$

$$\text{or} \quad K_{ij} = -n \quad \text{and} \quad K_{ji} = -1 \; ; \; (\alpha^i)^2 = n(\alpha^j)^2$$

To illustrate this we consider the following diagram:

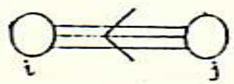


Fig. 12.4

It implies that $\alpha_j^2 = 3\alpha_i^2$

$K_{ij} = -1$ and $K_{ji} = -3$

(12.11) shows that the $E_{(ij)}$'s indeed satisfy the properties of the step operators. Moreover, they comprise all the step operators, since there are n(n-1) of them, just the number of roots. Note that $E_{(ij)}$ and $E_{(ji)}$ are the step operators for two equal and opposite roots. Further,

$$
\begin{aligned}
[E_{(ij)}, E_{(ji)}]_{k\ell}
&= \sum_m (E_{(ij)})_{km} (E_{(ji)})_{m\ell} - \sum_m (E_{(ji)})_{km} (E_{(ij)})_{m\ell} \\
&= \sum_m \delta_{ik}\delta_{jm}\delta_{jm}\delta_{i\ell} - \sum_m \delta_{jk}\delta_{im}\delta_{im}\delta_{j\ell} \\
&= \delta_{ik}\delta_{i\ell} - \delta_{jk}\delta_{j\ell}
\end{aligned}
$$

This shows that $[E_{(ij)}, E_{(ji)}]$ is a diagonal matrix with the i th diagonal entry 1 and j th diagonal entry -1, the rest being all zero. Such a matrix has trace zero and thus belongs to the Cartan subalgebra. Denoting the matrices h_i as

$$(h_i)_{k\ell} = \delta_{ik}\delta_{k\ell}$$

we have

$$[E_{(ij)}, E_{(ji)}] = h_i - h_j \tag{12.12}$$

This verifies that $E_{(ij)}$ and $E_{(ji)}$ are the step operators for two equal and opposite roots. By comparing with the relation

$$[E_\alpha, E_{-\alpha}] = \alpha\cdot H$$

we find that E_{ij} is associated with a root vector $e_i - e_j$ where e_i's are the unit vectors in the n-dimensional vector space, $(e_i)_k = \delta_{ik}$.

Since

$$(e_i - e_j)\cdot\sum_{k=1}^{n} e_k = 0 , \tag{12.13}$$

we see that the n(n-1) roots all lie in the (n-1) dimensional subspace perpendicular to the vector $\sum_{k=1}^{n} e_k$. This is as it should be since the root-

space of SU(n) has the dimension n-1 = rank of the group. We further notice that all the roots have equal length, namely $\sqrt{2}$.

After obtaining the root system of SU(n) we now proceed to find the simple roots. Define the following vectors in the n-dimensional space:

$$
\begin{aligned}
\alpha^1 &= e_1 - e_2 \\
\alpha^2 &= e_2 - e_3 \\
&\vdots \\
\alpha^i &= e_i - e_{i+1} \\
&\vdots \\
\alpha^{n-1} &= e_{n-1} - e_n
\end{aligned}
\tag{12.14}
$$

Then notice that any root $e_i - e_j$ with i<j can be written as

$$e_i - e_j = \alpha^i + \alpha^{i+1} \ldots\ldots + \alpha^j \tag{12.15}$$

Similarly any root $e_j - e_i$ with i<j can be written as

$$e_j - e_i = -\alpha^i - \alpha^{i+1} \ldots\ldots\ -\alpha^j . \tag{12.16}$$

This shows that the set $(\alpha^1, \ldots\ldots \alpha^{n-1})$ form a simple root basis: their coefficients in the expansion of any root are either all positive or all negative. These simple roots have the following properties:

1) $(\alpha^i)^2 = 2$ for all i

2) $\alpha^i\cdot\alpha^j = (e_i - e_{i+1})\cdot(e_j - e_{j+1})$

$$= 2\delta_{ij} - \delta_{i,j+1} - \delta_{i+1,j} .$$

Hence if i ≠ j, the only simple roots that are not perpendicular to α_i are α_{i+1} and α_{i-1}. Consequently

$$\frac{2\alpha_i\cdot\alpha_j}{\alpha_j^2} = \begin{cases} 0 & \text{if } |i-j| \neq 0,1 \\ -1 & \text{if } |i-j| = 1 , \end{cases} \tag{12.17}$$

and the Cartan matrix has the form

$$k = \underbrace{\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}}_{n-1} \left.\vphantom{\begin{pmatrix} 2 \\ -1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}}\right\} n-1 \qquad (12.18)$$

The associated Dynkin diagram is

$$D(SU(n)) = \overset{1}{0}\!-\!\!\!-\overset{2}{0}\!-\!\!\!-\overset{3}{0}\!-\!\!\!- \cdots\cdots\cdots \overset{n-3}{0}\!-\!\!\!-\overset{n-2}{0}\!-\!\!\!-\overset{n-1}{0}$$

Lecture 13

SO(n) - its Lie algebra and Dynkin diagrams:

The group SO(n) is formed out of real rotations in n-dimensions, i.e., from all proper orthogonal matrices in n-dimensional real vector space. Its elements are characterised by $n \times n$ matrices R with the property

i) $R = R^*$ (elements are real)

ii) $RR^T = 1$ (orthogonality)

iii) $\det R = 1$ (i.e. R a proper transformation, one that can be continuously built up from the unit matrix).

Let us take an element infinitesimally close to the identity element,

$$R = 1 + L \, , \qquad (13.1)$$

then we notice that

$$RR^T = (1+L)(1+L^T) = 1+L+L^T = 1$$

$$\text{i.e., } L = -L^T \qquad (13.2)$$

This means that L is antisymmetric.

The dimension of the Lie algebra of SO(n) is the dimension of the algebra of all real $n \times n$ antisymmetric matrices $= \frac{n(n-1)}{2}$.

For the basis of the algebra we consider a set of matrices $L_{(ij)}$ defined by

$$(L_{(ij)})_{k\ell} = -\delta_{ik}\,\delta_{j\ell} + \delta_{i\ell}\,\delta_{jk} \qquad (13.3)$$

$L_{(ij)}$ can be thought as a generator of rotations of vectors in the i-j plane of the n-dimensional vector space.

Now since

$$Tr(L_{(ij)}L_{(mn)}) = -2(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) \qquad (13.4)$$

we see that $IL_{(ij)}$'s form an orthogonal basis.

It is straightforward to see that

$$[L_{(ij)}, L_{(mn)}] = \delta_{im} L_{jn} - \delta_{in} L_{jm} - \delta_{jm} L_{in} + \delta_{jn} L_{im} \qquad (13.5)$$

(13.5) is the Lie algebra of SO(n). Finding the Cartan subalgebra is now not as simple as in SU(n) where the equivalent construction gave n-1 diagonal generators. In SO(n) none of the generators of (13.3) even has diagonal matrix elements. Instead we observe from (13.5) that the set of hermitian matrices

$$iL_{12},\ iL_{34},\ iL_{56},\ \ldots\ldots$$

commute between themselves and in that sequence the complete basis for the Cartan subalgebra can be obtained. At this stage we must distinguish between n even and n odd.

I) When n is even and equals 2r, r an integer, the Cartan subalgebra consists of r operators.

$$iL_{(12)},\ iL_{(34)},\ \ldots\ldots\ iL_{(2r-1,2r)}\quad ,$$

which is the maximum number of commuting generators.

II) When n is odd and equals 2r + 1, one would again have

$$iL_{(12)},\ iL_{(34)},\ \ldots\ldots\ iL_{(2r-1,2r)}$$

r maximally commuting generators.

Hence <u>SO(2r) and SO(2r+1) have the same rank r</u>.

For SO(2r) the number of roots is $\frac{2r(2r-1)}{2} - r = 2r(r-1)$.

For SO(2r+1) the number of roots is $\frac{2r(2r+1)}{2} - r = 2r^2$.

Let us now evaluate the root-systems for SO(2r) and SO(2r + 1). To do that we have to find the step operators.

We notice from (13.5) that

$$[L_{(12)}, L_{(13)}] = L_{(23)}$$
$$[L_{(12)}, L_{(23)}] = -L_{(13)} \qquad (13.6)$$

By analogy with the angular momentum we then have (with $\epsilon = \pm 1$)

$$[iL_{(12)}, L_{(13)} + i\epsilon\, L_{(23)}] = \epsilon\, (L_{(13)} + i\epsilon\, L_{(23)})$$

$$(13.7)$$

Similarly $[iL_{(12)}, L_{(14)} + i\epsilon\, L_{(24)}] = \epsilon\, (L_{(14)} + i\epsilon\, L_{(24)})$

One is thus tempted to consider such type of combinations as the step operators. But there is a snag. We notice that

$$[L_{(34)}, L_{(14)} + i\epsilon\, L_{(24)}] = -(L_{(13)} + i\epsilon\, L_{(23)}) \qquad (13.8)$$

which rules out the fact that $L_{(14)} + i\epsilon\, L_{(24)}$ is a step operator. Such a problem is however cured by considering the following combination

$$\bar{L} \equiv L_{(13)} + i\epsilon\, L_{(23)} + i\eta\, (L_{(14)} + i\epsilon\, L_{(24)})\quad .$$

where now ϵ and η can independently take values $\pm$ 1. It is easy to check now that

$$[iL_{(12)}, \bar{L}] = \epsilon\, \bar{L}$$

and

$$[iL_{(34)}, \bar{L}] = \eta\, \bar{L}$$

while

$$[iL_{(56)}, \bar{L}] = 0 \qquad (13.9)$$

etc.

With this prescription we find that the four roots corresponding to $\epsilon = \pm 1$ and $\eta = \pm 1$ are $(\pm 1, \pm 1, 0 \ldots\ldots)$. Proceeding in this way we find that SO(n) for both n even and odd have a set of roots

$\pm e_i \pm e_j$ ($i \neq j$ and $i,j = 1, \ldots r$), where the e_i are the unit vectors in the vector space of dimension r, the rank of the algebra. We notice that the number of these roots is equal to the number of ways any two of the r unit vectors can be picked up, multiplied by four (for all possible signs), i.e., $\binom{r}{2} \times 4 = 2r(r-1)$. Further we see that the length of all these roots are equal to $\sqrt{2}$.

For SO(2r) we have seen that the number of roots are precisely 2r(r-1) and therefore these are all the root SO(2r) can have. For SO(2r+1) others will be necessary.

A. SO(2r)

The first r-1 simple roots can be chosen as

$$\alpha^1 = e_1 - e_2$$

$$\alpha^2 = e_2 - e_3 \qquad (13.10)$$

$$\vdots$$

$$\alpha^{r-1} = e_{r-1} - e_r$$

and the rth one as

$$\alpha^r = e_{r-1} + e_r \qquad (13.11)$$

With this choice we notice that for $i < j$

$$e_i - e_j = e_i - e_{i+1} + e_{i+1} - e_{i+2} + \ldots\ldots e_j$$

$$= \alpha^i + \alpha^{i+1} + \ldots\ldots\ldots + \alpha^{j-1}$$

$$e_i + e_j = e_i - e_{i+1} + e_{i+1} - e_{i+2} + \ldots\ldots\ldots e_{j-1} - e_j$$

$$+ 2(e_j - e_{j+1}) + 2(e_{j+1} - e_{j+2}) + \ldots\ldots + 2(e_{r-2} - e_{r-1})$$

$$+ (e_{r-1} - e_r) + (e_{r-1} + e_r)$$

$$= \alpha^i + \alpha^{i+1} + \ldots\ldots \alpha^{j-1} + 2\alpha^j + 2\alpha^{j+1} + \ldots\ldots 2\alpha^{r-2} + \alpha^{r-1} + \alpha^r$$

Similarly for $i > j$ we will have

$$e_i - e_j = -\alpha^j - \alpha^{j+1} - \ldots\ldots - \alpha^{i-1}$$

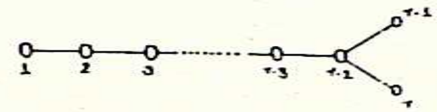
Further $-e_i - e_j = -\alpha^j - \alpha^{j+1} - \ldots -\alpha^{i-1} - 2\alpha^i - 2\alpha^{i+1} - \ldots\ldots -2\alpha^{r-2} - \alpha^{r-1} - \alpha^r$.

So all $e_i + e_j$ and $e_i - e_j$ (for $i < j$) are positive roots and the others are negative. The Dynkin diagram for SO(2r) follows easily from (12.10)

$$\alpha^i \cdot \alpha^{i+2} = 0 \text{ for } 1 \leq i \leq r-3 .$$

However $\alpha^{r-2} \cdot \alpha^r = (e_{r-2} - e_{r-1}) \cdot (e_{r-1} + e_r) = -1 = \alpha^{r-2} \cdot \alpha^{r-1}$. Further $\alpha^{r-1} \cdot \alpha^r = 0$. So the diagram looks like

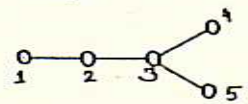
D(SO(2r)) =



The corresponding Cartan matrix is

$$K = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & & & \\ -1 & 2 & -1 & 0 & 0 & & & & \\ 0 & -1 & 2 & -1 & 0 & \cdots & & & \\ \cdots & & & & & & & & \\ 0 & 0 & \cdots & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 & 0 & 2 \end{pmatrix} \qquad (13.11)$$

It is the bottom right-hand corner of K which make SO(2r) different from SU(r+1). If one of the bifurcating arms at the end is removed the remaining part of the diagrams will resemble the Dynkin diagram of SU(r). For example from the Dynkin diagram of SO(10),

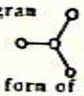


if the point 5 (or 4) were removed, one would get the diagram for SU(5). This means that the subalgebra generated by all the step operators corresponding to the roots formed out of α^1, α^2, α^3 and α^4 only and the Cartan subalgebra

H^1, H^2, H^3 and H^4 would be that of SU(5). Such an embedding of SU(5) in SO(10) plays an important role in grand unified theories. Later, in the context of spontaneous symmetry breaking in gauge theories we will discuss how SU(5) symmetry can be realized as the surviving symmetry from SO(10) by such deletion of one of the bifurcating arms. (See lecture 24). It is a useful generalization that if you remove a point from a Dynkin diagram you immediately can identify the Dynkin diagrams of a possible subalgebra.

The lower SO(2r) algebras are also of some interest.

i) SO(8) has the Dynkin diagram



which as can be guessed from the form of the diagram has some interesting extra symmetries.

ii) SO(6) has the Dynkin diagram  which is the same as that of SU(4). The two algebras are indeed the same.

iii) Similarly SO(4) which has the Dynkin diagram  has the same algebra as SU(2) x SU(2).

B. SO(2r+1):

In the previous discussion on SO(2r) we found the step operators for $2r(r-1)$ roots all of the form $\pm e_i \pm e_j$. However for SO(2r+1) the number of roots must be $2r^2$. So we have to find 2r number of extra step operators and the corresponding roots other than those already found earlier. We have now the extra label 2r+1 and consequently find from (13.5) that

$$[L_{12}, L_{1,2r+1}] = L_{2,2r+1}$$
$$[L_{12}, L_{2,2r+1}] = -L_{1,2r+1} \qquad (13.13)$$

Hence $[L_{12}, (L_{1,2r+1} \pm iL_{2,2r+1})] = \pm (L_{1,2r+1} \pm L_{2,2r+1})$.

and $[L_{34}, (L_{1,2r+1} \pm iL_{2,2r+1})] = 0$, etc. (13.13)

up to $[L_{2r-1,2r}, (L_{1,2r+1} \pm L_{2,2r+1})] = 0$

In this way we find that we have 2r extra step operators corresponding to the roots $(\pm 1, 0, \ldots)$, $(0, \pm 1, 0, \ldots)$ etc. This means that for SO(2r+1) the roots are

$\pm e_i \pm e_j$, $2r(r-1)$ of them, all having length $\sqrt{2}$

and also $\pm e_i$, 2r of them, all having length 1

So we have now a root system of unequal lengths - a set of long roots and a set of short roots.

The simple roots are taken to be

$$\alpha^1 = e_1 - e_2$$
$$\alpha^2 = e_2 - e_3$$
$$\cdots\cdots$$
$$\alpha^{r-1} = e_{r-1} - e_r$$

and $\alpha^r = e_r$. (13.14)

So for $i<j$ we have

$$e_i - e_j = \alpha^i + \alpha^{i+1} + \ldots + \alpha^{j-1}$$

Also $e_i + e_j = \alpha^i + \alpha^{i+1} + \ldots \alpha^{j-1} + 2\alpha^j + 2\alpha^{j+1} \ldots 2\alpha^r$

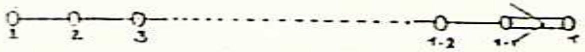
and $e_i = \alpha^i + \alpha^{i+1} + \ldots \alpha^r$.

These are the positive roots. The negative ones are $e_i - e_j$ for $i>j$, $-e_i - e_j$ and $-e_i$, obtained in a similar way with negative coefficients.

Given the simple roots as in (13.14), obtaining the Dynkin diagram is straightforward. We notice that $\alpha^1, \alpha^2 \ldots \alpha^{r-1}$ all have the same length, that

$$\alpha^i \cdot \alpha^{i+2} = 0, \text{ for all } i, \qquad (13.15)$$

and that $2(\alpha^r)^2 = (\alpha^{r-1})^2$. Thus

D(SO(2r+1)) = 

The corresponding Cartan matrix is

$$K = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & & & & \\ -1 & 2 & -1 & 0 & 0 & & & 0 & \\ 0 & -1 & 2 & -1 & 0 & & & & \\ & & & & -1 & 2 & -1 & 0 \\ & & 0 & & 0 & -1 & 2 & -2 \\ & & & & 0 & 0 & -1 & 2 \end{pmatrix} \qquad (13.16)$$

Notice that the only entry which differs from the Cartan matrix of SU(r+1) is $K_{r-1,r}$.

Sp(2r): One can in a similar way, obtain the Lie-algebra, root system, the simple roots and the Dynkin diagram for the algebra of the symplectic group Sp(2r) the group of all 2r x 2r matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$: A,B,C and D being rxr matrices with the property $B^T = B$, $C^T = C$ and $A^T = -D$.

The Dynkin diagram of Sp(2r) can be shown to be

D(Sp(2r)) = 

We have thus exhausted the Dynkin diagrams associates with the Lie algebras of all the "classical" groups, which were investigated by Cayley. Cartan showed, by the techniques discussed here, the existence of a few more Lie algebras (they are the exceptional groups, namely G_2, F_4, E_6, E_7 and E_8): these will exhaust all possible simple compact Lie algebras.

Lecture 14

Reconstruction of the root system from Dynkin diagrams:

In the last two lectures we described the method of obtaining the Dynkin diagrams corresponding to the classical simple Lie groups SU(n), SO(2r), SO(2r+1) and Sp(2r) by first constructing the root systems from the definition of these groups and their algebras and then finding the simple roots. We will now show that it is possible to do the reverse. Given the Dynkin diagram, i.e., the simple roots (lengths up to an overall scale) and their relative orientations, it is possible to reconstruct all the roots and hence the algebra. This enables one to completely classify the simple Lie algebras that would include, apart from SU(n), SO(2r), SO(2r+1) and Sp(2r), the exceptional groups. In the process we will be showing some further restriction on the allowed form of the Dynkin diagrams.

We first introduce the concept of the α-rootstring through β. If β and α are two distinct roots (i.e., $\beta \neq \pm\alpha$) we know from the lemma proved in the Lecture 10 that if $\alpha\cdot\beta > 0$ then $\beta - \alpha$ is a root while if $\alpha\cdot\beta < 0$ $\beta+\alpha$ must be a root. We can continue this process and obtain a set of roots forming the string. It is defined in the following way. Let p be the largest positive number for which $\beta + p\alpha$ is a root. Similarly let q be the largest positive number for which $\beta - q\alpha$ is a root. Then the set of vectors forming the α-rootstring through β --

$$\beta + p\alpha,\ \beta + (p-1)\alpha,\ \ldots\ldots\ \beta,\ \beta - \alpha,\ \beta - 2\alpha,\ \ldots\ldots\ \beta - q\alpha \qquad (14.1)$$

can all be shown to be roots. We show this by contradiction. For, if some of them are not, i.e., if the string is broken, say on the positive side, then there would be two positive numbers r,s with $p > r \geq s$, such that $\beta + (r+1)\alpha \in \Phi(g)$ while $\beta + r\alpha \notin \Phi(g)$ and $\beta + (s+1)\alpha \notin \Phi(g)$ while $\beta + s\alpha \in \Phi(g)$.

Now, if $\beta + r\alpha \notin \Phi$, $\alpha\cdot[\beta + (r+1)\alpha]$ cannot be greater than zero (according to the lemma of lecture 11), i.e., one must have

$$\alpha\cdot(\beta + (r+1)\alpha) \leq 0 \quad . \tag{14.2}$$

Similarly if $\beta + (s+1)\alpha \notin \Phi$, $\alpha\cdot[\beta + s\alpha]$ cannot be less than zero, i.e., one must have

$$\alpha\cdot(\beta + s\alpha) \geq 0 \tag{14.3}$$

But since $\alpha^2 > 0$ it follows from (14.2) and (14.3) that $s > r$, clearly contradicting the starting point $r > s$. Hence the root string is unbroken.

Since all the elements in (14.1) are roots, according to the arguments presented in Lecture 9, we will have

$$[E_\alpha, E_\beta] \sim E_{\beta+\alpha} \quad ,$$

$$[E_\alpha, E_{\beta+\alpha}] \sim E_{\beta+2\alpha} \quad ,$$

$$[E_\alpha, E_{\beta+(p-1)\alpha}] \sim E_{\beta+p\alpha} \quad ,$$

$$[E_\alpha, E_{\beta+p\alpha}] = 0 \quad ,$$

and similarly on the negative side with $E_{-\alpha}$ appearing in the commutator along with E_β, then $E_{\beta-\alpha}$, etc. Thus E_α and $E_{-\alpha}$ are respectively like raising and lowering operators along the root string. We now recall the operator $2T_3 = \frac{2\alpha\cdot H}{\alpha^2} = [E_\alpha, E_{-\alpha}]$. T_3 must have half integral eigenvalues. We notice that

$$[2T_3, E_{\beta+p\alpha}] = \frac{2\alpha\cdot(\beta+p\alpha)}{\alpha^2} E_{\beta+p\alpha} \tag{14.4}$$

i.e., $E_{\beta+p\alpha}$ is an eigenvector of $2T_3$ with the eigenvalue $\frac{2\alpha\cdot(\beta+p\alpha)}{\alpha^2}$. Similarly $E_{\beta-q\alpha}$ is another with the eigenvalue $\frac{2\alpha\cdot(\beta-q\alpha)}{\alpha^2}$. In between there are $(p+q-2)$ other eigenvectors with eigenvalues in integer steps. (?) Combined they form a $2j+1$ - dim multiple of the SU(2) subalgebra generated

by E_α, $E_{-\alpha}$ and T_3. We must, therefore, have

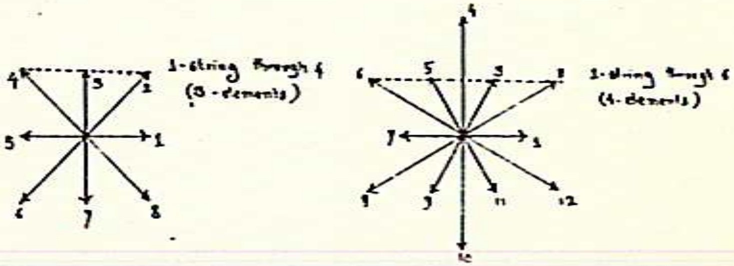
$$2j = \frac{2\alpha\cdot(\beta+p\alpha)}{\alpha^2} - \frac{2\alpha\cdot(\beta-q\alpha)}{\alpha^2} \tag{14.5}$$

or

$$q - p = \frac{2\alpha\cdot\beta}{\alpha^2} = 0, \pm 1, \pm 2, \pm 3,$$

from lecture 9. Note also that $2j = p + q$. So if one starts with the extreme end of a root-string, say $q = 0$, then the only possible values of p are 0, 1, 2 or 3.

Hence the number of roots in a string ≤ 4, or in other words $j \leq \frac{3}{2}$. One can check this result from the root diagrams of SO(5) and G_2



We will now classify (and hence enable ourselves to reconstruct) the roots in terms of height. Recall that any positive root β is written in terms of the simple roots α^i as $\Sigma n_i \alpha^i$, where the n_i are positive integers. One defines

$$\text{Height } \alpha = \Sigma n_i \quad . \tag{14.6}$$

Hence the height 1 roots are the simple roots.

Let us now look at the α^j string through the height 1 roots. Take two arbitrary simple roots α^i and α^j and consider the α^j root string through α^i.

Since $\alpha^i-\alpha^j$ cannot be a root we have q = 0. Hence from (14.5)

$$p = -\frac{2\alpha^i\cdot\alpha^j}{(\alpha^j)^2} = -K_{ij} \qquad (14.7)$$

The right hand side of (14.7) can be read off from the Dynkin diagram. Thus α^i, $\alpha^i+\alpha^j$, ... $\alpha^i - K_{ij}\alpha^j$ are the roots in the string. $\alpha^i+\alpha^j$ is a height 2 root.

For the height 3 roots consider the α^i string through a height two root $\alpha = \alpha^j+\alpha^k$, $j\neq k$. If $i \neq j$ or k, $\alpha^j+\alpha^k-\alpha^i$ cannot be a root and hence q = 0. In that case

$$p = -\frac{2\alpha^i\cdot(\alpha^j+\alpha^k)}{(\alpha^i)^2} = -(K_{ji}+K_{ki}) \qquad (14.8)$$

Again the right hand side is known from the Dynkin diagram and $\alpha^j+\alpha^k+\alpha^i$, ... $\alpha^j+\alpha^k-(K_{ji}+K_{ki})\alpha^i$ are the roots in the string. If $i = j$, then $\alpha^i+\alpha^k-\alpha^i=\alpha^k$. Hence q = 1. Therefore $p = -(K_{ij}+K_{kj})+1 = -1-K_{kj}$.

The process of finding the roots associated with all possible α^i strings through the positive roots of successive heights must stop some time since the number of roots is finite. Thus Dynkin diagrams completely specify the associated root-systems.

The next question is whether there exists simple Lie algebras that we have overlooked so far. In other words are there simple Lie groups and their algebras other than the classical ones? We have already seen the Dynkin diagram of G_2 and the method described above tells us how to find the algebra of G_2. Our task is now to enumerate all simple Lie algebras by enumerating all admissible Dynkin diagrams D(G).

We will, for the time being, ignore the arrows in D(G). Define the unit vectors in the direction of the simple roots —

$$\epsilon^i = \frac{\alpha^i}{\sqrt{(\alpha^i)^2}} \qquad (14.9)$$

Hence, associated with each D(G) there are r linearly independent unit vectors ϵ^i (i = 1,.....r) such that

$$2\epsilon^i\cdot\epsilon^j = \frac{2\alpha^i\cdot\alpha^j}{((\alpha^i)^2(\alpha^j)^2)^{1/2}} = -\sqrt{K_{ij}K_{ji}}$$

($\epsilon^i\cdot\epsilon^j$ is always negative)

$= -$(number of lines between i and j)$^{1/2}$

$= 0, -1, -\sqrt{2}$ or $-\sqrt{3}$. (14.10)

We will see that (14.10) imposes strong restriction on the structure of Dynkin diagrams.

We will use the following terms:

If the number of lines between i and j

a) > 0, i and j are said to be linked

b) = 1, the link is called single

c) = 2, the link is called double

d) = 3, the link is called triple

We note that any subset of an admissible D(g) has to be admissible.

Lemma 1: The number of links in D(g) is ≤ r - 1. (wrong)

Proof:

$$0 < (\Sigma\,\epsilon^i)^2 = r + 2\sum_{\text{pairs}} \epsilon^i\cdot\epsilon^j$$

$$= r - \sum_{\text{pairs}} (\text{no. of lines between i and j})^{1/2}$$

(from (14.10))

G_2 has 3 links and r=2 !!

Since at least one pair is linked the term within the sum is ≥ 1 and consequently

$$\sum_{\text{pair}} \text{(no. of lines between i and j) (= no. of links).}$$

$$\geq \sum_{\text{pair}} \text{(no. of lines between i and j)}^{1/2}$$

Hence we have

$$0 < r - \text{(no. of links)} .$$

Thus $r - \text{(no. of links)} \geq 1$ Q.E.D.

<u>Corollary</u>: <u>There exist no loops</u> in D(g), since in such a case the number of points in any loop will equal the number of links.

Hence Dynkin diagrams are <u>tree diagrams</u>.

<u>Lemma 2</u>: The number of lines joined to any vertex ≤ 3.

Proof: Let ε be the unit vector corresponding to a vertex and $\varepsilon^1, \varepsilon^2, \ldots \varepsilon^k$ correspond to the k points to which it is linked. Then because no loops are possible, i and j cannot be linked,

$$\varepsilon^i \cdot \varepsilon^j = 0 . \tag{14.11}$$

Using (14.11) we can now write

$$\varepsilon = \sum_{i=1}^{k} (\varepsilon\cdot\varepsilon^i)\varepsilon^i + (\varepsilon\cdot\varepsilon^0)\varepsilon^0 , \tag{14.12}$$

where the second term indicates the part of the vector ε not in the subspace spanned by the orthormal basis $\varepsilon^1, \ldots\ldots \varepsilon^k$; ε^0 is a unit vector perpendicular to the subspace. Squaring (14.12) we get

$$1 = \sum_{i=1}^{k} (\varepsilon\cdot\varepsilon^i)^2 + (\varepsilon\cdot\varepsilon^0)^2$$

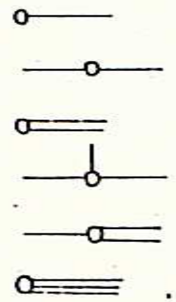
Hence $4 \geq 4 \sum_{i=1}^{k} (\varepsilon\cdot\varepsilon^i)^2 = \Sigma$ (the no. of lines joined to ε), by Eq.(14.10).

But the equality holds only when $\varepsilon\cdot\varepsilon^0 = 0$ which means from (14.12) that ε is a linear combination of ε^i's. That contradicts the starting proposition of the simple roots being linearly independent. Hence we have

The no. of lines joined to $\varepsilon < 4$ i.e., ≤ 3 Q.E.D.

The consequence of this lemma is that the part of the diagram around a certain vertex will look like one of the following:

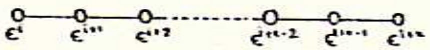
i) ii) iii) iv) v) vi)



<u>Corollary</u>: The only connected Dynkin diagram with a triple link is

This is G_2, which we have discussed earlier.

<u>Lemma 3</u>: If an admissible D(g) has a portion

$\Gamma =$ 

then a D'(g) obtained by <u>fusing</u> Γ (i.e., shrinking the chain to one point) is also admissible. D'(g) is thus obtained by considering a new vector

$$\varepsilon \equiv \sum_{\ell=i}^{i+k} \varepsilon^{\ell} \tag{14.13}$$

corresponding to a new simple root $\alpha \equiv \sum_{\ell=i}^{i+k} \alpha^{\ell}$, along with α^m's where $\{m\}\cap\{i,\ldots\ldots k\} = 0$. Squaring (14.13) we get

$$\varepsilon^2 = k + 2 \sum_{\text{pairs}} \varepsilon^{\ell}\cdot\varepsilon^{m}$$

In the sum only the contribution from the adjacent pairs survive, since the links exist only between them. Moreover, since they are all single links, the scalar product between adjacent pair is $-\frac{1}{2}$ (from(14.10)). Hence

$$\epsilon^2 = k + (k-1)(-1) = 1$$

So ϵ is indeed a unit vector.

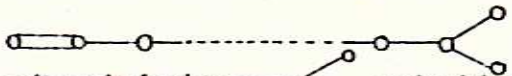
We also notice that the fusing can be performed even if one of the ϵ^i in Γ has an extra link with η which, by lemma 1, cannot be linked to any other member of the chain. Hence

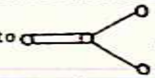
$$\eta\cdot\epsilon = \eta\cdot\epsilon^i \qquad \text{Q.E.D.}$$

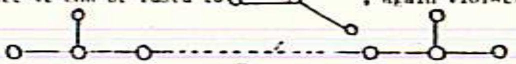
Corollary: The following diagrams (or subgraphs of diagrams) are not admissible.

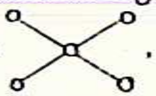
i) 

(Since it can be fused to , which violates lemma 2)

ii) 

(Since it can be fused to , again violating lemma 2)

iii) 

(Since it can be fused to , violating lemma 2)

We have thus narrowed down the choices of the admissible Dynkin diagrams. The possibilities are now

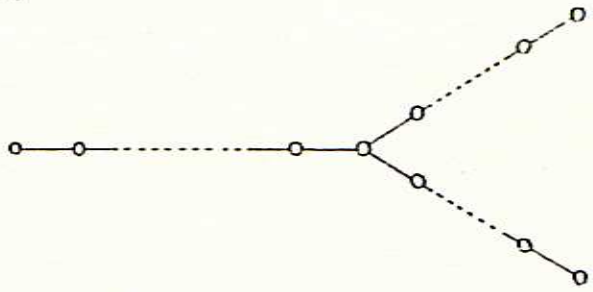
i)  (SU(r+1))

ii) 

(with only one such double link)

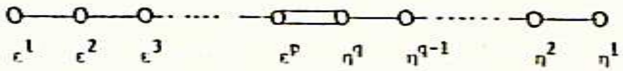
iii)  (G_2)

and iv)



Lecture 15

There are further constraints on the structure of admissible Dynkin diagrams beyond those discussed in the last lecture. Consider the diagram with a double-link. The only possibility is



Let $\epsilon = \sum_1^p j\epsilon^j$ and $\eta = \sum_1^q j\eta^j$. (15.1)

Then $(\epsilon)^2 = \sum_1^p j^2 - \sum_1^p j(j+1) = \frac{p(p+1)}{2}$ (15.2)

and similarly $(\eta)^2 = \frac{q(q+1)}{2}$. (15.3)

Further, since $-2\epsilon^p\cdot\epsilon^q = \sqrt{2}$ (doubly linked), we have

$$\epsilon\cdot\eta = pq\epsilon^p\cdot\epsilon^q = -\frac{pq}{\sqrt{2}} \qquad (15.4)$$

Now using Schwartz' inequality,

$$(\epsilon\cdot\eta)^2 = \frac{p^2q^2}{2} = (\epsilon)^2(\eta)^2\cos^2\theta < (\epsilon)^2(\eta)^2 \ .$$

($\cos\theta$ cannot be one since pq is an integer $\neq 0$)

Hence $\frac{p^2q^2}{2} < \frac{p(p+1)q(q+1)}{4}$. (15.5)

(15.5) implies

$$(p-1)(q-1) < 2 \ . \qquad (15.6)$$

But p and q are nonzero positive integers. Therefore the possible values of them are

i) $p = 2,\ q = 2$,

ii) $p = 1$, q any positive integer,

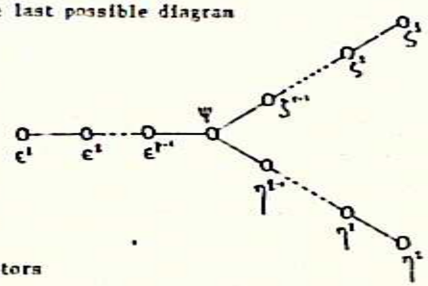
iii) $q = 1$, p any positive integer.

For case i) we have a diagram

which corresponds to an algebra of rank 4, not belonging to the classical set. The associated group is called F_4.

For the other two cases we have our familiar diagrams for SO(2r+1) or Sp(2r)

Next consider the last possible diagram



We define the vectors

$$\epsilon = \sum_1^{p-1} j\epsilon^j \ ,$$
$$\eta = \sum_1^{q-1} j\eta^j \qquad (15.7)$$
and
$$\zeta = \sum_1^{r-1} j\zeta^j \ .$$

As before we will have $(\epsilon)^2 = \frac{p(p+1)}{2}$, $(\eta)^2 = \frac{q(q-1)}{2}$ and $(\zeta)^2 = \frac{\zeta(\zeta-1)}{2}$.

Hence $\cos^2(\epsilon,\psi) = \frac{(\epsilon\cdot\psi)^2}{(\epsilon)^2(\psi)^2} = \frac{((p-1)\epsilon^{p-1}\cdot\psi)^2}{p(p-1)/2} = \frac{(\frac{p-1}{2})^2}{\frac{p(p-1)}{2}}$

$$= \frac{1}{2}(1 - \frac{1}{p}) \ . \qquad (15.8)$$

Similarly $\cos^2(\eta,\psi) = \frac{1}{2}(1-\frac{1}{q})$ and $\cos^2(\zeta,\psi) = \frac{1}{2}(1-\frac{1}{r})$. (15.9)

We now observe the inequality

$$\cos^2(\xi,\psi) + \cos^2(\eta,\psi) + \cos^2(\zeta,\psi) < 1. \qquad (15.10)$$

This follows from the argument of lemma 2 of lecture 14.

Since ξ, η and ζ are mutually perpendicular we can write

$$\psi = \cos(\xi,\psi)\xi + \cos(\eta,\psi)\eta + \cos(\zeta,\psi)\zeta + \cos(\zeta',\psi)\zeta' ,$$

where ζ' is a vector orthogonal to ξ, η and ζ, and $\cos(\zeta',\psi) \neq 0$ for the linear independence of ψ. Squaring both sides we obtain the inequality (15.10).

From (15.10) we obtain

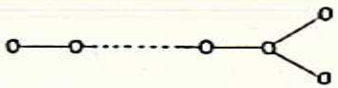
$$\frac{1}{2}\left(1 - \frac{1}{p}\right) + \frac{1}{2}\left(1 - \frac{1}{q}\right) + \frac{1}{2}\left(1 - \frac{1}{r}\right) < 1, \quad \text{or}$$

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1. \qquad (15.11)$$

Since p, q and r are nonzero positive integers, (15.11) sets restrictions on them. Without any loss of generality we can take $p \geq q \geq r$, and for the branching diagrams we must have all of them ≥ 2. Possible values are then

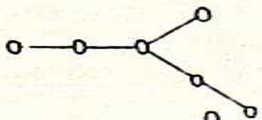
i) $(p,q,r) = (p,2,2)$, $p \geq 2$

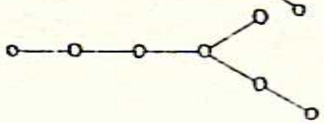
This corresponds to

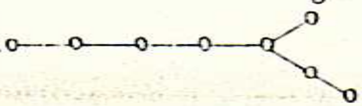


(SO(2r) diagrams)

ii) $(p,q,r) = (p,3,2)$; In such a case the only allowed values of p are 3, 4 and 5. The corresponding diagrams are

p = 3 ,  (E_6, a new algebra of rank 6)

p = 4 ,  (E_7, a new algebra of rank 7)

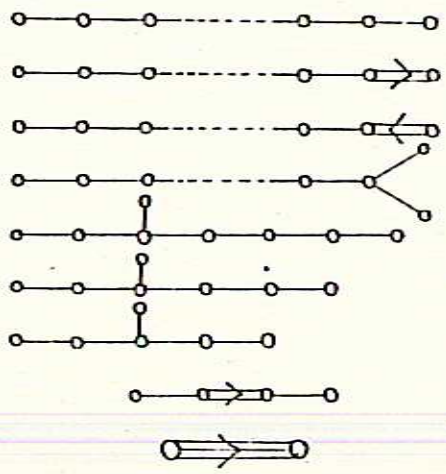
p = 5 ,  (E_8, a new algebra of rank 8)

iii) $(p,q,r) = (p,4,2)$; In such a case $p < 4$, but that violates the starting proposition $p \geq q$.

We have thus exhausted all the possibilities of choices of diagrams consistent with (15.11).

This completes the set of constraints on the Dynkin diagrams.

Summary of connected Dynkin diagrams associated with all possible simple Lie algebras:



i) SU(r + 1) or A_r

ii) SO(2r + 1) or B_r

iii) Sp(2r) or C_r

iv) SO(2r) or D_r

(i)–(iv): Classical Series

v) E_8

vi) E_7

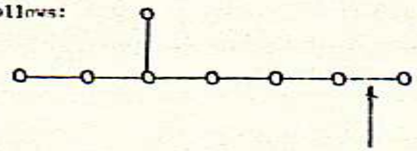
vii) E_6

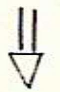
viii) F_4

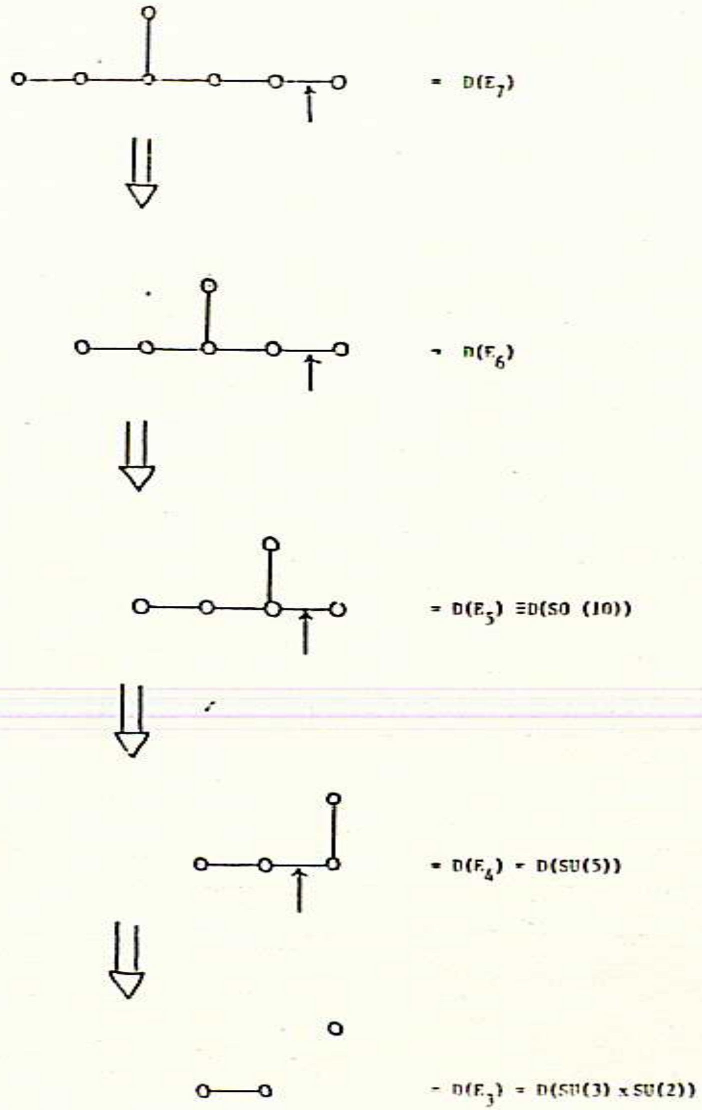
ix) G_2

(v)–(ix): "Exceptional Lie algebras"

Notice that amongst the exceptional groups only E_8, E_7 and E_6 have roots of equal length. Subalgebras are quite easy to obtain by deleting a simple root and its associated links. An interesting sequence of subalgebras starting from E_8 is as follows:

$D(E_8)$ = 





This particular sequence of groups is of special interest in the context of grand unified theories. [See Olive 1981 and Olive and West 1982 for more discussion and earlier references.]

Before we end the formal discussion on root systems and Dynkin diagrams we say a few words about the Chevalley basis which we have actually been using without explicitly mentioning so. This is the basis (i.e., a linear combination of the Cartan-Weyl basis) where the generators of the Cartan subalgebra form the $2T_3$ components of the various SU(2) subalgebras, i.e., $H_\alpha = \frac{2\alpha \cdot H}{\alpha^2}$, where the H_i form the Cartan-Weyl basis. The H_α's are mutually commuting.

Now since $[H_i, E_\beta] = \beta_i E_\beta$,

we get

$$[H_\alpha, E_\beta] = \frac{2\alpha \cdot \beta}{\alpha^2} E_\beta \; .$$

If one considers the step operators corresponding to simple roots only, then

$$[H_\alpha, E_\beta] = K_{\beta\alpha} E_\beta \qquad (15.12)$$

where the $K_{\beta\alpha}$ are the elements of the Cartan matrix. (15.12) and the other already known set (Eq. 11.6)

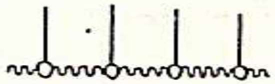
$$[E_\alpha, E_{-\beta}] = \delta_{\alpha\beta} H_\alpha \quad ,$$

will be extensively used, from the next lecture, in connection with the integrable systems associated with Lie algebra.

Even though the commutators between the step operators of opposite signs as in (11.6) are easy to evaluate those of the form $[E_\alpha, E_\beta]$ are not so simple. They were derived by Chevalley. The structure constants are again integers given by simple rules in terms of the root strings. Since these can be reconstructed from the Dynkin diagram (Lecture 14), so therefore can the complete Lie algebra.

Having enumerated the simple Lie algebras I should next study their finite dimensional representations. I want to motivate this step better by first treating some apparently diverse physical applications of the theory so far. The first is recent and quite surprising.

After World War II, Fermi, Pasta and Ulam [1955] performed some numerical experiments on a computer to check whether large non-linear systems showed ergodic behaviour, i.e. eventually filled the available phase space. They used a model of pendula linked by non-linear springs;



initially displacing the right-most mass and observing the evolution of the system. They were surprised to find that after a certain (finite) time, the initial configuration was repeated, demonstrating non-ergodic behavior and hinting at the existence of extra conserved quantities or symmetries.

Next, Toda [1967, 1975, 1981] observed that if one takes an exponential form for the spring tension

$$T = e^{\Delta x}$$

where Δx is its (algebraic) extension, the equations can be solved analytically. One actually needs an infinite or periodic lattice to obtain a stable equilibrium configuration. However the finite Toda lattice equations do arise in gauge theories, as we shall see later. For these equations of motion for the particles are [Moser, 1975]

$$\begin{aligned} \ddot{x}_0 &= e^{x_1-x_0} \\ \ddot{x}_1 &= e^{x_2-x_1} - e^{x_1-x_0} \\ \ddot{x}_i &= e^{x_{i+1}-x_i} - e^{x_i-x_{i-1}} \end{aligned} \tag{16.1}$$

where x_i is the position of the i^{th} particle and we have taken unit mass. The extension of the i^{th} spring is

$$y_i = x_i - x_{i-1} \tag{16.2}$$

and in terms of these variables we find

$$\begin{aligned} \ddot{y}_1 &= e^{y_2} - 2e^{y_1} \\ \ddot{y}_2 &= e^{y_3} - 2e^{y_2} + e^{y_1} \\ \ddot{y}_i &= e^{y_{i+1}} - 2e^{y_i} + e^{y_{i-1}} \end{aligned} \tag{16.3}$$

which in matrix form is

$$\begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \\ \vdots \\ \ddot{y}_r \end{pmatrix} = -\begin{pmatrix} 2 & -1 & 0 & \cdots \\ -1 & 2 & -1 & \cdots \\ 0 & -1 & 2 & \cdots \\ & & & \ddots \\ & & & \end{pmatrix} \begin{pmatrix} e^{y_1} \\ e^{y_2} \\ \\ \vdots \\ e^{y_r} \end{pmatrix} \tag{16.4}$$

for a lattice with $r+1$ points. The $r \times r$ matrix is actually the Cartan matrix for $SU(r+1)$, and there is an obvious resemblance between the lattice and the Dynkin diagram of the $SU(r+1)$ algebra. There are similar equations corresponding to all the algebras we classified involving their Cartan matrices K_{ij};

$$\frac{d^2 y_i}{dt^2} = -K_{ij} e^{y_j} \tag{16.5}$$

which we call the Toda molecule equations.

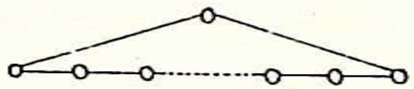
For the Toda lattice, corresponding to $SU(\infty)$, Toda found soliton solutions and Henon [1974] and Flaschka [1974] found an infinite number of conserved quantities. In fact the continuum limit of this case is the Korteweg-de Vries equation, which is well known to be an integrable system.

Another way of obtaining a stable system is to close the lattice-identifying x_n and x_0

This system is called the periodic Toda lattice and corresponds to (6.5) where

$$K_{ij} = \begin{pmatrix} 2 & -1 & 0 & & & -1 \\ -1 & 2 & -1 & & & \\ 0 & -1 & 2 & & & \\ & & & & & -1 \\ -1 & & & & -1 & 2 \end{pmatrix} \tag{16.6}$$

This is actually the Cartan matrix, not of a Lie algebra but of a Kac-Moody algebra, with Dynkin diagram which again looks just like the



lattice. In fact you can use the Cartan matrix for any Kac-Moody algebra and obtain a consistent integrable system. The solution for the Toda molecule (which we shall show later) uses the global structure of the Lie group. For Kac-Moody algebras the corresponding global group structure is not yet understood, and so the solutions here are more difficult to obtain. Nevertheless the methods of the inverse scattering problem have been applied.

<u>The Construction of a Lax Pair for the Toda Molecule Equations</u>

If we define ϕ by $y_a = \sum_{b=1}^{r} K_{ab}\phi_b$, then in terms of ϕ (16.5) reads (K_{ab} is nonsingular in this case)

$$\ddot{\phi}_a = -e^{K_{ab}\phi_b} \tag{16.7}$$

Consider now two generators A,B in the appropriate Lie algebra,

$$A = \tfrac{1}{2}(\dot{\phi}_a H_a + e^{\frac{1}{2}K_{ab}\phi_b}(E_a + E_{-a}))$$
$$B = \frac{i}{2}(e^{\frac{1}{2}K_{ab}\phi_b}(E_a - E_{-a})) \tag{16.8}$$

where the repeated indices are summed over 1, . . . rank (G) and G is the algebra. The E_a's are the step operators for the simple roots - with H_a they satisfy

$$[H_a, H_b] = 0$$
$$[H_a, E_b] = E_b K_{ba} \tag{16.9}$$
$$[E_a, E_{-b}] = \delta_{ab} H_b$$

Notice that both A and B are hermitian.

Then A and B obey the <u>Lax pair equation</u>; [Bogoyavlensky 1976, Leznov and Saveliev 1979]

$$\frac{dA}{dt} - i[B, A] = 0 \tag{16.10}$$

which is superficially like a Heisenberg equation of motion with B a Hamiltonian. To show this, evaluate

$$\frac{dA}{dt} = \frac{1}{2}\left[\ddot{\phi}_a H_a + \frac{1}{2}K_{ac}\dot{\phi}_c\, e^{\frac{1}{2}K_{ab}\phi_b}(E_a + E_{-a})\right]$$

$$-i[B,A] = \tfrac{1}{4}(e^{\frac{1}{2}K_{ab}\phi_b}\left[E_a - E_{-a}, \dot{\phi}_c H_c + e^{\frac{1}{2}K_{cd}\phi_d}(E_c + E_{-c})\right]$$

$$= \tfrac{1}{4}\, e^{\frac{1}{2}K_{ab}\phi_b}(-K_{ac}\dot{\phi}_c\,(E_a + E_{-a}) + 2\delta_{ac}\, e^{\frac{1}{2}K_{ad}\phi_d} H_c)$$

The sum of these is

$$\frac{dA}{dt} - i[B,A] = \tfrac{1}{2}H_a(\ddot{\phi}_a + e^{K_{ab}\phi_b}) \tag{16.11}$$

which is zero if the Toda equation holds. This construction converts a non-linear equation (16.7) into a linear one (16.10). It also shows the original equation is not ergodic. To see this, define

$$I_n = \mathrm{Tr}A^n \tag{16.12}$$

which is an n-th order polynomial in $\dot{\phi}_a$, and evaluate

$$\frac{dI_n}{dt} = \mathrm{Tr}(nA^{n-1}\dot{A})$$
$$= i\mathrm{Tr}(nA^{n-1}[B,A])$$
$$= i\mathrm{Tr}([B,A^n]) = 0 \tag{16.13}$$

by the cyclic property of the trace. The I_n are thus conserved, and so severely restrict the phase space of the system. In fact the number of functionally independent I_n turns out to be equal to the rank of the algebra involved which is the number of dynamical variables. This defines a completely <u>integrable</u> system as long as the Poisson brackets between the I_n vanish;

$$\{I_n, I_s\} = 0 \tag{16.14}$$

If (16.14) is true, as it is for all the Toda equations [Olive and Turok 1983b], the I_n can be chosen as canonical momenta which are constant and the conjugate variables θ obey the simple equation

$$-\dot{\theta} = \{H(I_n), \theta\} = \omega(I_n) \tag{16.15}$$

where H is the Hamiltonian.

In fact, in the Toda molecule case since $A^+ = A$, the I_n are all real and in particular,

$$I_2 = \mathrm{Tr}A^2 = \mathrm{Tr}AA^+ > 0$$

So I_2 at least is non-zero and being quadratic in $\dot{\phi}_a$ turns out to be the Hamiltonian for the system (16.7).

Lecture 17

The Lax pair A, B introduced in the last lecture has some very special properties which may be used to aid solution of the system. [Lax 1968].

Consider the eigenvalue problem for the Lax matrix A, which is finite and Hermitian,

$$A(t)|\psi_\lambda(t)\rangle = \lambda(t)|\psi_\lambda(t)\rangle \tag{17.1}$$

Then we have the isospectral property:

Theorem 17.

$$\frac{\partial\lambda}{\partial t} = 0 \tag{17.2}$$

which clearly implies that $\mathrm{Tr}A^n = \Sigma\, \lambda^n = $ constant. Hence the theorem is not surprising in view of our previous discussion. The proof uses the Lax equation (16.10);

$$\frac{\partial}{\partial t}(A|\psi_\lambda\rangle) = \dot{A}|\psi_\lambda\rangle + A|\dot{\psi}_\lambda\rangle = \dot{\lambda}|\psi_\lambda\rangle + \lambda|\dot{\psi}_\lambda\rangle \tag{17.3}$$

Multiplying on the left with $\langle\psi_\lambda|$, and using $\langle\psi_\lambda|A = \langle\psi_\lambda|\lambda$ (A is hermitian), we see

$$\begin{aligned}\dot{\lambda}\langle\psi_\lambda|\psi_\lambda\rangle &= \langle\psi_\lambda|i[B,A]|\psi_\lambda\rangle \\ &= \langle\psi_\lambda|(iB\lambda - \lambda iB)|\psi_\lambda\rangle \\ &= 0\,, \qquad\qquad \text{QED.}\end{aligned}$$

Now define U(t), the time evolution operator, by

$$\frac{dU}{dt} = iB(t)U(t)\ ; \qquad U(o) = 1 \tag{17.4}$$

Dyson's solution is formally the path-ordered exponential [Dyson 1949]

$$U(t) = \bar{T}\exp i\int_o^t B(t)dt \tag{17.5}$$

where T means that later time is moved to the left. Also,

$$\frac{dU^{-1}}{dt} = -U^{-1}\frac{dU}{dt}U^{-1} = -iU^{-1}B$$

so

$$U^{-1}(t) = T\exp -i\int_0^t B(t)dt \qquad (17.6)$$

where T means that later time is moved to the right. Next, observe that A(t) conjugated with U(t) obey

$$\frac{d}{dt}\left(U^{-1}(t)A(t)U(t)\right) = U^{-1}(-iB\dot{A} + A + iAB)U = 0 \qquad (17.7)$$

So

$$U^{-1}(t)A(t)U(t) = A(o)$$

or

$$A(t) = U(t)A(o)U^{-1}(t) \qquad (17.8)$$

which also shows that TrA^N is constant, and that

$$U^{-1}(t)A(t)U(t)|\psi_\lambda(o)\rangle = A(o)|\psi_\lambda(o)\rangle = \lambda|\psi_\lambda(o)\rangle \quad ,$$

so

$$A(t)U(t)|\psi_\lambda(o)\rangle = \lambda U(t)|\psi_\lambda(o)\rangle$$

or

$$|\psi_\lambda(t)\rangle = U(t)|\psi_\lambda(o)\rangle. \qquad (17.9)$$

So U really is the time evolution operator.

Obviously the various statements above are equivalent ways of seeing the same thing. In different systems, either one may be more appropriate. In the Lax Pair for some equations A is a differential operator, for instance in the Korteweg-de Vries equation $A = -\frac{d^2}{dx^2} + u(x,t)$. Then $Tr(A^N)$ is difficult (but still possible) to define. In this case however, the eigenvalue

approach is much easier;

$$\left(-\frac{d^2}{dx^2} + u(x,t)\right)\psi = \lambda\psi, \qquad (17.10)$$

which is just the Schroedinger equation describing a scattering problem when λ is positive.

Let us introduce now a formalism for path integrals which will be useful in solving the Toda molecule equations. The first thing we do is introduce an extra dimension x, upon which nothing depends, i.e. $\frac{dA}{dx} - \frac{dB}{dx} = 0$. Then we write

$$\frac{dA}{dt} - \frac{dB}{dx} = i[B,A]$$

or

$$\left[\frac{\partial}{\partial t} - iB,\ \frac{\partial}{\partial x} - iA\right] = 0 \qquad (17.11)$$

So if we write $B = W_t$ and $A = W_x$, where W_μ is a two dimensional gauge potential

$$[D_t, D_x] = -iF_{tx} = 0 \qquad (17.12)$$

where F_{tx} is the only component of the field strength in two dimensions, and $D_\mu = \partial_\mu - iW_\mu$ is the covariant derivative.

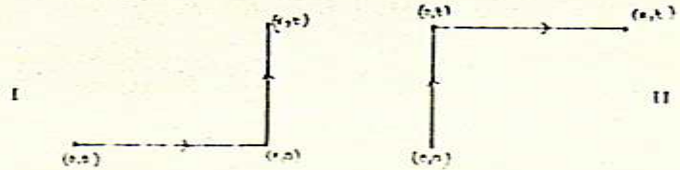
As discussed in Lecture 6, this means that W_μ is a pure gauge, i.e.,

$$-iW_\mu = g^{-1}\partial_\mu g \qquad (17.13)$$

from which one can solve for g:

$$g(x,t) = T\exp -i\int_{(o,o)}^{(x,t)} W_\mu dx^\mu \qquad (17.14)$$

taking $g(o,o) = 1$. The integral is path-independent because the curvature F_{xt} vanishes. Two possibilities for the contour are



From I,

$$g(x,t) = e^{-iA(o)x}\,U^{-1}(t)$$

while from II

$$g(x,t) = U^{-1}(t)\,e^{-iA(t)x}$$

and equality of these yields

$$A(t) = U(t)A(o)U(t)^{-1} \tag{17.15}$$

as before. The above method (using a zero curvature condition) is more powerful than the Lax method which may be regarded as a special case.

The point I want to make here is that it has become increasingly clear in recent years that the treatment of integrable dynamical system is often helped by the introduction of both the gauge theory concepts of lectures 1-6 and the Lie algebra theory of lectures 7-15.

Lecture 18

We devote this lecture to a glimpse of some of the results known so far about Toda systems. A full proof of these results requires knowledge of representation theory, a topic covered later in the course.

The first set of results for the Toda molecule equations

$$\ddot{\gamma}_a = -K_{ab}\,e^{\gamma b} \quad , \tag{18.1}$$

where K_{ab} is the Cartan matrix of a simple Lie algebra, is due independently to Olshanetsky and Perelomov [1979] and Kostant [1979]. They showed that the solution actually corresponds to geodesic motion on a symmetric space, such as a sphere, and the only complication comes in the projection onto the Cartan subalgebra of the isometry group.

This arises because the generators iH_i and $iE_{\pm\alpha}$, generate a noncompact <u>real form</u> of the Lie algebra G, called the normal form G^N. The compact real form is generated by H_i, $E_\alpha + E_{-\alpha}$, $\frac{E_\alpha - E_{-\alpha}}{i}$. We show G^N is noncompact by evaluating the Killing form:

$$
\begin{aligned}
&(1)\quad Tr(H_iH_j) = \delta_{ij} > 0 \\
&(2)\quad Tr((E_\alpha + E_{-\alpha})^2) = 2\,Tr(E_\alpha E_{-\alpha}) > 0 \\
&(3)\quad Tr((E_\alpha - E_{-\alpha})^2) = -2\,Tr(E_\alpha E_{-\alpha}) < 0
\end{aligned}
\tag{18.2}
$$

So (3) are the compact generators, (1) and (2) the noncompact ones.

In fact this construction is associated with an involution σ called a Cartan involution (an involution is an automorphism of order two);

$$
\begin{aligned}
\sigma(H_\alpha) &= -H_\alpha \\
\sigma(E_\alpha) &= -E_{-\alpha}
\end{aligned}
\tag{18.3}
$$

which clearly satisfies $\sigma^2 = 1$.

To see σ is an automorphism, which means it preserves the commutation relations, notice

$$[H_i, E_\beta] = K_{\beta i}E_\beta \ \overset{\sigma}{\to}\ [H_i, E_{-\beta}] = -K_{\beta i}E_{-\beta}$$

- the other commutation relations are easily checked to be preserved under σ.

The significance of having an involution σ of a Lie algebra is that since σ has eigenvalues ± 1 the algebra can be split into odd and even parts L_- and L_+:

$$\sigma(L_\pm) = \pm L_\pm \ .$$

If $\varepsilon, \eta = \pm 1$, we have using this and the invariance property,

$$\sigma([L_\varepsilon, L_\eta]) = [\sigma(L_\varepsilon), \sigma(L_\eta)] = \varepsilon\eta [L_\varepsilon, L_\eta] \ .$$

Hence

$$[L_+, L_+] \subset L_+ ; \quad [L_+, L_-] \subset L_- ; \quad [L_-, L_-] \subset L_+$$

Thus L_+ generates a subalgebra and the quotient of the group generated by $L_+ + L_-$ divided by the group generated by L_+ is a "symmetric space". [Helgason 1978]

For the σ defined by (18.3) the symmetric space is G^N/K where K is the maximal compact subalgebra of G^N and has generators $i(E_\alpha - E_{-\alpha})$. Olshanetsky and Perelomov [1979] showed that each solution to the Toda molecule equations (16.5) yielded a geodesic on G^N/K, essentially because $A \in L_-$. They use this result to construct an explicit solution (see lecture 29).

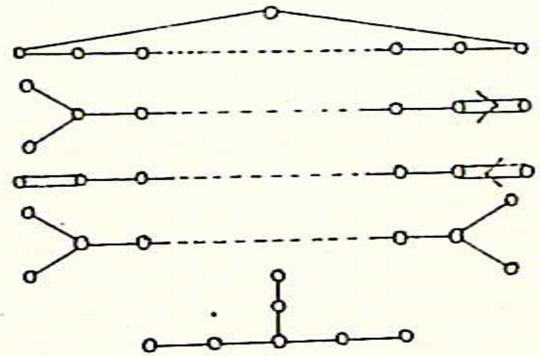
The second set of results on Toda systems relates to the two dimensional version

$$\partial^2 y_a = \frac{\partial^2}{\partial t^2} y_a - \frac{\partial^2}{\partial x^2} y_a = - K_{ab}\, e^{y_b} \qquad a = 1 \ldots r \qquad (18.4)$$

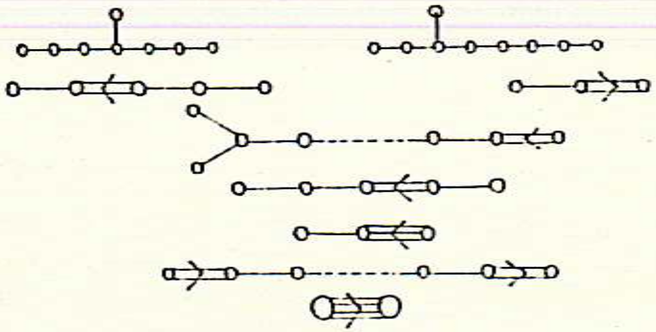
Where K_{ab} is an ordinary Cartan matrix, Leznov and Saveliev [1979] found a zero curvature condition (17.11) where there is now non-trivial x-dependence and used it to find the general solution in terms of r arbitrary functions of x + t and r arbitrary functions of x − t. The connection of these results to the geometric results above is not yet clear, but in my Poiana Brasov lectures [Olive 1982] I have given a derivation of their results somewhat similar to my treatment in lecture 29.

The third set of results concerns extending (18.1) to the case where K_{ab} is an extended or generalized Cartan matrix corresponding to a (infinite dimensional) Euclidean Kac-Moody algebra (which will nevertheless have finite rank and a finite number of simple roots). Integrability depended on the commutation relations (6.9), so this is equivalent to extending these relations. [Leznov and Saveliev 1979, Drinfeld and Sokolov 1981, Wilson 1981.] The appropriate Dynkin diagrams turn out to be:

(A)



(B)



A) are called extended Dynkin Diagrams,

B) are called generalized Dynkin Diagrams.

Both types are obtained by adding a single point to an ordinary Dynkin diagram. Olive and Turok [1983a] showed how all those of type (B) could be obtained by a 'reduction' procedure from those of type (A).

The above diagrams are the complete catalogue of known integrable equations of the form (16.5). The simplest extended Toda lattice equation comes from  which yields the equations

$$\partial^2\rho_1 = -2e^{\rho_1} + 2e^{\rho_2}$$

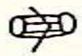
$$\partial^2\rho_2 = -2e^{\rho_2} + 2e^{\rho_1}$$

Since $\partial^2(\rho_1 + \rho_2) = 0$, if we may set $\rho_1 + \rho_2 = 0$ or $\rho_1 = -\rho_2 = \rho$ to obtain

$$\partial^2\rho = -2e^{\rho} + 2e^{-\rho} = -4 \sinh \rho\ , \qquad (18.5)$$

the sinh-Gordon equation. Setting $\rho = i\phi$, we obtain the sine-Gordon equation

$$\partial^2\phi = -4 \sin\phi\ . \qquad (18.6)$$

Actually K corresponding to the extended or generalized Dynkin diagrams is always singular, and the corresponding equations involve one less independent variable than the number of rows of K. The generalized diagram  yields

$$\partial^2\rho_1 = -2e^{\rho_1} + 4e^{\rho_2}$$

$$\partial^2\rho_2 = -2e^{\rho_2} + e^{\rho_1}$$

and so $\partial^2(\rho_1 + 2\rho_2) = 0$ (corresponding to the null vector of K). If we set $\rho_1 + 2\rho_2 = 0$, or $\rho_2 = \rho$, $\rho_1 = -2\rho$, we obtain

$$\partial^2\rho = e^{-2\rho} - 2e^{\rho} \qquad (18.7)$$

which is the Bullough-Dodd equation, [1977] known previously to possess many interesting properties connected to its integrability.

Finally I want to make a general comment on the fact that constructing an explicit solution to the equation (16.5) is more difficult to establish than the zero curvature or Lax condition. The reason is that the latter is a statement about the infinitesimal evolution of the system and therefore related to the underlying Lie algebra which deals with the infinitesimals of the Lie group. A solution on the other have depended on the global time evolution and correspondingly depends on the global properties of the group associated with the algebra in question. This is fairly well understood for simple Lie algebras but not for Kac-Moody algebras. As we shall see in lecture 29, the explicit solution of the Toda molecule equations depends on new notions like symmetric spaces, Iwasawa decompositions and the representation theory of the algebra.

Lecture 19

Symmetry Breaking in Gauge Theories

We again rapidly change subject here - to discuss spontaneous symmetry breaking in gauge theories. This will give us another immediate application of the Lie algebra theory above, and lead to a discussion of topological objects in spontaneously broken gauge theories - monopoles and vortices. Finally we shall have an opportunity to discuss the connection between the Toda equations of the previous lectures and self-dual magnetic monopoles.

One of the first problems encountered with gauge theories is how to give the gauge particles masses. One might try a term in the Lagrangian such as

$$\frac{1}{2} m^2 W_\mu W^\mu \quad , \tag{19.1}$$

but this is not gauge invariant and would spoil the good properties of gauge theories e.g., renormalizability. However $D_\mu = \partial_\mu - ieW_\mu$ is gauge covariant, so formally at least $\mathrm{Tr}(D_\mu^2)$ is gauge invariant and includes a W_μ^2 term. We introduce therefore a new scalar field ϕ and add to the Lagrangian the gauge-invariant term

$$(D_\mu\phi)^+ (D^\mu\phi) \text{ if } \phi \text{ is a complex field} \tag{19.2}$$

or $$\frac{1}{2}(D_\mu\phi)^+ (D_\mu\phi) \text{ if } \phi \text{ is real.}$$

The group indices are implicitly contracted. The mass squared given to the W_μ field will then be of order $e^2|\langle\phi\rangle|^2$ if ϕ has a non-zero vacuum expectation value $\langle\phi\rangle$. This process is called the Higgs Mechanism [Higgs 1964].

Let us see how this works in detail, for a complex field ϕ. The real case differs by a factor of $\frac{1}{2}$ everywhere. If $\langle\phi\rangle$ = constant, independent of space or time,

$$(D_\mu\langle\phi\rangle)^+ (D^\mu\langle\phi\rangle) = e^2\langle\phi\rangle^+ W^\mu W_\mu \langle\phi\rangle \quad . \tag{19.3}$$

and recalling that $W_\mu = T_a W_\mu^a$ where T_a are the generators of the gauge group,

$$\begin{aligned}(D_\mu\langle\phi\rangle)^+ (D^\mu\langle\phi\rangle) &= e^2\langle\phi\rangle^+ T_a T_b \langle\phi\rangle W_\mu^a W^{b\mu} \\ &= \frac{e^2}{2}\langle\phi\rangle^+ \{T_a, T_b\} \langle\phi\rangle W_\mu^a W^{b\mu} \\ &= \frac{1}{2} M_{ab}^2 W^{a\mu} W_\mu^b/\hbar^2\end{aligned} \tag{19.4}$$

where we have replaced $T_a T_b$ by its symmetric part since it is contracted with a symmetric term $W_\mu^a W^{\mu b}$. We have inserted $/\hbar^2$ in (9.4) to account for the fact that $-\hbar^2\partial^2 = p^2 = M^2$ where M is the physical mass of the gauge particle associated with the gauge field. Thus $M = m\hbar$ where m is the parameter occurring in the classical field Lagrangian (19.1). The mass squared matrix is therefore

$$(M^2)_{ab} = e^2\hbar^2\langle\phi\rangle^+ \{T_a, T_b\} \langle\phi\rangle \quad , \tag{19.5}$$

which is basically our earlier estimate with some group theoretical factors.

Let's now see how this works in the simplest model:

The U(1) Higgs model.

For this model, the Lagrangian is

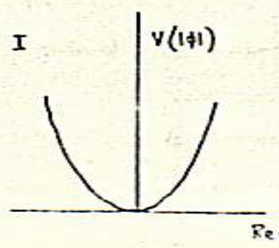
$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu\phi)^+ D^\mu\phi - V(|\phi|) \tag{19.6}$$

where ϕ is the complex scalar field. Under gauge transformations, ϕ, A_μ and $F_{\mu\nu}$ transform as

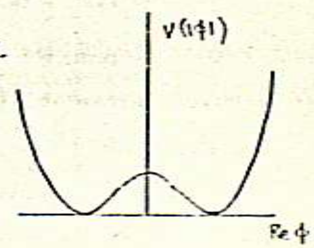
$$\begin{aligned}&\phi(x) \to e^{-iey(x)}\phi(x) \qquad A_\mu(x) \to A_\mu(x) + \partial_\mu\chi(x) \\ &F_{\mu\nu}(x) \to F_{\mu\nu}(x).\end{aligned} \tag{19.7}$$

Consider two possible potentials $V(|\phi|)$.

I



II



Both depend only on $|\phi|$ and so are actually paraboloids of revolution if we add a third axis Imϕ. In case I, $\langle\phi\rangle = 0$ in the vacuum but in case II, this point is not stable and the set of values $|\langle\phi\rangle| = \phi_o \neq 0$ would be chosen as the vacuum of the theory. Of course the set of all such $\langle\phi\rangle$ are just $\phi_o\, e^{i\theta}$, for any angle θ, and so correspond to a circle, all points of which can be seen from (19.7) to be related by gauge transformations. The circle is isomorphic to U(1).

Now let us see how the theory behaves around a particular constant vacuum, which we take as ϕ_o. We set

$$\phi = (\phi_o + \lambda)\, e^{i\theta} \tag{19.8}$$

where λ and θ are two new real fields. Note that if there were no gauge field, the kinetic part of the Lagrangian would be

$$\begin{aligned} |D_\mu\phi|^2 \to |\partial_\mu\phi|^2 &= |\partial_\mu\lambda e^{i\theta} + i(\phi_o + \lambda)\partial_\mu\theta\, e^{i\theta}|^2 \\ &= (\partial_\mu\lambda)^2 + (\phi_o + \lambda)^2\, (\partial_\mu\theta)^2 \end{aligned} \tag{19.9}$$

and the potential would depend on λ only. So we obtain from a single complex field two real fields λ and θ. One of them, θ, is massless, and corresponds to zero frequency oscillations around the possible set of vacua, a circle in this case. θ is called a <u>Goldstone particle</u> [Goldstone 1961].

However something special happens to θ in a gauge theory;

$$\begin{aligned} D_\mu\phi &= (\partial_\mu - ieA_\mu)\,(\phi_o + \lambda)e^{i\theta} \\ &= (\partial_\mu\lambda)e^{i\theta} - ie(A_\mu - \frac{\partial_\mu\theta}{e})\,(\phi_o + \lambda)e^{i\theta} \\ &= (\partial_\mu\lambda)e^{i\theta} - ie\, B_\mu\,(\phi_o + \lambda)e^{i\theta} \end{aligned} \tag{19.10}$$

where we define a new gauge potential $B_\mu = A_\mu - \dfrac{\partial_\mu\theta}{e}$. This is valid because it is simply a gauge transformation and so leaves $F_{\mu\nu}$ invariant.

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu B_\nu - \partial_\nu B_\mu$. Then θ disappears from the Lagrangian completely.

$$L = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + (\partial_\mu\lambda)^2 + e^2(B_\mu)^2\,(\phi_o + \lambda)^2 - V(\phi_o + \lambda)\,. \tag{19.11}$$

Expanding the third term,

$$e^2(B_\mu)^2\,(\phi_o + \lambda)^2 = e^2(B_\mu)^2\,\phi_o^2 + 2e^2(B_\mu)^2\phi_o\lambda + e^2(B_\mu)^2\lambda^2 \tag{19.12}$$

we see we have obtained a massive gauge field B_μ, with $m^2 = 2e^2\phi_o^2$, and a massive real scalar field λ interacting through Yukawa (linear) and quadratic couplings with B_μ. The would-be Goldstone particle θ has been 'eaten' by the gauge field which acquires a mass.

In the ground state of the theory, one expects

$$D_\mu\phi = 0 \tag{19.13}$$

which minimizes the kinetic term in ϕ. This implies

$$[D_\mu, D_\nu]\phi = ie\,F_{\mu\nu}\,\phi = 0 \tag{19.14}$$

which in the U(1) model tells you that $F_{\mu\nu} = 0$, i.e., the electromagnetic field vanishes. In fact the U(1) Higgs model is just the Landau-Ginzberg theory of superconductivity and this effect is just the Meissner effect. Magnetic field lines are forced into filaments whose energy is proportional to their length. Current theories of quark confinement are based on this idea.

Englert and Brout [1964], and Kibble [1967] extended the Higgs mechanism to more general compact gauge groups, taking the Lagrangian

$$L = -\frac{1}{4}\,\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) + (D_\mu\phi)^+\,(D_\mu\phi) - V(\phi) \tag{19.15}$$

We now have an arbitrary choice available for the representation ϕ is in – in fact the general problem of symmetry breaking is still unsolved and we only know the results in some special cases.

In vacuo, we take

$$D_\mu\langle\phi_o\rangle = 0\ ,\qquad V(\langle\phi_o\rangle) = 0 \tag{19.16}$$

$V(\phi)$ is gauge-invariant, so

$$V(\phi) = V(D(g)\phi)\ ,\qquad g \in G \tag{19.17}$$

where G is the gauge group. Say a particular value $\langle\phi_o\rangle \neq 0$ minimizes $V(\phi)$ and define

$$H_o = \{g \in G \;:\; D(g)\langle\phi_o\rangle = \langle\phi_o\rangle\} \tag{19.18}$$

$$= \text{subgroup of } G \text{ leaving } \phi_o \text{ invariant} .$$

H_o is called the <u>little group</u> or <u>stability group</u> of $\langle\phi_o\rangle$. It is the gauge symmetry of the vacuum and Lagrangian – that is the remaining exact symmetry of nature. In our U(1) example, G was completely broken, so $H_o = 1$.

Now $D_\mu\langle\phi_o\rangle = 0$ implies as before that $D(F_{\mu\nu})\langle\phi_o\rangle = 0$ where D is the representation ϕ is in. We also know that for any infinitesimal h in H_o, $h = e^{i\epsilon T} = 1 + i\epsilon T$, with ϵ small, we have

$$D(1 + i\epsilon T)\langle\phi\rangle = \langle\phi\rangle \Rightarrow D(T)\langle\phi\rangle = 0 .$$

So the generators of H_o are precisely those which annihilate $\langle\phi_o\rangle$. Thus $D(F_{\mu\nu})\langle\phi_o\rangle = 0$ implies that $F_{\mu\nu}$ is a generator of H_o, i.e., an H_o gauge field strength. These are the only fields which avoid the nonabelian Meissner effect.

Consider the <u>orbits</u> of $\langle\phi_o\rangle$ – the set of all $\phi = D(g)\langle\phi_o\rangle$ for any g in G. These all minimize the potential $V(\phi)$ by its gauge invariance property. Now two elements g_1 and g_2 correspond to ϕ_1 and ϕ_2 which are equal if

$$\begin{aligned} & D(g_1)\langle\phi_o\rangle = D(g_2)\langle\phi_o\rangle \\ \Rightarrow\; & D(g_2^{-1}g_1)\langle\phi_o\rangle = \langle\phi_o\rangle \\ \Rightarrow\; & g_2^{-1}g_1 = h \in H_o \\ \Rightarrow\; & g_1 = g_2 h . \end{aligned} \tag{19.19}$$

So g_1 and g_2 lie in the same <u>coset</u> with respect to H_o. We write the coset as G/H_o – it corresponds to the manifold of vacua of $V(\phi)$ obtained by gauge transformations of $\langle\phi_o\rangle$. Unless there is 'accidental symmetry' of $V(\phi)$ this will give the complete manifold of all vacuum states of the theory.

<u>Lecture 20</u>

<u>The Mass Term for Gauge Particles.</u>

Recall that for a complex Higgs field ϕ the mass squared matrix was given by

$$M^2_{ab} = e^2\hbar^2\langle\phi_o\rangle^+ D(\{T_a,T_b\})\langle\phi_o\rangle \tag{19.5}$$

M^2_{ab} is clearly real since the generators T_a are hermitian, and symmetrical. Thus it can be diagonalized by a real orthogonal transformation of the gauge fields W^a_μ. Such transformations automatically leave invariant the kinetic term for the gauge fields, $F^a_{\mu\nu} F^{a\mu\nu}$, so we can always diagonalize M^2_{ab} in a safe manner.

For the generators T_b of the exact symmetry H_o, we have

$$D(T_b)\langle\phi_o\rangle = 0 , \tag{20.1}$$

so the b'th row and column of M^2_{ab} vanish. When M^2_{ab} is diagonalized, this tells us that to each generator of H_o there corresponds a massless gauge field.

M^2_{ab} also turns out to be positive semidefinite, so its eigenvalues are non-negative. This is because for any real vectors x

$$\begin{aligned} x^a M^2_{ab} x^b &= 2e^2\hbar^2\langle\phi_o\rangle^+ (x\cdot T)^2 \langle\phi_o\rangle \\ &= 2e^2\hbar^2 |x\cdot T\langle\phi_o\rangle|^2 \geq 0 . \end{aligned} \tag{20.2}$$

So the gauge particles always get real masses.

If ϕ is in a real representation – for instance the adjoint representation – then

(i) We have an extra factor of $\frac{1}{2}$ in the kinetic energy term for the Higgs field and hence in M^2_{ab}.

(ii) $D(T)$ is pure imaginary for all generators T (elements of the group representation $e^{i\theta\cdot T}$ are all real), and since it is hermitian it must be antisymmetric.

Then

$$\langle\phi_o\rangle^T D(T) \langle\phi_o\rangle = 0 \tag{20.3}$$

However we can write

$$\langle\phi_o\rangle^T \frac{1}{2}\{T_a,T_b\}\langle\phi_o\rangle = \langle\phi_o\rangle^T (T_aT_b - \frac{1}{2}[T_a,T_b])\langle\phi_o\rangle \tag{20.4}$$

and the commutator term gives $if_{abc}T_c$ which vanishes by (20.3). Thus for ϕ in a real representation,

$$M^2_{ab} = e^2\hbar^2\langle\phi_o\rangle^T D(T_aT_b)\langle\phi_o\rangle \quad . \tag{20.5}$$

Let's look at some <u>examples</u>; for simplicity we take $G = SU(2)$.

1) ϕ is a <u>doublet</u>.

In the doublet representation (which is complex), we have

$$D(T_a) = \frac{1}{2}\tau_a \ , \tag{20.6}$$

τ_a being the Pauli matrices. Then

$$\begin{aligned} M^2_{ab} &= e^2\hbar^2\langle\phi_o\rangle^\dagger\{\tfrac{1}{2}\tau_a, \tfrac{1}{2}\tau_b\}\langle\phi_o\rangle \\ &= \frac{e^2\hbar^2}{2}\delta_{ab}|\langle\phi_o\rangle|^2 \end{aligned} \tag{20.7}$$

and we see that all three gauge particles acquire the same mass $\dfrac{e\hbar|\langle\phi_o\rangle|}{\sqrt{2}}$. There are no massless particles, and no exact symmetry group H_o remaining, so

$$G/H_o = SU(2)/1 = SU(2) \quad . \tag{20.8}$$

If the potential depends only on $|\phi|$, which is actually the only invariant available in this case, then the vacuum manifold is alternatively described by

$$|\langle\phi_o\rangle|^2 = (\mathrm{Re}\langle\phi_o\rangle_1)^2 + (\mathrm{Im}\langle\phi_o\rangle_1)^2 + (\mathrm{Re}\langle\phi_o\rangle_2)^2 + (\mathrm{Im}\langle\phi_o\rangle_2)^2 = \text{constant}.$$

This describes a three-sphere S_3. Actually it is well known that $SU(2)$ is isomorphic to S_3, so all this confirms our earlier analysis, that the vacuum manifold is G/H_o.

2) ϕ is a <u>real triplet</u>.

This is the adjoint representation of $SU(2)$, and corresponds to a three-component vector under rotations. For the adjoint representation, we know that

$$D_{ij}(T_a) = i f_{iaj} = i\epsilon_{iaj} \tag{20.9}$$

So

$$\begin{aligned} M^2_{ab} &= -e^2\hbar^2\langle\phi_o\rangle_i\,\epsilon_{iaj}\,\epsilon_{jbk}\langle\phi_o\rangle_k \\ &= -e^2\hbar^2\langle\phi_o\rangle_i(\delta_{ib}\delta_{ak} - \delta_{ik}\delta_{ab})\langle\phi_o\rangle_k \\ &= e^2\hbar^2(\langle\phi_o\rangle^2\delta_{ab} - \langle\phi_o\rangle_a\langle\phi_o\rangle_b) \quad , \end{aligned} \tag{20.10}$$

and we see that components of W^a_μ parallel to $\langle\phi_o\rangle$ have $M^2 = 0$. Components perpendicular to $\langle\phi_o\rangle$ have $M^2 = e^2\hbar^2\langle\phi_o\rangle^2$. Thus one component is massless and two massive. In particular, if we rotate $\langle\phi_o\rangle$ so that

$$\langle\phi_o\rangle_a = |\langle\phi_o\rangle|\delta_{a3} \tag{20.11}$$

then $M_{33} = 0$, $M_{11} = M_{22} = e\hbar|\langle\phi_o\rangle|$. Thus H_o has a single generator T_3 corresponding to rotations about the 3 axis and so is just $U(1)$, or $SO(2)$. This gauge theory was once a candidate for the weak interactions, but has only one neutral current - the electromagnetic (H) one - and is so ruled out. We have

$$G/H_o = SU(2)/U(1) = S_2 \tag{20.12}$$

which again is seen to be the same manifold as $|\phi|^2$ = constant. Recalling ϕ is a three vector, the vacuum manifold is thus the surface of a two-sphere.

Constructing Unified Theories

We have yet to observe Higgs particles and they may therefore be regarded as ugly features of the theory. However we do observe an exact symmetry group H_0, just Maxwell's U(1) (neglecting colour). We infer G from the fermion content, using the symmetries of their interactions to tell us which representations of which groups they lie in. In fact the fermions tend to be chiral and so massless if G is exact, and they too (like the gauge particles) get their masses from Higgs fields.

In the Salam-Weinberg model' of the electro-weak interactions, G is SU(2) x U(1). The electron and neutrino form an SU(2) doublet. One only needs a single Higgs field to give one of the fermions and three gauge particles a mass (leaving the photon and neutrino massless), breaking $SU(2) \times U(1) \to U(1)$.

We shall not pursue phenomenology further here, but turn to an unexpected feature of spontaneously broken gauge theories - the existence of topologically stable objects: monopoles, vortex lines or strings, and domain walls, whose masses are again determined by the Higgs' fields.

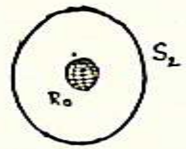
Lecture 21

Topological objects in Spontaneously Broken Gauge Theories

There exist topologically stable objects in 3+1 dimensional Minkowski spacetime, but only in spontaneously broken gauge theories, not in unbroken theories as we saw occur in 4 Euclidean dimensions. Their classification is surprisingly simple, and requires only knowledge of the homotopy groups of the vacuum manifold G/H_0.

Suppose at time t_0 we have a field configuration in which $D_\mu\phi = 0$, and ϕ is in the vacuum manifold G/H_0 everywhere except in a region R_0. For our first example, take

1) R_0 is a ball, a three-dimensional region.



The value of the Higgs field ϕ on the two-dimensional sphere S_2 surrounding R_0 provides a map from S_2 to G/H. Consider the homotopy classes of all possible ϕ's (two configurations $\phi(x)$ and $\phi'(x)$ are said to be in the same homotopy class if they are smoothly deformable into one another, that is if there exists a smooth function $F(x,t)$ such that $F(x,0) = \phi(x)$ and $F(x,1) = \phi'(x)$). Obviously any stretching of S_2 does not alter the homotopy class of ϕ. In fact its homotopy class is independent of

(1) Choice of S_2 provided it encloses R_0

(2) Choice of gauge, provided no non-singular gauge transformations are made

1) (continued)

(1) time, as long as $\phi(x)$ evolves smoothly via some equation of motion.

The set of all homotopy classes of $\phi: S_2 \to G/H_o$ actually forms a finite group, called $\pi_2(G/H)$, which is the group of conserved topological 'quantum' numbers of these objects under addition as in Lecture 6. [Tyupkin et. al. (1975), Monastyrsky and Perelomov (1975)]

An example of a theory in which $\pi_2(G/H_o)$ is nontrivial is just the model where $G = SU(2)$ and ϕ is a triplet which breaks it to $U(1)$ described in (20.12). Here $\pi_2(G/H_o) = \pi_2(S_2) = \mathbb{Z}$, the group of integers, corresponding to wrapping around G/H_o 0, ±1, ±2,...±n... times as one wraps around the sphere S_2 once. [In fact for an n-sphere $\pi_n(S_n) = \mathbb{Z}$. The simplest case is the circle S_1, where we may have $\phi_n = e^{in\theta}$ for any integer n, and such ϕ_n's are not deformable into each other]. This theory therefore allows spherical topologically stable objects which turn out to be magnetic monopoles of charge n, as first discovered by t'Hooft and Polyakov [1974]. The conserved quantum numbers n are called <u>winding numbers</u>.

To see how this is possible, recall that in the vacuum, $D_i\phi = 0$ so $D(F_{ij})\phi = 0$. Inside R_o, however, Maxwell's theory breaks down. You only have clearly defined magnetic flux outside R_o. Choose a gauge where $W_r = 0$ (which is always possible). Then outside R_o,

$$D_i\phi = (\partial_i - ieD(W_i))\phi = 0$$

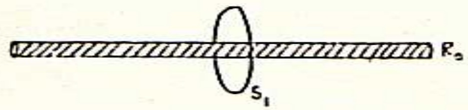
$$\Rightarrow \quad \partial_r\phi = 0$$

while $\frac{1}{r}\frac{\partial}{\partial\theta}\phi = ieD(W_\theta)\phi$ for a transverse derivative. (21.1)

1) (continued)

Thus we need $W_\theta \sim \frac{1}{er}$ and $B_r = \partial_r W_\theta \sim \frac{1}{er^2}$, which describes an inverse square law U(1) field of a magnetic monopole with charge $g \sim 1/e$. It turns out to obey Dirac's quantisation condition precisely. For more details see the review article Goddard and Olive [1978]. For our second example of a topologically stable object, consider

2) R_o is a long thin cylinder



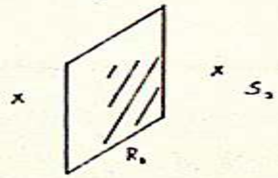
Consider a circle S_1 surrounding R_o. As before, ϕ is a map from S_1 to G/H. The homotopy classes are labelled by $\pi_1(G/H_o)$ and in this case the objects are called vortex lines or strings.

An example of a model with vortex lines is the Abelian Higgs model discussed previously. This was seen by Nielsen and Olesen [1973]. Here $G = U(1)$, $H_o = 1$ so

$$G/H_o = U(1) = S_1$$

and $\pi_1(G/H_o) = \pi_1(S_1) = \mathbb{Z}$ as before.

The last case is where R_o is a two-dimensional surface;



Then S_o is the zero dimensional sphere, consisting of two points, one on either side of the 'wall' R_o (to see S_o is just two points, notice that S_n is given by the manifold $x_1^2 + \ldots x_n^2 = a^2$, so S_o is $x^2 = a^2$ or $x = \pm a$). The homotopy classes relevant here are $\pi_o(G/H_o)$.

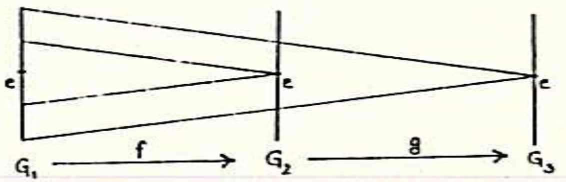
2) (continued)

We end this lecture by mentioning a major mathematical theorem which simplifies the task of working out homotopy classes considerably. It says that there exists an exact sequence of homotopy groups:

$$\rightarrow\pi_2(H_o)\rightarrow\pi_2(G)\rightarrow\pi_2(G/H_o)\rightarrow\pi_1(H_o)\rightarrow\pi_1(G)$$

$$\rightarrow\pi_1(G/H_o)\rightarrow\pi_o(H_o)\rightarrow\pi_o(G)\rightarrow\pi_o(G/H_o) \qquad (21.2)$$

(actually the sequence extends indefinitely beyond π_2 as well). An exact sequence is one where at each stage there is a homomorphism between the two groups, and the image of any homomorphism is isomorphic to the kernel of the subsequent one. Diagrammatically:



The diagram shows how the image of f (the part of G_2 onto which G_1 is mapped) is equal to the kernel of g (the part of G_2 mapped to the identity of G_3).

Now for any Lie group, $\pi_2(G) = 1$, and if G is semisimple, $\pi_1(G)$ is a finite discrete group. If we consider as G a simply connected group, then $\pi_1(G) = 1$, and (21.2) tells us (with a little group theory) that

$$\pi_2(G/H_o) = \pi_1(H_o) \qquad (21.3)$$

2) (continued)

This is one way of classifying magnetic monopoles (due to Lubkin [1963], and Wu and Yang [1975]). In our example, the t'Hooft-Polyakov monopole, we saw $\pi_2(G/H_o) = \mathbb{Z}$, and $\pi_1(H_o) = \pi_1(U(1)) = \pi_1(S_1) = \mathbb{Z}$, so (21.3) is verified in this case. Likewise, for vortex lines if $\pi_1(G) = 1$ (which it is not in the Nielsen-Olesen example), and $\pi_o(G) = 1$ (which means G is connected), then

$$\pi_1(G/H_o) = \pi_o(H_o) \qquad (21.4)$$

Some examples of this situation were discussed by Olive and Turok [1982] in the context of theories of the grand unified type.

Lecture 22

The possibilities of symmetry breaking seem endless, since the Higgs field may be chosen in any representation, and in any direction within that representation. There is no general theory at present, so we'll restrict ourselves here to discussing a particular case where the answer is known, and which is relevant in physics.

Higgs Fields in the Adjoint Representation

Recall the definition of the adjoint representation, whose matrices are $d_{ij}(g)$:

$$gT_ig^{-1} = T_jd_{ji}(g) \tag{3.1}$$

and recall also that (eqns. (7.4) and (7.10))

$$d(T_a)_{ij} = if_{iaj} \tag{7.4}$$

where $[T_i, T_j] = if_{ijk}T_k$. We want to find the generators of H_o, the exact symmetry group, given a Higgs field vacuum expectation value $\langle\phi\rangle$. Any generator T of H_o must satisfy

$$\begin{aligned} 0 &= (d(T)\langle\phi\rangle)_i \\ &= d_{ij}(\beta_aT_a)\langle\phi\rangle_j \\ &= i\beta_af_{iaj}\langle\phi\rangle_j \\ &= i\beta_af_{aji}\langle\phi\rangle_j \end{aligned} \tag{22.1}$$

using the total antisymmetry of f_{ijk}. But (22.1) is true if and only if

$$[\beta_aT_a, \langle\phi\rangle_jT_j] = i\beta_af_{aji}\langle\phi\rangle_jT_i = 0 \tag{22.2}$$

So defining

$$\Phi = \langle\phi\rangle_jT_j \tag{22.3}$$

we have

$$[T,\Phi] = 0 \iff T \text{ is a generator of } H_o \ . \tag{22.4}$$

One solution for T is obviously Φ itself, which clearly commutes with all other generators of H_o. Thus H_o is not semisimple (as it is not in nature!) and Φ generates an invariant (self-conjugate) U(1) subgroup of it,

$$U(1)_\Phi = e^{i\theta\Phi} \tag{22.5}$$

We can therefore write

$$H = "U(1)_\Phi \times K" \tag{22.6}$$

where " " indicates the local structure is $U(1)_\Phi \times K$. However we don't yet know whether this $U(1)_\Phi$ is compact or noncompact - that is if it is a circle ((22.5) periodic in θ) or a real line ((22.5) not periodic in θ). One might think that since G is taken to be compact (for positive definiteness of the Hamiltonian) in a Grand Unified Theory, it could not have a noncompact U(1) subgroup. But this is not true, as we shall see later, and it has to be shown that $U(1)_\Phi$ is compact in order both for monopoles to exist and charges to be quantised.

Now Φ has a simple interpretation in the theory - it is simply the generator of electric charge. Recall the covariant derivative is defined as

$$D_\mu = \partial_\mu - ieD(T_a)W^a_\mu \tag{2.11}$$

If A_μ is taken to be the component of the gauge field along Φ, i.e.

$$A_\mu = \frac{\langle\phi\rangle^aW^a_\mu}{a} \ , \quad a^2 = \langle\phi\rangle^2, \tag{22.7}$$

Then A_μ is the $U(1)_\Phi$ gauge field and its coefficient in D_μ will be

$$-ie\,\frac{D(T^a)\langle\phi\rangle^a}{a} = -ie\,\frac{D(\Phi)}{a} \tag{22.8}$$

The covariant U(1) derivative is

$$\partial_\mu - \frac{iD(Q)}{h}A_\mu \tag{22.9}$$

with Q the charge matrix, and we see that

$$D(Q) = \frac{e\hbar\, D(\hat{Q})}{a} \qquad (22.10)$$

So if $U(1)_Q$ is compact, the group generated by Q is compact. That is there is some nonzero angle α such that

$$e^{i\alpha Q} = 1 \qquad (22.11)$$

Then if the eigenvalues of Q are q, we must have $\alpha q = 2\pi n$, so electric charge is quantised if $U(1)_Q$ is compact [Yang 1970].

As an example, consider SU(2). Any compact generator may be rotated into T_3 which generates a U(1) subgroup, and since

$$T_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} / 2 \qquad (22.12)$$

we see that its eigenvalues are quantised. Thus all U(1) subgroups of SU(2) are compact.

This is not the case for SU(3). The diagonal Gell-Mann matrices are

$$\lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \qquad (22.13)$$

The U(1) subgroup generated by either one of these is compact since all eigenvalues are integer multiples of each other. However $\lambda_3 + \lambda_8$, which is an equally good generator of H_o, has eigenvalues in ratios $1 + 1/\sqrt{3}$: $-1 + 1/\sqrt{3}$: $-2/\sqrt{3}$ which are irrational. Hence exponentiating $\lambda_3 + \lambda_8$ (or $\alpha\lambda_3 + \beta\lambda_8$ apart from special values of α and β) gives a noncompact U(1) group. What happens is that $e^{i\theta(\lambda_3+\lambda_8)}$ winds around a torus (which is the direct product of $U(1)_{\lambda_3}$ and $U(1)_{\lambda_8}$) and never comes back to the same point. In fact the U(1)'s generated by λ_3 and λ_8 are seen to be very special cases in this respect.

The important thing however is that this never happens if K is semisimple, for then it is compact and since H_o is compact, it follows that $U(1)_Q$ is compact. We see that charge is quantised if G and K are both semisimple - but not necessarily if only G is simple.

Now one only obtained $\mathbb{Z}$ magnetic monopoles in the theory if (21.3)

$$\pi_2(G/H_o) = \pi_1(H_o) = \pi_1(U(1)_Q \times K) = \mathbb{Z} \qquad (22.14)$$

which is true if and only if $U(1)_Q$ is compact (Goddard and Olive [1981a]). Thus charge quantisation and the existence of magnetic monopoles in grand unified theories are closely related, as they were in Dirac's earliest considerations [1931]. The quantisation condition (Corrigan and Olive [1976]) relating electric to magnetic charge turns out to depend on K and is simply that

$$e^{ig\,Q/\hbar} = k, \text{ an element of } K. \qquad (22.15)$$

In Dirac's case, K and hence k was 1 and he simply obtained $gQ = 2\pi n\hbar$. We shall see how this generalises to larger unbroken gauge groups K in the next lecture. For a proof of (22.15) see the review [Goddard and Olive 1978].

Lecture 23

We saw in the last lecture how a higgs field Φ in the adjoint representation breaks the original compact gauge group G to

$$H = \text{"}U(1)_Q \times K\text{"} \qquad (22.6)$$

and that if the color group K is semisimple, the charge Q is quantised. We also noted that the magnetic charge

$$g = \int ds_i B^i_a \phi^a / |\phi|$$

obeys the generalized quantisation condition

$$e^{igQ/\hbar} = k \in K. \qquad (23.1)$$

Let us examine the consequences of (23.1). If a particle state $|\psi\rangle$ is a singlet under K, then

$$e^{igQ/\hbar}\,|\psi\rangle = k|\psi\rangle = |\psi\rangle$$

and if $Q|\psi\rangle = q|\psi\rangle$ defines its electric charge, then (23.1) implies

$$e^{igq/\hbar} = 1$$

So gq obeys Dirac's quantisation condition. To analyse states which are not K singlets, we note from (23.1) k is both in the $U(1)_Q$ subgroup of H and in the K subgroup. Because it is in the $U(1)_Q$ subgroup it must commute with all elements of K and so must in fact be in the center of K, Z(K). For example, if the colour group K is SU(3), by Schur's lemma k must be a multiple of the identity so

$$k = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

However for k to satisfy det k = 1, we see that the only possibilities are $\lambda = 1,\ e^{2\pi i/3},\ e^{4\pi i/3}$. These three numbers are the elements of the cyclic group of order three, Z_3, to which the center of SU(3) is, as we have shown, isomorphic.

In fact it is a general property of Lie groups that for K semisimple Z(K) is a finite abelian group.

To continue with our SU(3) example, if we took a quark state $|q\rangle$, then

$$k|q\rangle = \omega|q\rangle$$

where $\omega = 1,\ e^{2\pi i/3}$ or $e^{4\pi i/3}$, and so by the definition of k, (23.1),

$$e^{igq/\hbar} = e^{2\pi i/3},\ e^{4\pi i/3} \text{ or } 1$$

where if $e^{2\pi i/3}$ is selected may be translated as

$$gq = 2\pi\hbar\,(n + 1/3) \qquad (23.2)$$

rather than $2\pi\hbar n$ in Dirac's condition. Thus quark states can have fractional charges and still obey the quantisation condition. Note that the q charge has to do with color, not flavor, which has to do with a global symmetry. One may easily calculate $|Z(K)|$, the number of elements of Z(K) for more general K by

<u>Theorem 23.1:</u>

If K is simply connected then the number of elements of Z(K) is

$$|Z(K)| = \det \kappa$$

where κ is its Cartan matrix. (See equation (12.1)).

For SU(3), as may be seen from the Dynkin diagram ✕——✕ the Cartan matrix is

$$\kappa = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and $\det \kappa = 3$.

Finally let me explain the quotation marks in H = "$U_Q(1) \times K$". In fact H is not a true direct product, as evidenced above by the existence with Q the charge matrix, and we see that

$$D(Q) = \frac{e\hbar\, D(\text{[illegible]})}{a} \tag{22.10}$$

So if $U(1)_Q$ is compact, the group generated by Q is compact. That is there is some nonzero angle α such that

$$e^{i\alpha Q} = 1 \tag{22.11}$$

Then if the eigenvalues of Q are q, we must have $\alpha q = 2\pi n$, so electric charge is quantised if $U(1)_Q$ is compact [Yang 1970].

As an example, consider $SU(2)$. Any compact generator may be rotated into T_3 which generates a $U(1)$ subgroup, and since

$$T_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}/2 \tag{22.12}$$

we see that its eigenvalues are quantised. Thus all $U(1)$ subgroups of $SU(2)$ are compact.

This is not the case for $SU(3)$. The diagonal Gell-Mann matrices are

$$\lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \tag{22.13}$$

The $U(1)$ subgroup generated by either one of these is compact since all eigenvalues are integer multiples of each other. However $\lambda_3 + \lambda_8$, which is an equally good generator of H_0, has eigenvalues in ratios $1 + 1/\sqrt{3} : -1 + 1/\sqrt{3} : -2/\sqrt{3}$ which are irrational. Hence exponentiating $\lambda_3 + \lambda_8$ (or $\alpha\lambda_3 + \beta\lambda_8$ apart from special values of α and β) gives a noncompact $U(1)$ group. What happens is that $e^{i\theta(\lambda_3+\lambda_8)}$ winds around a torus (which is the direct product of $U(1)_{\lambda_3}$ and $U(1)_{\lambda_8}$) and never comes back to the same point. In fact the $U(1)$'s generated by λ_3 and λ_8 are seen to be very special cases in this respect.

The important thing however is that this never happens if K is semisimple, for then it is compact and since H_0 is compact, it follows that $U(1)_Q$ is compact. We see that charge is quantised if G and K are both semisimple - but not necessarily if only G is simple.

Now one only obtained $\mathbb{Z}$ magnetic monopoles in the theory if (21.3)

$$\pi_2(G/H_0) = \pi_1(H_0) = \pi_1(U(1)_Q \times K) = \mathbb{Z} \tag{22.14}$$

which is true if and only if $U(1)_Q$ is compact (Goddard and Olive [1981a]). Thus charge quantisation and the existence of magnetic monopoles in grand unified theories are closely related, as they were in Dirac's earliest considerations [1931]. The quantisation condition (Corrigan and Olive [1976]) relating electric to magnetic charge turns out to depend on K and is simply that

$$e^{ig\,Q/\hbar} = k, \text{ an element of } K. \tag{22.15}$$

In Dirac's case, K and hence k was 1 and he simply obtained $gQ = 2\pi n\hbar$. We shall see how this generalises to larger unbroken gauge groups K in the next lecture. For a proof of (22.15) see the review [Goddard and Olive 1978].

Lecture 23

We saw in the last lecture how a higgs field Φ in the adjoint representation breaks the original compact gauge group G to

$$H = \text{"}U(1)_Q \times K\text{"} \tag{22.6}$$

and that if the color group K is semisimple, the charge Q is quantised. We also noted that the magnetic charge

$$g = \int ds_i B^i_a \Phi^a / |\Phi|$$

obeys the generalized quantisation condition

$$e^{igQ/\hbar} = k \in K. \tag{23.1}$$

Let us examine the consequences of (23.1). If a particle state $|\phi\rangle$ is a singlet under K, then

$$e^{igQ/\hbar}|\phi\rangle = k|\phi\rangle = |\phi\rangle$$

and if $Q|\phi\rangle = q|\phi\rangle$ defines its electric charge, then (23.1) implies

$$e^{igq/\hbar} = 1$$

So gq obeys Dirac's quantisation condition. To analyse states which are not K singlets, we note from (23.1) k is both in the $U(1)_Q$ subgroup of H and in the K subgroup. Because it is in the $U(1)_Q$ subgroup it must commute with all elements of K and so must in fact be in the center of K, Z(K). For example, if the colour group K is SU(3), by Schur's lemma k must be a multiple of the identity so

$$k = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

However for k to satisfy det k = 1, we see that the only possibilities are $\lambda = 1, e^{2\pi i/3}, e^{4\pi i/3}$. These three numbers are the elements of the cyclic group of order three, Z_3, to which the center of SU(3) is, as we have shown, isomorphic.

In fact it is a general property of Lie groups that for K semisimple Z(K) is a finite abelian group.

To continue with our SU(3) example, if we took a quark state $|q\rangle$, then

$$k|q\rangle = \omega|q\rangle$$

where $\omega = 1, e^{2\pi i/3}$ or $e^{4\pi i/3}$, and so by the definition of k, (23.1),

$$e^{igq/\hbar} = e^{2\pi i/3}, e^{4\pi i/3} \text{ or } 1$$

where if $e^{2\pi i/3}$ is selected may be translated as

$$gq = 2\pi\hbar\,(n + 1/3) \tag{23.2}$$

rather than $2\pi\hbar n$ in Dirac's condition. Thus quark states can have fractional charges and still obey the quantisation condition. Note that the q charge has to do with color, not flavor, which has to do with a global symmetry. One may easily calculate $|Z(K)|$, the number of elements of Z(K) for more general K by

<u>Theorem 23.1:</u>

If K is simply connected then the number of elements of Z(K) is

$$|Z(K)| = \det \kappa$$

where κ is its Cartan matrix. (See equation (12.1)).

For SU(3), as may be seen from the Dynkin diagram ×——× the Cartan matrix is

$$\kappa = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and $\det \kappa = 3$.

Finally let me explain the quotation marks in $H = \text{"}U_Q(1) \times K\text{"}$. In fact H is not a true direct product, as evidenced above by the existence of k, a common element of $U(1)_Q$ and K. The correct global structure is

$$H = U(1)_Q \times K/Z \tag{23.3}$$

where Z is some cyclic subgroup of the centre of K, assumed semisimple. For example if K = SU(3) we can have

$$H = \frac{U(1)_Q \times SU(3)}{Z_3} = U(3), \tag{23.4}$$

so H is just the unitary group in three dimensions.

Gauge particle masses from a Higgs field in the adjoint representation

In this section we generalise the example in lecture 20 (Eq. 20.10) to an arbitrary semisimple gauge group. Since the adjoint representation is real, we can use

$$M^2_{ab} = e^2\hbar^2 \langle\phi_o\rangle_i \, d_{ij}\,(T_aT_b)\langle\phi_o\rangle_j \tag{20.5}$$

Henceforth we drop the expectation value notation for ϕ.

In the adjoint representation, (see equation (7.4))

$$d_{ij}(T_a) = if_{iaj} \tag{7.4}$$

So

$$M^2_{ab} = (ie\hbar)^2 \phi_i f_{iaj} f_{jbk} \phi_k$$
$$= (ie\hbar)^2 \phi_i f_{aij} f_{jkb} \phi_k \tag{23.5}$$

by the total antisymmetry of f_{ijk}. However

$$ie\phi_i f_{aij} = e\, d(T_i\phi_i)_{aj}$$

and we have $e\hbar\, T_i\phi_i = a\,Q$ from (22.10), where a is the length of $\langle\phi_o\rangle$.

So

$$M^2_{ab} = a^2\, d_{aj}(Q)\, d_{jb}(Q)$$
$$= a^2\, d^2_{ab}(Q). \tag{23.6}$$

So the charge squared matrix is the mass squared matrix. Suppose Q has eigenvalues q_1, q_2 ... Then the mass eigenstates are charge eigenstates and have masses

$$M = |a\,q| \tag{23.7}$$

Note that $\hbar$ has cancelled in the relation between the physical mass and U(1) charge.

Now if K is semisimple, q is quantised and so therefore are the masses. However, this isn't terribly important in physical cases, where one finds n = 0, ± 1 only, as we shall see later.

Monopole Masses

Let us compare the gauge particle masses (23.7) with the masses of magnetic monopoles. We can find a lower bound on these as follows: The Hamiltonian is given by

$$E = \frac{1}{2}\int d^3x \; \{E_i^{a2} + B_i^{a2} + (D_i\phi)^2 + (D_o\phi)^2 + V(\phi)\} \tag{23.8}$$

where $V(\phi) \geq 0$ and vanishes in the ground state. To estimate monopole masses, [Bogomolny,1976] write

$$(B_i^a)^2 + (D_i\phi^a)^2 = (B_i^a \mp D_i\phi^a)^2 \pm 2\, B_i^a\,(D_i\phi^a)$$

(which is only possible for ϕ in the adjoint representation). Hence

$$E \geq \pm \int d^3x\, B_i^a D_i \phi^a$$

and integrating by parts we see

$$\pm\int d^3x\,[\partial_i(B_i^a\phi^a) - (D_iB_i)^a\phi^a]$$

The second term vanishes by the Bianchi identity and the first is a surface term. So

$$E \geq \pm \int dS_i\, B_i^a\phi^a = \pm\, ag = |ag| \tag{23.9}$$

taking the + or − sign as g is positive or negative respectively.

This is an important result, stating that the Higgs field sets the energy scale. For a monopole at rest, we have

$$M_{mon} \geq |ag|.$$

Compared to the gauge particle mass

$$M_x = |aq|$$

Thus $M_{mon} \geq M_x\,(g/q)$ and if $gq \sim 2\pi\hbar$ we have

$$M_{mon} \geq M_x\, 2\pi\hbar/q^2 = M_x/\alpha \sim 137\, M_x \qquad (23.10)$$

since α is just the fine structure constant. Can one saturate this lower bound? If $E_i = D_0\phi = 0$ then

$$E = \frac{1}{2}\int d^3x[(B_i^a \mp D_i\phi^a)^2 + V(\phi)] \pm ag$$

and if $V(\phi) = 0$ the equality $M_{mon} = |ag|$ is satisfied as long as

$$B_i^a = \pm D_i\phi^a \qquad (23.11)$$

Taking $V(\phi) = 0$ is called the <u>Prasad-Sommerfield</u> limit [1975]; (23.11) is called the <u>Bogomolny</u> equation [1976]. For configurations satisfying it we have

$$M_{mon} = |ag| \text{ and } M_x = |aq|$$

which we can unify in a single formula invariant under rotations between q and g, namely (Montonen and Olive (1977)).

$$M = |a|\sqrt{q^2 + g^2}. \qquad (23.12)$$

This in fact also holds for electrically charged monopoles, called <u>dyons</u>. Of course the surviving scalars have $M = g = q = 0$. So in fact <u>all</u> particles satisfy this formula for any G which is compact and semisimple, providing the Higgs field lies in the adjoint representation.

<u>Lecture 24</u>

<u>Extra Dimensions</u>

How is it that the universal mass formula (23.12) comes about? As we shall see it has a natural interpretation in terms of extra dimensions. However let us first see how the Bogomolny equations (23.11) relate to the self or anti-self duality equations (5.8). This will explain the terminology 'self-dual' magnetic monopoles.

The higgs field ϕ, lying in the adjoint representation, may be thought of as an extra component to the gauge field,

$$\phi^a = W_5^a \qquad (24.1)$$

where $\partial_5 \equiv \frac{\partial}{\partial x_5} = 0$ on everything in the theory. We have four extra components to the field strength;

$$F_{\mu 5} = \partial_\mu W_5 - \partial_5 W_\mu - ie[W_\mu, W_5]$$

$$= D_\mu W_5 = D_\mu\,\phi$$

and a new term in the Lagrangian

$$-\frac{1}{4}\sum_{\substack{\mu,\nu\\=0,1\ldots 5}} (F_{\mu\nu}^a)^2 = -\frac{1}{4}\sum_{\substack{\mu,\nu\\=0,1\ldots 4}} (F_{\mu\nu}^a)^2 + (D_\mu\phi)^2 \qquad (24.2)$$

which has the right sign to be the kinetic term for the ϕ field if the extra dimension is spacelike. Notice the absence of a potential $V(\phi)$ term - which would not be gauge invariant. With this interpretation of ϕ we can write the Bogomolny equations

$$D_i\phi = \pm B_i = \pm\frac{1}{2}\epsilon_{ijk}F_{jk}$$

as

$$F_{i5} = \pm\epsilon_{5ijk}F_{jk}/2 \qquad (24.3)$$

where ϵ_{ijkl} is the totally antisymmetric tensor with four spatial Euclidean components 1,2,3,5. (24.3) is obviously a self or anti-self duality equation like the one for instantons in Lecture 5.

What is the Poincaré group in five dimensions? It acts as

$$[P_\mu, \psi(x)] = i\hbar\partial_\mu\psi(x)$$

in a non-gauge theory, but in a gauge theory $\psi(x)$ is gauge covariant and P_μ being a physical quantity should be gauge invariant. This means we should have

$$[P_\mu, \psi(x)] = i\hbar D_\mu\psi(x) \qquad (24.4)$$

and in our case, for the fifth dimension

$$\begin{aligned} [P_5, \psi(x)] &= i\hbar(\partial_5 - ie\phi_a D(T_a))\psi \\ &= e\hbar\phi_a D(T_a)\psi \\ &= aD(Q)\psi \end{aligned} \qquad (24.5)$$

So $P_5 = aQ$. The mass formula (23.12) reads

$$M^2 = P_o^2 - P_1^2 - P_2^2 - P_3^2 = a^2(g^2 + q^2) = P_5^2 \qquad (24.6)$$

if for the moment we take $g = o$. Thus all states satisfying the formula are light-like in five dimensions. If g is not zero, one can introduce in an analogous way a $P_6 = ag$, but only if the theory has extended supersymmetry. A ten dimensional theory is also of interest from the point of view of extended supersymmetry. (For a review see Olive 1982b)

Symmetry breaking via a Higgs field in the adjoint representation

Let us now give a complete discussion of this especially interesting case (from the point of view of magnetic monopoles) for all simple Lie groups.

Recall from above that if

$$\underline{\phi} = \sum_{a=1}^{\dim G} T^a\phi^a$$

then the generators of the exact symmetry group H_o are those that commute with $\underline{\phi}$. In particular $\underline{\phi}$ itself does, since $\underline{Q}$ is parallel to $\underline{\phi}$ we have $H_o =$ "$U(1)_Q \times K$". We will be particularly interested in cases where K is semisimple, because then charge quantization occurs. Our general aim here concerns the possible choices for K given G.

First we will use a powerful theorem from Lie group theory which says that any generator of a compact Lie group can be rotated into the Cartan subalgebra, i.e., for any $\underline{\phi}$ there is a $g \in G$ such that

$$\underline{\phi}' = g\phi g^{-1} = \sum_{i=1}^{r} \phi_i H_i \qquad (24.7)$$

The components ϕ_i are clearly not unique (as is seen in SO(3) - any vector can be rotated to the z axis but with a directional ambiguity), and in fact the ambiguity is given by the Weyl group, as seen from the formula

$$S_\alpha(\Sigma\phi_i H_i)S_\alpha^{-1} = \sum_i \sigma_\alpha(\phi)_i H_i \quad . \qquad (9.25)$$

By the properties of Weyl chambers however, there always exists a unique element of the Weyl group taking the chamber containing ϕ_i into the positive one, so that after the gauge transformation (see lecture 12)

$$\sum_{i=1}^{r} \phi_i\alpha_i \geq 0 \qquad \text{for all positive roots } \alpha \qquad (24.8)$$

The Weyl group is effectively a subgroup of the gauge group so this just corresponds to a choice of gauge. Given (24.8), the ϕ_i are uniquely determined.

Let us look at the generators of G commuting with $\underline{\phi}$. Clearly all the H_i do, so the rank of the exact symmetry group H_o is equal to the rank of the original group G. Next, notice that

$$[E_\alpha, \underline{\phi}] = -(\alpha\cdot\phi)E_\alpha \tag{24.9}$$

using the commutation relations (8.11) and (24.7). Thus E_α is a generator of H_o if and only if $(\alpha\cdot\phi) = 0$. The same applies to $E_{-\alpha}$.

Now expand α in terms of simple roots α_a (see lecture 11)

$$\alpha = \sum_{a=1}^{r} n_a\alpha_a \qquad n_a \text{ positive integers}$$

Then

$$[E_{-\alpha}, \underline{\phi}] = \sum_{a=1}^{r} (n_a\alpha_a\cdot\phi)E_\alpha \tag{24.10}$$

and each term in the sum is positive. For the right hand side to vanish, we must have

$$n_a\alpha_a\cdot\phi = 0 \qquad \forall\, a = 1 \ldots r \tag{24.11}$$

Now we can enumerate the possibilities.

1) Suppose $\alpha_a\cdot\phi > 0$ for all $a = 1 \ldots r$. Then (24.11) implies that $n_a = 0$ for all a i.e., $\alpha = 0$. This means no E_α's generate H_o, and so the exact symmetry group, generated by the H_i only is

$$H_o = (U(1))^r = U(1) \times U(1) \ldots U(1) \quad r \text{ times}$$

that is, $K = (U(1))^{r-1}$, obviously not semisimple.

2) Suppose one $\alpha_a\cdot\phi$, say $\alpha_1\cdot\phi = 0$ and the others are all > 0. Then by (24.11), $n_2 = n_3 = \ldots = n_r = 0$ so $\alpha = n_1\alpha_1 = \pm\alpha_1$ only (2β is never a root if β is). So $E_{\pm\alpha}$ generate H_o. Along with the H_i they generate

$$H_o = \text{"}SU(2) \times U(1)^{r-1}\text{"}$$

So $K = \text{"}SU(2) \times U(1)^{r-2}\text{"}$, also not semisimple unless $r = 2$, in which case our analysis is completed.

3) Suppose two of the $\alpha_a\cdot\phi = 0$, and the others are positive, say

$$\alpha_1\cdot\phi = \alpha_2\cdot\phi = 0$$

Then $n_3 = n_4 = n_r = 0$ and so $\alpha = n_1\alpha_1 + n_2\alpha_2$. If one thinks of the Dynkin diagram of G, the subdiagram consisting of the points corresponding to α_1 and α_2 only is the diagram for K and tells you which such α are actually allowed roots. In fact one sees that

$$H_o = \text{"}U(1)^{r-2} \times J\text{"}$$

where J is obtained by deleting the points 3,4 ... r from the Dynkin diagram of G.

4) Generally, with k of the $\alpha_a\cdot\phi = 0$ and the rest positive one gets

$$H_o = \text{"}U(1)^{r-k} \times J\text{"}$$

where J has rank k. For semisimple K we want $r - k = 1$ i.e., $k = r-1$. So we need $\alpha_a\cdot\phi = 0$ for $r-1$ simple roots and $\alpha_b\cdot\phi > 0$ for the one remaining. Then

$$[E_\alpha, \underline{\phi}] = 0 \qquad \text{if } \alpha = \sum_{\substack{a=1\\a\neq b}}^{r} n_a\alpha_a$$

so that α belongs to the root system constructed from the simple roots $\alpha_1 \ldots \alpha_r$ not including α_b. Then

$$H_o = \text{"}U(1) \times K\text{"}$$

where K has rank $r - 1$, is semisimple and its Dynkin diagram is obtained by deleting the point $\alpha_b \equiv \alpha_\phi$; there are clearly r points of the Dynkin diagram of G and so r possibilities.

Let us look at some examples.

A) $G = SU(3)$

The Dynkin diagram is 

If $\alpha_\phi = \alpha_1$, $K = SU(2)$ 

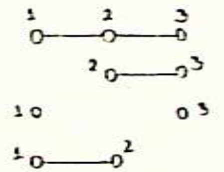
If $\alpha_\phi = \alpha_2$, $K = SU(2)$ 

B) $G = SU(4)$.

If $\alpha_\phi = \alpha_1$, $K = SU(3)$

If $\alpha_\phi = \alpha_2$, $K = SU(2) \times SU(2)$

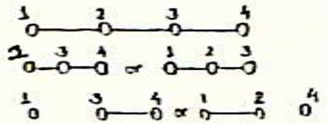
If $\alpha_\phi = \alpha_3$, $K = SU(3)$



C) $G = SU(5)$

If $\alpha_\phi = \alpha_1$ or α_4 , $K = SU(4)$

If $\alpha_\phi = \alpha_2$ or α_3 , $K = SU(2) \times SU(3)$



The latter example is the so-called minimal SU(5) Grand Unified Theory broken by an adjoint higgs field to $SU(3)_{colour} \times SU(2)_{weak\ isospin} \times U(1)_{hypercharge}$

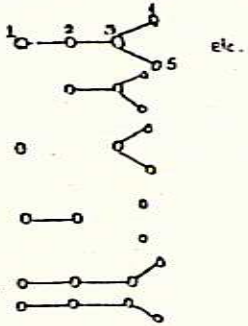
D) $G = SO(10)$

If $\alpha_\phi = \alpha_1$, $K = SO(8)$

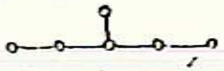
If $\alpha_\phi = \alpha_2$, $K = SU(2) \times SU(4)$

If $\alpha_\phi = \alpha_3$, $K = SU(3) \times SU(2) \times SU(2)$

If $\alpha_\phi = \alpha_4$ or α_5 , $K = SU(5)$

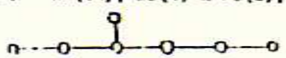


E) $G = E_6$



As easily seen, $K = SO(10)$, $SU(5) \times SU(2)$, $SU(3) \times SU(3) \times SU(2)$ or $SU(6)$.

F) $G = E_7$

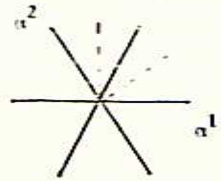


Here, $K = E_6$, $SO(10) \times SU(2)$, $SU(5) \times SU(3)$, $SU(4) \times SU(3) \times SU(2)$ etc.

G) $G = E_8$, can be broken so $K = E_7$ etc.

We have therefore completely solved the problem of enumerating possible symmetry breaking for a higgs field in the adjoint representation. For higgs fields in other representations the problem is much more difficult and as yet unsolved. [E. Weinberg, 1980, Goddard and Olive, 1981a, Slansky 1981]

The special directions for the Higgs fields for which $G \to U(1) \times K$ are also closely connected with the quantisation of the U(1) charge. We illustrate it in the example of SU(3) broken to $U(1) \times SU(2)$ by the octet Higgs fields. Since SU(2) is simple, the only allowed directions of ϕ (within the positive Weyl chamber) are the two dotted lines in the diagram, perpendicular to the



simple roots α^1 and α^2. In the language of SU(3) flavor symmetry in elementary particle physics, it means that the matrix ϕ should be either proportional to λ^8 (when $\phi \perp \alpha^1$) or to $\frac{\sqrt{3}}{2}\lambda_3 + \frac{1}{2}\lambda_8$ (when $\phi \perp \alpha^2$). In the Gell-Mann basis

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \quad .$$

and

$$\frac{\sqrt{3}}{2}\lambda_3 + \frac{1}{2}\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

These are the only directions in the positive Weyl chamber where the corresponding combination of λ_3 and λ_8 yields a possible Higgs field whose little group is U(1) times a semisimple group. U(2).

Lecture 25

Charge Quantization Revisited

Recall that the electric charge operator corresponding to the exact U(1) gauge symmetry is given by (equation 22.10)

$$Q = \frac{e\hbar}{a}\,\phi\cdot H_o \qquad (25.1)$$

That its eigenvalues are quantized follows from the following analysis.

$$[Q,E_\alpha] = \frac{e\hbar\phi\cdot\alpha}{a}\,E_\alpha \qquad (25.2)$$

Now any positive root α can be written as (lecture 11)

$$\alpha = \sum_1^r n_i\alpha^i\ , \quad n_i \geq 0\ .$$

For K to be semisimple we recall from Lecture 24 that

$$\phi\cdot\alpha^i = \delta_{i\phi}\,\phi\cdot\alpha^\phi \qquad (25.3)$$

i.e., ϕ must be perpendicular to all the simple roots except one which we denote by α^ϕ. We then have

$$\frac{e\hbar\phi\cdot\alpha}{a} = n_\phi\left(\frac{e\hbar\phi\cdot\alpha_\phi}{a}\right) \qquad (25.4)$$

where n_ϕ is an integer, positive if α is positive and negative if α is negative. On the other hand $\frac{e\hbar\phi\cdot\alpha}{a}$ is precisely an eigenvalue of Q (from (25.2)) and thus the electric charge is quantized in units $e\hbar\phi\alpha_\phi/a$. Here Q refers to the charge of the gauge particles and it will be interesting to see the allowed values of n_ϕ, i.e., the possible charges of the gauge particles.

The special directions of ϕ which break the original group G to U(1) x K, with K semisimple, are what are called in the mathematical literature the fundamental dominant weights. They are the vectors λ_i (i = 1, ... r) defined by

$$\frac{2\lambda_i\cdot\alpha^j}{(\alpha^j)^2} = \delta_{ij} \qquad (25.5)$$

i.e., the i^{th} one is perpendicular to all the simple roots except α^i.

So for semisimple K the Higgs field ϕ must be parallel to one of the fundamental dominant weights once it is gauge transformed into the positive Weyl chamber.

Masses of the Gauge Fields

We have shown that the mass of the gauge particle is given by

$$M = |aq| \qquad (25.6)$$

So by using (25.4) we have

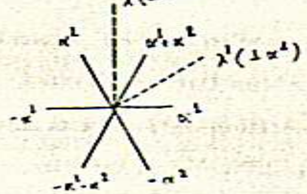
$$M = |n_\phi| q_o a \qquad (25.7)$$

where $\quad q_o = e\hbar\phi\cdot\alpha_\phi/a\ . \qquad (25.8)$

If $|n_\phi| \geq 2$, a peculiar situation could arise: A gauge particle of exactly double the mass of the minimum ones ($|n_\phi| = 1$) would be on the verge of instability for decay into a pair of the minimum ones. Such a possibility may persist in quantum theory if one deals with supersymmetry.

One remedy for this type of instability is that the Higgs fields has some preferred direction, i.e., it is not parallel to any arbitrary fundamental weight, but only to some special ones.

For example let us look at SU(3) -



So here we see that all the roots have coefficient 0 or ±1 when written in terms of the simple roots. Hence $|n_\phi|$ cannot be greater than 1.

In general for SU(n), no matter which fundamental weight the Higgs field is parallel to, n_ϕ will be always 0, ±1. Hence the instability will not occur.

For an arbitrary simple Lie algebra, to find those preferred fundamental weights, λ_ϕ we first notice that

$$2\alpha\cdot\lambda_\phi = n_\phi(\alpha_\phi)^2 \tag{25.9}$$

Therefore we must choose λ_ϕ such that $|n_\phi| \le 1$ for all roots α. To do that we require the concept of the <u>highest root</u> ψ. Given a positive Weyl chamber, there exists for any arbitrary Lie algebra, a positive root ψ with maximum height, i.e.,

$$\psi - \text{any other root} = \sum_i^r m_i\alpha^i, \text{ with all } m_i \ge 0$$

(For SU(3), this is clearly $\alpha_1+\alpha_2$; the proof of the existence of such unique highest root will follow from work in lecture 27).

Now

$$2\lambda_\phi\cdot\psi - 2\lambda_\phi\cdot\alpha = \sum_i^r m_i(2\lambda_\phi\cdot\alpha^i) = m_\phi(\alpha_\phi)^2 \ge 0$$

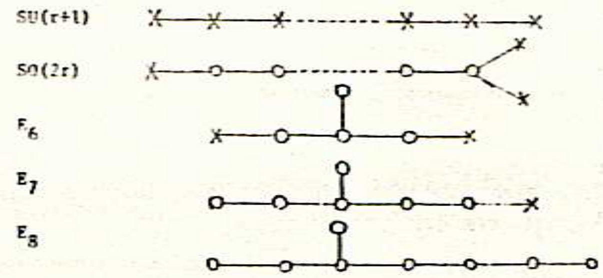
On the other hand

$$2\lambda_\phi\cdot\psi = k_\phi(\alpha_\phi)^2 \quad \text{and} \quad 2\alpha\cdot\lambda_\phi = n_\phi(\alpha_\phi)^2$$

where the k_i are defined by $\psi = \sum_i k_i\alpha_i$. Hence

$$k_\phi \ge n_\phi \tag{25.10}$$

Therefore, the way to avoid the instability is to have $k_\phi \le 1$. This means that we should look for those simple roots whose associated coefficients (when ψ is written as $\Sigma k_i\alpha^i$) are equal to 0 or 1. In the Dynkin diagrams they are marked with x and those for which $k_i > 1$ are marked with 0. Below we list them for the Lie algebras of roots of equal length



This means that while for SU(r+1) all the fundamental weights are all right, for SO(2r) only three of them, for E_6 only two, for E_7 only one and for E_8 none are all right for the purpose of the stability of the gauge particles. It also shows that while for SU(n) all the possible symmetry breakings will be all right, for SO(2r) we can only break from the end (thus SO(10) → SU(5) is o.k.) and E_6 can only break to SO(10), E_7 can only break to E_6 and E_8 cannot be broken at all if one wants to have single mass for the gauge particles. [Goddard and Olive 1981b]

The number of x points in the Dynkin diagrams is also closely related to the center of the corresponding groups. It can be shown that the order (number of elements) of the center of the group $|Z| = 1 +$ no. of x points. For example:

Group	Center
SU(r+1)	Z_{r+1}
SO(2r)	Z_4 for r odd; $Z_2\times Z_2$ for r even
E_6	Z_3
E_7	Z_2
E_8	1

(It is also true that $|Z| = \det k$, where k is the Cartan matrix)

Dual Lie Algebras

Starting with the root system α corresponding to a Lie algebra $\underline{g}$, let us now define a new root system $\alpha^\vee$ defined as

$$\alpha^\vee = \frac{\alpha}{\alpha^2} \tag{25.11}$$

obviously if $|\alpha_1| > |\alpha_2|$, then $|\alpha_1^\vee| < |\alpha_2^\vee|$.

Since the set $\alpha^\vee$ satisfies all the properties of the root-system, one can associate a Lie algebra to it. This is called the dual Lie algebra and is denoted by $\underline{g}^\vee$.

If $\underline{g}$ has roots of equal length, $\alpha^\vee$ is just a scaled version of α and hence $\underline{g}^\vee$ is identically equal to $\underline{g}$. In that sense SU(n), SO(2r), E_6, E_7 and E_8 are self-dual. Otherwise $\underline{g}^\vee$ is different from $\underline{g}$. The $\alpha^\vee$'s are called the coroots of $\underline{g}$. Coroots of a Lie algebra are the roots of its dual Lie algebra.

Now $$(\alpha^\vee)^2 = \frac{\alpha^2}{(\alpha^2)^2} = \frac{1}{\alpha^2}$$

Hence

$$(\alpha^\vee)^\vee = \frac{\alpha^\vee}{(\alpha^\vee)^2} = \frac{\alpha}{\alpha^2}\,\alpha^2 = \alpha \ .$$

So

$$(\underline{g}^\vee)^\vee = \underline{g} \tag{25.12}$$

It can be shown that the simple roots of $\underline{g}^\vee$ are

$$\alpha^{i\vee} = \frac{\alpha^i}{(\alpha^i)^2} \tag{25.13}$$

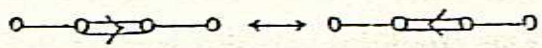
Then

$$(K_{ij})^\vee = \frac{2\alpha^{i\vee}\cdot\alpha^{j\vee}}{(\alpha^{i\vee})^2} = \frac{2\alpha^i\cdot\alpha^j}{(\alpha^j)^2} = K_{ji} \tag{25.14}$$

So the Cartan matrix for $\underline{g}^\vee$ is transpose to that for $\underline{g}$. This makes it easy to identify the dual Lie algebra from the Dynkin diagrams.

$D_{SO(2r+1)}$ =  = $D_{Sp(2r)}$

D_{G_2} =  = D_{G_2}

D_{F_4} =  = D_{F_4}

Hence all except SO(2r+1) and Sp(2r), r > 2, are self-dual. SO(2r+1) and Sp(2r) are dual to each other.

The fundamental weights $\lambda_i^\vee$ for $\underline{g}^\vee$ must be such that

$$2\,\frac{\lambda_i^\vee\cdot\alpha^{j\vee}}{(\alpha^{j\vee})^2} = \delta_{ij} \tag{25.15}$$

i.e.,

$$2\lambda_i^\vee\cdot\alpha^j = \delta_{ij}$$

or

$$\frac{2\lambda_i^\vee(\alpha^i)^2\alpha^j}{(\alpha^j)^2} = \delta_{ij}$$

But since $$2\,\frac{\lambda_i\cdot\alpha^j}{(\alpha^j)^2} = \delta_{ij}$$

we have

$$\lambda_i^\vee = \frac{\lambda_i}{(\alpha^i)^2} \tag{25.16}$$

These are also called the fundamental co-weights of $\underline{g}$.

Returning to the highest root ϕ which is written as

$$\phi = \alpha_\phi k_\phi + \sum_{i\neq\phi} \alpha_i k_i \ .$$

We see that k_ϕ can be written as

$$k_\phi = \frac{2\phi\cdot\lambda_\phi}{(\eta_\phi)^2}$$

$$= 2\phi\cdot\lambda^\vee_\phi$$

$$= \frac{2\phi^\vee\cdot\lambda^\vee_\phi}{(\phi^\vee)^2} \qquad (25.17)$$

Therefore the condition for single mass of gauge particle is now

$$\frac{2\phi^\vee\cdot\lambda^\vee_\phi}{(\phi^\vee)^2} = 0 \text{ or } 1 \qquad (25.18)$$

Such a $\lambda^\vee_\phi$ is called the <u>minimal</u> <u>co-weight</u> of <u>g</u>.

In general the <u>minimal</u> weight λ is defined to be the one for which

$$2\,\frac{\lambda\cdot\alpha}{\alpha^2} = 0,\ \pm 1 \quad \text{for } \underline{\text{any}} \text{ root } \alpha\ . \qquad (25.19)$$

Minimal weights and the corresponding co-weights play important roles in the following three apparently different problems.

1) Single mass of the gauge particles (as discussed in detail above).
2) Brandt-Neri-Coleman condition for the stability of magnetic monopoles [Brandt and Neri 1979, Coleman 1982, Goddard and Olive 1981b].
3) The constituent fermion multiplets in grand unified theories (we will discuss this in the next couple of lectures).

<u>Lecture 26</u>

<u>Representation Theory</u>

We now review and renew the discussion on representation theory of simple Lie algebras. We gave an outline in the Lecture 8 on the context of SU(2) but postponed the general discussion as we were yet to develop the idea of quantized weights for general Lie algebras.

It is assumed that the reader is familiar with the representation theory for finite groups [Hamermesh, 1962]. The essential point is that the elements of the group can be written as matrices with the property that the multiplication of these matrices obey the multiplication laws of the group elements. Block-diagonal matrices like $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ are <u>reducible</u> and are the direct product of two <u>irreducible</u> representations A and B provided A and B are not themselves block diagonal.* For finite groups the number of irreducible inequivalent representations (i.e., not related to each other by similarity transformations) is finite. This is ~~also~~ not true for compact simple Lie groups.

For Lie algebras we want the finite dimensional irreducible representations ("<u>irreps</u>."), i.e., finite dimensional matrices (or linear operators) acting on a finite dimensional vector space. The basis of this vector space can be taken to be the eigenstates of H_i's, the generators of the Cartan subalgebra. Each element of the basis is therefore the simultaneous eigenstate of all the H_i's. Thus for a basis state $|\mu\rangle$, we will have

$$H_i|\mu\rangle = \mu_i|\mu\rangle,\ i = 1 \ldots r \quad (r = \text{rank of the algebra}) \qquad (26.1)$$

Therefore the basis is characterized by an r-component vector $\{\mu_i\}$ with components called <u>weights</u>. Vectors labelled by weights can be degenerate, i.e., there can be more than one vector with the same weight. (In SU(3), for example, π^0 and η both have the same weight).

---

* Strictly speaking the block-diagonal representations are 'completely reducible'. The general definition of 'reducible representation' is when it has a form $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, in which case A and B are independently representations. However for compact groups any representation which is reducible is also completely reducible (Hamermesh, 1962).

The quantization of the weights arises from restrictions similar to those we observed earlier for the root-systems. Recall that $H_\alpha = \frac{2\alpha\cdot H}{\alpha^2}$ has only integer eigenvalues. Now

$$H_\alpha|\mu\rangle = \frac{2\mu\cdot\alpha}{\alpha^2}|\mu\rangle \tag{26.2}$$

Hence $\frac{2\mu\cdot\alpha}{\alpha^2}$ must be integer for each root α (26.3)

This is the only condition on the allowed weights.

Notice that the roots themselves have the same restriction as the weights μ. All roots are therefore weights but the reverse is not true.

Let us now take the simple roots α^i; according to (26.3) we must have, for any weight μ,

$$2\frac{\mu\cdot\alpha^i}{(\alpha^i)^2} = m_i \tag{26.4}$$

If $\{\lambda^i\}$ are the fundamental weights i.e., $2\frac{\lambda^i\cdot\alpha^j}{(\alpha^j)^2} = \delta^{ij}$, then

$$2(\mu - \sum_{i=1}^{r} m_i\,\lambda^i)\cdot\alpha^j/(\alpha^j)^2 = 0 \tag{26.5}$$

Therefore the vector $\mu - \sum_{i=1}^{r} m_i\lambda^i$ must be either perpendicular to α^j or zero. But since the simple roots form the basis of the r-dimensional vector space, the first possibility is ruled out.

Thus
$$\mu = \sum_{i=1}^{r} m_i\lambda^i \tag{26.6}$$

(26.6) shows that each weight is an integer sum over the fundamental weights. Such a quantized spectra of the weights is a remarkable feature of compact Lie groups.

(26.6) shows that the allowed weights lie on a lattice Λ, called the <u>weight lattice</u>. Not only all the weights lie on the lattice, any point lying on the lattice defined by the fundamental weights λ^i, will now be shown

to correspond to a weight. In other words if a vector μ is of the form

$$\mu = \sum_{j=1}^{r} \ell_i\lambda^i \text{ , with } \ell_i\text{'s integer.}$$

it is a weight.

Then we have to show that

$$\frac{2\mu\cdot\alpha}{\alpha^2} \text{ is an integer.}$$

To show that we recall that the coroots $\frac{\alpha}{\alpha^2}$ can be written as an integral sum of the simple co-roots $\frac{\alpha^i}{(\alpha^i)^2}$ with the coefficients either all positive or all negative.

$$\frac{\alpha}{\alpha^2} = \sum_{i=1}^{r} m_i\frac{\alpha^i}{(\alpha^i)^2}$$

Hence

$$\frac{2\mu\cdot\alpha}{\alpha^2} = \sum_{i,j=1}^{r} \ell_i m_j \frac{2\lambda^i\cdot\alpha^j}{(\alpha^j)^2} = \sum_{i=1}^{r} \ell_i m_i = \text{integer}$$

since

$$\frac{2\lambda^i\cdot\alpha^j}{(\alpha^j)^2} = \delta^{ij}$$

We have also seen that roots are weights and hence belong to the weight lattice. The lattice Λ_r formed out of the simple roots α^i, called the <u>root lattice</u>, form a sublattice of Λ, i.e.,

$$\Lambda_r \in \Lambda .$$

The lattice Λ can be thought of as an abelian group of translations such that Λ_r forms an invariant subgroup. The quotient group, Λ/Λ_r turns out to be $Z(\tilde{G})$, i.e. the center of the covering group. We will not prove it here but shortly demonstrate it in two simple examples.

Now, if the representation is irreducible, then starting from any arbitrary element of the basis, say $|\mu\rangle$, one would be able to obtain any other by a sequence of step operations. In other words $E_{\alpha^1} E_{\alpha^2} \ldots E_{\alpha^l} |\mu\rangle$ is a state of an irreducible representation. Clearly it has a weight $(\mu+\alpha^1+\alpha^2+\ldots \alpha^l)$. Thus the weights in an irreducible representation differ only by an integer sum of roots, implying that they all lie in the same coset Λ/Λ_r. Corresponding to this is a unique element of the center Z(G). (In SU(3) this is the notion of triality).

Example 1. For SU(2), $\alpha = 1$ is the only positive root. Λ_r therefore consists of translation by integral steps

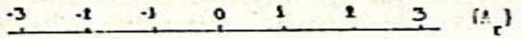


Fig. 26.1

or in short Λ_r = {integers}.

The fundamental weight λ must be such that

$$2\frac{\lambda\cdot\alpha}{\alpha^2} = 1, \text{ i.e., } 2\lambda = 1, \text{ i.e., } \lambda = \frac{1}{2}$$

Hence $\Lambda = \{\frac{\text{integers}}{2}\}$

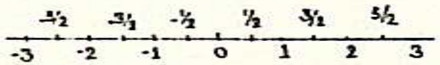


Fig. 26.2

Clearly $\Lambda_r \in \Lambda$ and $\frac{\Lambda}{\Lambda_r} = Z_2 = Z\,(SU(2))$

Example 2. For SU(3) the root lattice Λ_r can be generated by the simple roots α^1 and α^2 of the root system

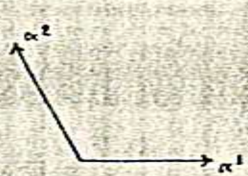


Fig. 26.3

to

Λ_r =

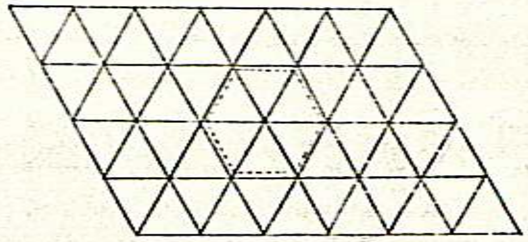


Fig. 26.4

The fundamental weights, corresponding to the simple roots α^1 and α^2 with coordinates (1,0) and $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ respectively, are obtained by solving the equations

$$\frac{2\alpha^i\cdot\lambda^j}{(\alpha^i)^2} = 2\alpha^i\cdot\lambda^j = \delta^{ij}$$

Hence $\lambda^1_x = \frac{1}{2}$, $-\frac{1}{2}\lambda^1_x + \frac{\sqrt{3}}{2}\lambda^1_y = 0$

i.e., $\lambda^1 = (\frac{1}{2}, \frac{1}{2\sqrt{3}})$

clearly showing that λ^1 corresponds the centroid of the triangle Δ^1

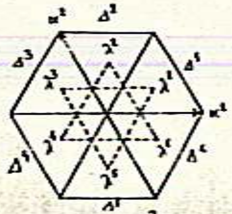


Fig. 26.5

Similarly λ^2 is the centroid of Δ^2. They belong to two different irreducible representations of SU(3), namely the triplet formed out of the weights λ^1, λ^3 and λ^5 and the antitriplet formed out of λ^2, λ^4 and λ^6.

The weight lattice is therefore obtained from the pair λ^1, λ^2 and the origin by adding to these all points of the root lattice. Thus the weight lattice of SU(3) consists of the points of the root lattice together with the centroids.

The quotient Λ/Λ_r means the points of Λ identified modulo Λ_r. For SU(3) Λ/Λ_r possesses three elements which can be thought of as 0, λ_1 and λ_2 since all points of Λ differ from one or other of these by an element of Λ_r. Since from Fig. 26.5 $\lambda_1+\lambda_2=\alpha_1+\alpha_2$ which is a root and $3\lambda_1=\alpha_2+2\alpha_1$ which is again in Λ_r we see that these points form the cyclic group Z_3 which is indeed the centre of SU(3) as we saw in lecture 23.

Lecture 27

Finite Dimensional Irreducible Representations

The compact Lie groups have the property that the volume of the group space is finite. In other words the sum over the group elements (actually integration, since the elements constitute a topological space) must converge. As a consequence it can be shown that just as for finite groups, [Hamermesh, 1962]

1) A finite dimensional representation of a compact group is equivalent to a unitary one, and

2) Unitary representation can be decomposed into unitary irreps.

These two features tell us that the irreps. are the building blocks for any finite dimensional representation of a compact group. Thus as far as the representation theory is concerned it suffices to find all the irreps. (Notice that for noncompact groups irreps. do not exhaust the set. Representation theory for noncompact groups is beyond the scope of this course.)

The crucial feature for an irrep. is the highest weight state. This state $|\lambda\rangle$ is defined to be one with weight λ such that $\lambda+\alpha$ is not a weight for all positive roots α. In other words, there exists, for every irreducible representation, a state $|\lambda\rangle$ of weight λ, such that

$$E_\alpha|\lambda\rangle = \quad \text{for all } \alpha > 0. \tag{27.1}$$

The existence of such a state follows from the finite dimensionality. We shall show that such a state is unique.

Let us now construct other states by sequential application of $E_{-\beta^1}, \ldots\ldots E_{-\beta^k}$, where $\beta^1, \ldots\ldots \beta^k$ belong to the set of positive roots of the algebra. (The number of such lowering operators $=\frac{1}{2}(\dim G-r)$). From the commutator

$$[H_i, E_{-\alpha}] = -\alpha E_{-\alpha} \quad ,$$

we obtain

$$H_i E_{-\alpha} = E_{-\alpha}(H_i-\alpha_i) \tag{27.2}$$

(27.2) can be used to find the weight of a typical state $|\mu\rangle$ defined by

$$|\mu\rangle = E_{-\beta^k} \cdots\cdots E_{-\beta^1}|\lambda\rangle \ . \qquad (27.3)$$

$$H_i|\mu\rangle = H_i E_{-\beta^k} \cdots E_{-\beta^1}|\mu\rangle$$

$$= E_{-\beta^k}H_i \cdots E_{-\beta^1}|\lambda\cdot - \beta^k|\mu\rangle$$

$$= (\lambda - \beta^k \cdots -\beta^1)_i|\mu\rangle \ . \qquad (27.4)$$

Therefore the components of μ are always less than those of λ by integer combinations of simple roots.

It is possible to show that the states $|\mu\rangle$ obtained as above exhaust the basis states of the particular representation. In other words, for any arbitrary, root β, $E_\beta|\mu\rangle$ is always a linear combination of the various $|\mu\rangle$'s. To prove this statement all one needs to know, apart from (27.1) is

$$E_\beta E_{-\alpha} = E_{-\alpha}E_\beta + [E_\beta, E_{-\alpha}] \qquad (27.5)$$

and, $[E_\beta, E_{-\alpha}] = 0$ if $\beta - \alpha \neq 0$ and $\beta - \alpha \notin \Phi$

$\sim H_\beta$ if $\beta - \alpha = 0$

$\sim E_{\beta-\alpha}$ if $\beta - \alpha \in \Phi$

(if $\beta - \alpha$ is a negative root, the proof is obvious and if $\beta - \alpha$ is positive, the process is to be continued).

The basis of irreps, obtained as above, has weights which differ from the highest weight by positive integer multiples of positive roots. It follows that there is only one highest weight state $|\lambda\rangle$ and further that it is not degenerate.

The highest weight λ is <u>dominant</u>, i.e., it lies in the positive Weyl chamber (recall that a vector x lies in the positive Weyl chamber if and only if $\alpha\cdot x \geq 0$ for all positive roots). To see this we notice that if λ is a weight, so must be its Weyl reflection $\sigma_\alpha(\lambda) = \lambda - \frac{2\lambda\cdot\alpha}{\alpha^2}\alpha$ since S_α is a gauge transformation and the irrep is unitary. However, since by definition

λ is the highest weight, $\sigma_\alpha(\lambda)$ will be a lesser weight and that means

$$\frac{2\lambda\cdot\alpha}{\alpha^2} \geq 0 \qquad (27.7)$$

i.e. $\lambda\cdot\alpha \geq 0$, Q.E.D.

We now state an important theorem without proof.

<u>Theorem 27.1:</u> There exists a unique irrep. (upto equivalence) with a highest weight λ for each λ of the weight lattice in the positive Weyl chamber, i.e., irreps. can be labelled by points of $\Lambda^+ = \{\Lambda, \lambda\cdot\alpha \geq 0, \alpha > 0\}$.

An irrep. is therefore typically denoted as $D^\lambda_{\mu\mu'}(g)$ or $D^{(n_1,\ldots,n_r)}_{\mu\mu'}(g)$ where the superscript represents the highest weight λ or its components (positive integers since it is dominant) along the r fundamental weights (i.e. $\lambda = \sum_{i=1}^{r} n_i\lambda_i$ where $n_i = \frac{2\alpha_i\cdot\lambda}{\alpha_i^2} \geq 0$).

<u>Example</u> SU(2): $\alpha = 1$

$$H_\alpha = \frac{2\alpha\cdot H}{\alpha^2} = 2H = 2J_3$$

For the highest weight ℓ,

$$J_3|\ell\rangle = \ell|\ell\rangle$$

$$J_+|\ell\rangle = 0$$

then

$$H_\alpha|\ell\rangle = 2\ell|\ell\rangle$$

2ℓ must be integers and that means

$$\ell = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots.$$

For every value of ℓ there will be an irrep.

$\ell = 0$, scalar representation (spin 0)

$\ell = \frac{1}{2}$, spinor representation (spin $\frac{1}{2}$)

$\ell = 1$, vector representation (spin 1)

and these are recognized to be the usual angular momentum representations.

Example: SU(3):

The fundamental weights are λ_1 and λ_2. The representation $D^{(1,0)}$ and $D^{(0,1)}$ are the triplet and antitriplet representations respectively. $D^{(1,1)}$ is the adjoint representation. Hence the highest root is the highest weight of the adjoint representation.

Another way of defining a representation, i.e., the highest weight, is to assign positive integers to every point of the Dynkin diagram of the associated Lie algebra. For example for SU(3) the three representations mentioned above can be alternatively denoted by

D(1,0) D(0,1) D(1,1)

Notice, however, that labelling the irreducible representations by the highest weight does not yet tell us anything about the matrices. Nevertheless, it is possible to obtain the characters, related to the trace of the matrices of the representation, from the knowledge of the highest weight, and often they turn out to be sufficient. This will be outlined in the next lecture.

Lecture 28

For the adjoint representation the weights are nothing but the roots of the algebra. Since we have

$$[H_i, H_j] = 0$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha$$

We observe that the non-zero weights are non-degenerate (i.e., the α's are all distinct), whereas all the zero weights are r-fold degenerate. That is why in the SU(3) octet we had η and π° (or Λ and Σ°) having the same weight.

Quadratic Casimir Operator

This is defined to be

$$C = g^{ij} T_i T_j \tag{28.1}$$

where $\quad g^{ij} = (g^{-1})_{ij}$, (28.2)

g being the Killing form. In the Chevalley basis g is diagonal and therefore

$$C = \sum_{i=1}^{r} H_i H_i + \sum_{\alpha>0} \frac{\alpha^2}{2} (E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha) \tag{28.3}$$

It is quite straightforward to verify that $[C, H_i] = [C, E_\alpha] = 0$; Casimir operators, by definition, commute with all the generators and hence, according to Schur's lemma, are multiples of the unit matrix in any irrep. In particular for the highest weight state $|\lambda\rangle$ we have, since $E_\alpha|\lambda\rangle = 0$, $\alpha > 0$;

$$C|\lambda\rangle = \left(\sum_{i=1}^{r} H_i^2 + \sum_{\alpha>0} \frac{\alpha^2}{2} [E_\alpha, E_{-\alpha}] \right) |\lambda\rangle$$

$$= \left(\sum_{i=1}^{r} \lambda_i^2 + \sum_{\alpha>0} \frac{\alpha^2}{2} H_\alpha \right) |\lambda\rangle$$

$$= (\lambda^2 + \lambda \cdot \sum_{\alpha>0} \alpha) |\lambda\rangle \qquad \left(\text{since } H_\alpha = \frac{2\alpha \cdot H}{\alpha^2}\right)$$

Hence

$$C = \lambda\cdot(\lambda + 2\delta) \quad = ((\lambda+\delta)^2 - \delta^2) \tag{28.4}$$

where

$$\delta = \frac{1}{2}\sum_{\alpha>0} \alpha \tag{28.5}$$

Example: for SU(2), $\alpha = 1$, $\delta = \frac{1}{2}$

and so $C = \lambda(\lambda+1)$

where λ's take half-integral values. In the usual notation the quadratic Casimir operator is J^2, which has eigenvalues $j(j+1)$.

Character

As in finite group theory, almost all the useful information about the irreps of a compact Lie group can be obtained from their characters, which are defined as

$$\chi^\lambda(g) = \mathrm{Tr}(D^\lambda(g)) = \sum_\mu D^\lambda_{\mu\mu}(g) \tag{28.6}$$

Note that

$$\chi^\lambda(pgp^{-1}) = \mathrm{Tr}(D^\lambda(p)D^\lambda(g)D^\lambda(p^{-1})) \tag{28.7}$$

$$= \chi^\lambda(g)$$

implying that two elements of the group conjugate to each other will have the same character.

In the case of angular momentum we know that rotations by an angle θ about any two axes are conjugate to each other. In other words the angle of rotation defines the conjugacy classes and hence the character. Thus

$$\chi^J(\text{rotation by } \theta) = \chi^J(e^{i\theta J_3}) \qquad \text{(by choosing the rotation axis to be the 3-axis)}$$

$$= \sum_{m=-J}^{J} e^{i\theta m}$$

The terms in the sum are in a geometric progression and hence we obtain

$$\chi^J(\text{rotation by } \theta) = \frac{e^{i(J+\frac{1}{2})\theta} - e^{-i(J+\frac{1}{2})\theta}}{e^{i\theta/2} - e^{-i\theta/2}} \tag{28.8}$$

The concept worked out in the SU(2) case (or SO(3)) can be generalized to any compact Lie group. In the latter case the conjugacy classes can be identified by r 'rotation' parameters θ_i ($i = 1, \ldots, r$), such that any element of the group can be written as

$$g = p\, e^{i\theta\cdot H}\, p^{-1} \tag{28.9}$$

However, we should notice that two group elements with different set of θ's can be conjugate to each other. To see this consider the 'gauge' conjugated element of $e^{i\theta\cdot H}$,

$$S_\alpha\, e^{i\theta\cdot H}\, S_\alpha^{-1} = e^{i\sigma_\alpha(\theta)\cdot H} \tag{28.10}$$

where S_α is the Weyl reflection associated with the root α. Thus, a group element parameterized by θ's and those parameterized by $\sigma_\alpha(\theta)$'s are conjugate and hence have the same character. In the SO(3) case it implies that a rotation by θ and a rotation by $-\theta$ are conjugate. Fortunately the ambiguity ends here. In other words, the conjugacy class involves r - number of 'rotation' parameters θ_i and their Weyl reflections.

An important result was obtained by Weyl in [1935]. The character of an element associated with r parameters θ in the representation connected with the highest weight λ is given by

$$\chi^\lambda(\theta) = \mathrm{Tr}(D^\lambda(\theta)) = \frac{\sum_{\sigma\in W} (\det\sigma)\, e^{i\sum_j \sigma_j(\lambda+\delta)\theta_j}}{e^{i\delta\cdot\theta}\prod_{\alpha>0}(1 - e^{-i\alpha\cdot\theta})} \tag{28.11}$$

Here δ is as in Eq. (28.5) and $\det\sigma = \pm 1$ according to whether the number of reflections is even or odd. This is known as the Weyl character formula.

We will not prove it beyond SU(2), (28.8) but point out some of the consequences.

a) For the scalar representation, i.e., for $\lambda = 0$, all the elements of of the group are mapped into the identity. In such a case, since

$$D^{\circ}(g) = 1 ,$$

we have

$$\chi^{\circ}(g) = 1 \qquad (28.12)$$

Consequently in right hand side of the character formula the numerator equals the denominator –

$$e^{i\delta\cdot\theta} \prod_{\alpha>0} (1-e^{-i\alpha\cdot\theta}) = \sum_{\sigma\in W} (\det \sigma)\, e^{i\sigma(\delta)\cdot\theta} . \qquad (28.13)$$

This is called the Weyl denominator formula. It relates a product to a sum. Such relations are, in general, quite nontrivial. For SU(2) it is, however, a trivial identity since

$$\text{L.H.S.} = e^{i\theta/2}(1-e^{-i\theta}) = e^{i\theta/2} - e^{-i\theta/2} = \text{R.H.S.}$$

b) The denominator formula also applies to Kac-Moody algebras where λ's remain discrete and real but have $r+1$ components and the number of roots become infinite. Consequently the denominator formula relates an infinite product to an infinite sum. For the SU(2) Kac-Moody algebra, this relation is the well-known Jacobi identity for the elliptic functions. For other algebras one would get its generalization. One can well imagine that the transition from finite dimensional Lie algebras to infinite dimensional Kac-Moody algebras introduces convergence problems in formulae like (28.13). These are neatly handled by the theory of Kac-Moody algebras which provide an algebraically consistent regularization method. [MacDonald 1981]

c) Using the denominator formula we have an alternative structure of the Weyl character formula

$$\chi^{\lambda}(\theta) = \frac{\sum_{\sigma\in W} (\det \sigma)\, e^{i\sigma(\lambda+\delta)\cdot\theta}}{\sum (\det \sigma)\, e^{i\sigma(\delta)\cdot\theta}} \qquad (28.14)$$

Numerator and denominator will each have N! terms for SU(N), because $W_{SU(N)}$ is the permutation group of N objects.

d) From definition we have

$$\chi^{\lambda}(\theta) = \mathrm{Tr}(D^{\lambda}(e^{i\theta\cdot H}))$$

$$= \sum_{\mu} d_{\mu} e^{i\theta\cdot\mu} , \qquad (28.15)$$

where d_{μ} is the degeneracy for the μ^{th} weight. Equating this expression with the character formula one obtains a remarkable set of identities.

e) The character formula also tells us about the dimensionality of the representation. This is obtained from the relation

$$\text{Dimensionality} = \mathrm{Tr}(1) = \mathrm{Tr}(e^{i\theta\cdot H})\Big|_{\theta=0} = \chi^{\lambda}(0) \qquad (28.16)$$

For SU(2), from (28.8) we have

$$\chi^{J}(0) \xrightarrow[\theta\to 0]{} 2J + 1 \quad \text{(by using L'Hôpital's rule)} ,$$

which is the well-known result. For any arbitrary compact Lie group, the Weyl dimensionality formula is given by

$$\dim D^{\lambda} = \chi^{\lambda}(0) = \frac{\prod_{\alpha>0} (\lambda+\delta,\alpha)}{\prod_{\alpha>0} (\delta,\alpha)} \qquad (28.17)$$

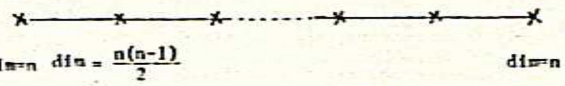
It is remarkable that the right hand side is always an integer.

We will end this discussion on representation theory with a few comments.

i) In physical situations the most important representations are a) the adjoint (where the highest weight is the highest root) and b) the fundamental (where the highest weight is one of the fundamental weights) .

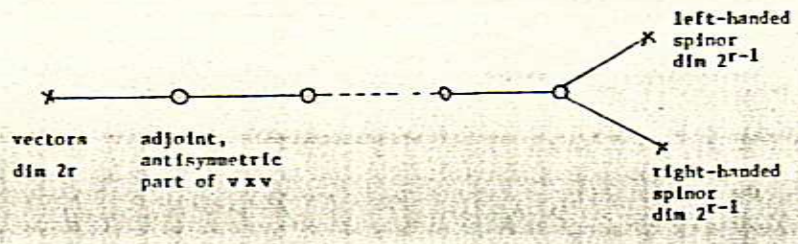
We have already seen the importance of the adjoint representation in the context of gauge theories. The fundamental ones play the important role for constituent fermions. Unlike the adjoint one, which is unique, there can be r different fundamental representations. They can be, therefore, identified with the different points of the Dynkin diagram.

1) SU(n)



These fundamental representations correspond to the antisymmetrical representation of Young tableau notation.

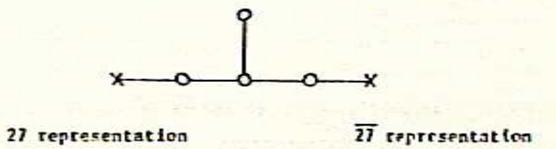
2) SO(2r)



3) SO(2r+1)

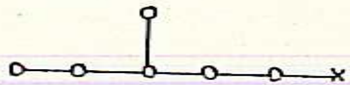
(there is only one spinor representation as the SO(2r+1) symmetric fermions cannot have chiral invariance)

4) E_6



(seems to be associated with fermions)

5) E_7

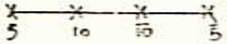


56 representation

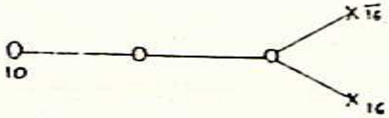
(seems to be associated with fermions)

We have already seen that the points associated with x always refer to minimal weights. The corresponding representations are called minimal representations. It may be that these are somewhat special as the G.U.T. models prefer them for defining the constituent fermions. For example -

i) SU(5) has fermions in 5, $\overline{10}$.



ii) SO(10) has fermions in 16.



iii) E_6 has fermions in 27.

iv) E_7 has fermions in 56.

Besides being relevant in grand unified theories these minimal representations play a special role in the theory of magnetic monopoles [Brandt and Neri (1979), Coleman (1982), Goddard and Olive (1981b), Olive (1981)].

Lecture 29

The character formulae discussed in the last lecture seem to be closely connected to the solutions of some nonlinear differential equations. The one associated with the SU(2) Kac-Moody algebra can be recognized to be the elliptic function solution of the finite oscillation pendulum equation ($\ddot{\theta} = -k^2\sin\theta$). Are all the others, both for Lie algebras as well as for the Kac-Moody algebras, related to some recognizable forms of nonlinear differential equation? The answer is in the affirmative. The corresponding equations are the one-dimensional Toda molecule (for the Lie algebra) and the periodic Toda Lattice (for the Kac-Moody algebra). Since Kac-Moody algebra is beyond the scope of this course we will not consider the periodic Toda lattice here.

The Toda molecule problem, apart from its close connection with Lie algebras, also has the remarkable property that it is exactly soluble, and further it leads to the solutions of some spherically symmetric monopoles in gauge theories of arbitrary simple compact Lie group [Leznov and Saveliev 1978, Ganoulis, Goddard and Olive 1982].

We will recall from Lecture 16 the Toda molecule equation:

$$\ddot{y}_\alpha = -\sum_{\beta=1}^{r} K_{\alpha\beta} e^{y_\beta}, \qquad \alpha = 1, \ldots\ldots r \tag{16.5}$$

Alternatively if we define

$$y_\alpha = K_{\alpha\beta}\phi_\beta$$

then

$$\ddot{\phi}_\alpha = -e^{\sum_\beta K_{\alpha\beta}\phi_\beta} \tag{16.7}$$

K is the familiar Cartan matrix which is nonsingular for simple Lie algebras (but singular for Kac-Moody algebras).

Define the Lie algebra valued pair of objects (the Lax pair) by

$$A = \frac{1}{2}(\dot{\psi}_\alpha(t)\, H_\alpha + e^{\frac{1}{2}K_{\alpha\beta}\psi_\beta(t)}\, (E_\alpha + E_{-\alpha})) \qquad (16.8)$$

$$B = \frac{i}{2}\, e^{\frac{1}{2}K_{\alpha\beta}\psi_\beta(t)}\, (E_\alpha - E_{-\alpha})$$

where H_α and $\hat{E}_\alpha$ are all expressed in the Chevalley basis. The algebra of the operators H_α, E_α etc. can then be used to show

$$\frac{dA}{dt} - i[B,A] = \sum_{\alpha=1}^{r} \frac{1}{2} H_\alpha\, (\ddot{\psi}_\alpha + e^{K_{\alpha\beta}\psi_\beta}) \qquad (16.11)$$

Since the H_α's are linearly independent, the left hand side will be zero if and only if the ψ_α's satisfy (16.7).

The remarkable thing here is that ψ_α satisfying (16.7) also implies (by taking the trace of the l.h.s. of (16.11))

$$\frac{d}{dt}\,\mathrm{Tr}(A^N) = 0 \qquad (16.13)$$

i.e. <u>$\mathrm{Tr}(A^N)$ is a constant of motion.</u>

So far one has only used the behavior of the field locally in time, as that is what exactly the equation of motion directly tells us. The global behavior, on the other hand, can be formed only after solving the equation, i.e., by integrating. Such an integration can be done in a straightforward way for the SU(2) case (K = 2):

$$\ddot{\psi} = -e^{2\psi} \qquad (29.1)$$

A direct integration gives us

$$\dot{\psi}^2 + e^{2\psi} = 2E \qquad (29.2)$$

Where 2E is the integration constant (energy). Integrating once more we obtain

$$t = \int \frac{d\psi}{\sqrt{2E - e^{2\psi}}} + C \; .$$

Substitute $e^{\psi} = \frac{\sqrt{2E}}{\phi}$, $(d\psi = -\frac{d\phi}{\phi})$

so that
$$t = -\int \frac{d\phi}{\phi\sqrt{2E(1-\frac{1}{\phi^2})}} + C = -\frac{1}{\sqrt{2E}} \int \frac{d\phi}{\sqrt{\phi^2-1}} + C$$

or
$$t = -\frac{1}{\sqrt{2E}}\, \cosh^{-1}\phi + C \; .$$

If at $t = t_o$ we set $\phi = 1$, then

$$\phi = \cosh[\sqrt{2E}\,(t-t_o)]$$

or
$$e^{-\psi} = \frac{1}{\sqrt{2E}} \cosh[\sqrt{2E}\,(t-t_o)]$$

or
$$\psi = -\ln[\frac{1}{\sqrt{2E}} \cosh \sqrt{2E}\,(t-t_o)] \qquad (29.3)$$

Globally we see that ψ has a solitonic behavior. What we will see now is that such global features are true for all Lie algebras. The method depends on the group theory associated with the algebra [Kostant (1979), Olshanetsky and Perelomov (1979), Leznov and Saveliev (1979)].

<u>Lax Pair as a Zero-Curvature Condition:</u>

Although the equation of motion (29.1) typically describes a (0 + 1) dimensional field theory, from the geometrical point of view it is more advantageous to consider it as a space-independent configuration of a (1 + 1) dimensional field theory. Introduce two gauge potentials

$$A_x = A \tag{29.4}$$

$$iA_t = B \tag{29.5}$$

The vanishing of the curvature means the vanishing of the field tensor $F_{xt} = [D_x, D_t]$, i.e., of the "electric field". When there is no explicit x-dependence,

$$0 = [D_x, D_t] = [\partial_x + A_x, \partial_t + A_t]$$

$$= [A_x, \partial_t + A_t] = \frac{\partial A_x}{\partial t} - i[A_t, A_x] \tag{29.6}$$

In this way the equation of the Lax pair can be recast into a geometrical language that makes its association with the global structure of the group more transparent.

This zero-curvature condition was used by Leznov and Saveliev to solve the (1+1) dimensional Toda molecule exactly. We will not discuss that solution. Nevertheless the (0+1) dimensional solution can be derived from the zero-curvature condition in a quite simple and elegant way.

Instead of the x and t variables it is more convenient to use the 'light-cone' variables

$$u = \frac{x+t}{2}, \quad v = \frac{x-t}{2} \tag{29.7}$$

so that

$$\partial_u = \partial_x + \partial_t, \quad \partial_v = \partial_x - \partial_t, \quad \partial_x^2 - \partial_t^2 = \partial_u \partial_v .$$

$$A_u = A_x + A_t, \quad \text{and } A_v = A_x - A_t \tag{29.8}$$

In this co-ordinate system the gauge potentials take a symmetrical form,

$$A_u = \sum_\alpha \frac{1}{2}(\partial_t \phi_\alpha) H_\alpha + e^{\frac{1}{2}K_{\alpha\beta}\phi_\beta} E_\alpha \tag{29.9}$$

$$A_v = \sum_\alpha \frac{1}{2}(\partial_t \phi_\alpha) H_\alpha + e^{\frac{1}{2}K_{\alpha\beta}\phi_\beta} E_{-\alpha} \tag{29.10}$$

Next comes a very non-obvious step. Define an operator p that satisfies

$$p^{-1} \frac{dp}{dt} = - A_v . \tag{29.11}$$

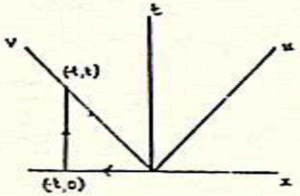
i.e. $\quad D_v p^{-1} = 0$

This can be integrated directly to obtain

$$p(t) = p(o)\, P[\exp \int^v A_v dv'] \tag{29.12}$$

Where P stands for path ordering.

The zero curvature condition can now become useful. Because of that (29.12) holds for any path between v'=0 and v'=v, since by Stokes' theorem $F_{\mu\nu} = 0$ implies $\oint A_\mu dx^\mu = 0$. We then choose the path shown in the following diagram.



$$p(t) = p(o)[P \exp \int_0^{-t} A_x dx][P \exp\int_0^t -A_t dt]$$

$$= p(o) e^{-tA_x}\big|_{t=0}\, U(t) \tag{29.13}$$

Where $\quad U(t) = P \exp\int_0^t -A_t dt . \tag{29.14}$

We now notice from (29.5) that A_t is antihermitian and consequently U is unitary. Hence

$$p(t) p^\dagger(t) = p(o) e^{-2tA_x}\big|_{t=0}\, p^\dagger(o) . \tag{29.15}$$

This is the most crucial result in the method of obtaining the final solution to the nonlinear differential eqn. (16.7). The time dependence has been made quite explicit for the expression of $p^\dagger(t)p(t)$. This result was obtained by Olshanetsky and Perelomov in a geometrical way where they showed that the trajectory of $pp^\dagger$ is a geodesic on the symmetric space (the coset space G/K, G being the group space and K being the space for the maximal compact subgroup of G) associated with the Lie algebra as mentioned in lecture 18.

Viewed from this angle the trajectories of the Toda Molecule equations are projections of those corresponding to free motions in higher dimensional manifolds (the symmetric spaces associated with the corresponding Lie algebras). The constants of motion and hence the integrability can be realized from these free motions. Olshanetsky and Perelomov (1981) further conjectured that <u>all</u> integrable theories are projected versions of free motion in different types of manifolds.

In order to obtain $\phi(t)$ from $p(t)$ we start with the equation of motion of p

$$p^{-1}\dot{p} = -(\tfrac{1}{2}\dot{\Psi} + e^{-\frac{1}{2}\Psi}(\Sigma E_{-\alpha})\, e^{\frac{1}{2}\Psi}) \,, \qquad (29.16)$$

where

$$\Psi \equiv \Sigma\phi_\alpha H_\alpha \,. \qquad (29.17)$$

Now, p, being an element of the noncompact group generated by E_α's and H, can be always written as, (Helgason 1978),

$$p = N\,A\,K \quad \text{(Iwasawa decomposition)}$$

Where N, A and K are obtained respectively by exponentiating $E_{-\alpha}$, H_α and $(E_\alpha - E_{-\alpha})$ (see Lecture 18). In the present context however one confines to the symmetric space i.e., where K is just the identity element. In other words

$$p = NA \qquad (29.18)$$

Then
$$p^{-1}\dot{p} = A^{-1}N^{-1}\frac{d}{dt}(NA)$$
$$= A^{-1}\dot{A} + A^{-1}N^{-1}\dot{N}A \qquad (29.19)$$

It is clear that the first term involves H_α's only and the second term $E_{-\alpha}$. Thus comparing with (29.16) we obtain

$$A^{-1}\dot{A} = -\tfrac{1}{2}\dot{\Psi}$$

i.e.
$$A = e^{-\frac{1}{2}\Psi(t)} \qquad (29.21)$$

Therefore
$$p(t) = N(t)\, e^{-\frac{1}{2}\Psi(t)} \qquad (29.22)$$

Since comparing (29.19) with (29.16) shows N(t) satisfies

$$N^{-1}\dot{N} = -e^{-\Psi(t)}(\Sigma E_{-\alpha})e^{\Psi(t)} \,,$$

which, since it is an homogeneous equation in N, shows that N(t) can be consistently chosen to have initial value

$$N(t=0) = 1$$

Consequently

$$p(t)p^\dagger(t) = N(t)\, e^{-\Psi(t)}\, N^\dagger(t)$$
$$= p(o)\, e^{-2tA\dot{x}|_{t=0}}\, p^\dagger(o)$$

15
(from (29.21))

$$= e^{-\frac{1}{2}\Psi(o)}\, e^{-2tA\dot{x}|_{t=0}}\, e^{-\frac{1}{2}\Psi(o)} \,.$$

Therefore

$$e^{-\Psi(t)} = N^{-1}(t)\, e^{-\frac{1}{2}\Psi(o)}\, e^{-2tA\dot{x}|_{t=0}}\, e^{-\frac{1}{2}\Psi(o)}\, N^\dagger(t)^{-1} \qquad (29.23)$$

This complicated looking expression can be simplified by taking its expectation value in the highest weight state $|\lambda_\alpha\rangle$ of any of the r fundamental representations mentioned in lectures 27 and 28.

$$E_\beta|\lambda_\alpha\rangle = 0 \qquad \beta > 0 \,. \qquad (29.24)$$

$$H_\beta|\lambda_\alpha\rangle = \frac{2\beta\cdot H}{\beta^2}|\lambda_\alpha\rangle = \frac{2\beta\cdot\lambda_\alpha}{\beta^2}|\lambda_\alpha\rangle$$
$$= \delta_{\alpha\beta}|\lambda_\alpha\rangle \qquad (29.25)$$

(29.25) implies $\quad \langle\lambda_\alpha|E_{-\beta} = 0$

$$\text{and} \quad E^\dagger_{-\beta}|\lambda_\alpha\rangle = 0 .$$

Consequently, since N contains $E_{-\alpha}$'s only

$$\langle\lambda_\alpha|N^{-1} = \langle\lambda_\alpha| \quad , \quad N^\dagger|\lambda_\alpha\rangle = |\lambda_\alpha\rangle .$$

Thus we finally obtain

$$e^{-\phi_\alpha(t)} = e^{-\phi_\alpha(o)} \langle\lambda_\alpha|e^{-2tA_x}\big|_{t=0}|\lambda_\alpha\rangle \tag{29.26}$$

which is the final solution to the Toda equations (16.7). Actual evaluation of the matrix element for an arbitrary group is quite hard. Nevertheless the t-dependence of the solution is quite clear.

Example: SU(2) K=2 Here the Chevalley basis generators in the fundamental (spinor) representation are:

$$H = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad E_\alpha = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad E_{-\alpha} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

Since the rank is 1, there is only one fundamental representation, the spinor representation

$$|\lambda\rangle = \begin{pmatrix}1\\ 0\end{pmatrix} .$$

Without any loss of generality we can take $\dot{\phi} = 0$ at $t = 0$.

Then $\quad 2A_x\big|_{t=0} = e^{\phi(o)} \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = e^{\phi(o)}\sigma_1 .$

Hence $\quad e^{-2tA_x}\big|_{t=0} = \cosh(t\, e^{\phi(o)}) - \sigma_1 \sinh(t\, e^{\phi(o)}) .$

Using (29.26) we obtain

$$e^{-\phi(t)} = e^{-\phi(o)} \cosh(t\, e^{\phi(o)})$$

If we denote $\quad e^{\phi(o)} = \sqrt{2E}$ we find

$$e^{-\phi(t)} = \frac{1}{\sqrt{2E}} \cosh(\sqrt{2E}\, t)$$

which is exactly what we obtained before by direct integration, equation (29.3).

Lecture 30

The method discussed in the previous lecture for solution of the one-dimensional Toda molecule equation possessed two basic steps: (i) the construction of the quantity $\rho\rho^\dagger$ with a simple time evolution (eq. (29.15), and (ii) the reconstruction of $\phi_\alpha(t)$ from $\rho\rho^\dagger$.

The structure of the argument resembles that of the inverse scattering method as applied to the KdV equation, say. There the eigenvalue equation for the Lax A reads as a time independent Schrödinger equation defining a scattering problem with the KdV solution u acting as scattering potential. Using the Lax pair or zero curvature condition one establishes that the scattering data have a simple time evolution. This is step (i) with the scattering data analogous to $\rho\rho^\dagger$. Step (ii) is the reconstruction of u(t) from the scattering data using inverse scattering theory. This is somewhat like the use of the Iwasawa decomposition above. It would be instructive to understand these parallels better.

Comments on the form of the solution:

Eqn. (29.26) can be written in a simpler form by noticing the fact that by a suitable gauge transformation $A_x\big|_{t=0}$, which is hermitian, can be made diagonal. In other words there exists a unitary matrix D such that A_x can be transformed to lie in the Cartan subalgebra,

$$D\, A_x\big|_{t=0} D^{-1} = W\cdot H$$

with $\quad D^{-1} = D^\dagger$

Consequently

$$e^{-\phi_\alpha(t)} = e^{-\phi_\alpha(0)} \langle\lambda_\alpha|D^\dagger e^{-2tW\cdot H} D|\lambda_\alpha\rangle$$

$$= e^{-\phi_\alpha(0)} \sum_\mu |\langle\lambda_\alpha|D|\mu\rangle|^2 e^{-2tW\cdot\mu} \tag{30.1}$$

(after introducing unity as a sum over a complete set of eigenstates of $|\mu\rangle$ of H_i).

This shows that the solution is describable in terms of a sum of exponentials of time times the weights of the n^{th} fundamental representation. Notice that each term in the sum is positive. Further evaluation needs the knowledge of the matrix D which is obviously connected with the initial configuration of ϕ and hence nothing in general can be said. However, when the Toda molecule equation is the self-duality equation for a maximally embedded spherically symmetric monopole case (we will shortly show the connection), the boundary condition at the origin (i.e. the condition for finite energy) implies a vanishing of $e^{-\phi}$ and hence a cancellation in (30.1). This is possible because radius plays the role of an imaginary time and the coefficients in (30.1) are no longer positive. Another way of solving the Toda molecule equation for imaginary time is to start with the general solution given by Leznov and Saveliev [1979] for the equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) \phi_\alpha = -e^{K_{\alpha\beta}\phi_\beta}$$

and look for the t-independent solutions [Ganoulis, Goddard and Olive, 1982].

Application to spherically symmetric self-dual monopoles:

We have seen in Lecture 23 that Yang-Mills-Higgs theories with all the fields in the adjoint representation give rise to monopole-like configurations. In the self-dual limit (i.e., when the lower bounds of the energy of such configurations are saturated) the relevant equation to solve is much simpler to tackle than the original Maxwell-like equations. The self-duality equation (the Bogomolny equation) reads

$$B_i = \pm D_i\phi \qquad (23.11)$$

which is satisfied by the Yang-Mills and Higgs fields in the limit of vanishing Higgs self-coupling, no time-dependence and no electric field. Originally this was solved for the SU(2) Y-M field broken down to U(1) for a spherically symmetric single monopole configuration. (Prasad and Sommerfield 1975) Generalization can be made in two different directions.

i) Multimonopole solutions sticking to SU(2) and abandoning spherical symmetry.

ii) Spherically symmetric monopole solutions for larger simple Lie groups broken down by Adjoint Higgs to U(1) x K where K is semisimple.

We will discuss the second point in the remaining part of this course. There are, of course, several types of spherically symmetric solutions depending upon various SU(2) subgroup structures of the larger groups. The Toda equation appears in one of them.

Unlike the normal situation for spherical symmetry where the variables would depend only on r, for gauge-variant objects the spherical symmetry does bring in an angular dependence; however it can be gauge-rotated with the SU(2) generators of the gauge-group to remove the angular dependence. In other words the relevant generators for angular momentum for the purpose of defining spherical symmetry can be taken to be

$$J_{o,k} = -i(\vec{r} \times \vec{\nabla})_k + t_k \qquad (30.2)$$

where the t_j's are the generators of the gauge-rotation. Here the rotation always refers to the three-dimensional space. It was shown by Manton et al. that for finite energy configurations all definitions of generalized rotation operators are equivalent to (30.2). [Forgács and Manton 1980, Manton 1981]

Consider now the Higgs field $\phi(\vec{r})$, which is a scalar field in ordinary space but, being in the adjoint representation, transforms under gauge-rotation. The Higgs field will be spherically symmetric in the generalized sense,

$$[\vec{J}_o, \phi(\vec{r})] = 0 \quad . \qquad (30.3)$$

Similarly, for the gauge field, which is a vector in ordinary space, we will have in the context of spherical symmetry

$$[J_o^i, W^j(\vec{r})] = i\epsilon^{ijk} W^k(\vec{r}) \quad . \tag{30.4}$$

(30.3) and (30.4) completely take care of determining the angular dependence (transverse shifts along the surface of the sphere) while the Bogomolny equation takes care of the radial dependence. In fact under suitable choice of the t_k's we will see that the relevant equation determining the radial dependence is the Toda molecule equation (with $r \to it$).

Since $\vec{r}\cdot(\vec{r} \times \vec{\nabla}) = 0$, we obtain from (30.3) and (30.4),

$$[\vec{r}\cdot\vec{t}, \Phi(\vec{r})] = 0 \tag{30.5}$$

(showing $\vec{r}\cdot\vec{t}$ to be in the exact symmetry group H_0), and

$$[\vec{r}\cdot\vec{t}, W^j(r)] = ir^i\epsilon^{ijk} W^k \tag{30.6}$$

Since all directions are equivalent we may work on the 3 axis. Then the radial direction is the 3-direction, while the polar and azimuthal directions can be considered to be the 1 and 2 directions respectively. If we define M by

$$eW^{\pm}(r) = \mp i\,(M^{\pm}(\vec{r}) - t^{\pm})/r \quad , \tag{30.7}$$

where $\quad W^{\pm} = W^1 \pm i\,W^2$,

then (30.5) and (30.6) reduce to

$$[t_3, \Phi] = 0 \tag{30.8}$$

$$[t_3, M^{\pm}] = \pm M^{\pm} \tag{30.9}$$

In the gauge where $W_r = W^3 = 0$, we will have from the above definition of $M^{\pm}$ the following identities: (Wilkinson and Goldhaber 1977).

$$D_3\Phi = \frac{d\Phi}{dr}$$

$$D^{\pm}\Phi = \mp[M^{\pm}, \Phi]/r$$

$$eB_3 = (\tfrac{1}{2}[M^+, M^-] - t_3)/r^2$$

$$eB^{\pm} = \frac{1}{r}\frac{dM^{\pm}}{dr}$$

Consequently the Bogomolny equation becomes

$$er^2\,\frac{d\Phi}{dr} = \eta(\tfrac{1}{2}[M^+, M^-] - t_3) \tag{30.10}$$

$$\frac{dM^{\pm}}{dr} = \pm\,\eta[\Phi, M^{\pm}] \tag{30.11}$$

Where $\eta = +1$ for monopoles and -1 for antimonopoles.

Lecture 31

At this stage, one makes the substitutions

$$e\Phi = \eta(\phi + t_3/r) \tag{31.1}$$

and $H^{\pm} = \eta r N^{\pm}$. (31.2)

Equations (30.10,11) now reduce to the following pair of equations

$$\frac{d\phi}{dr} = \frac{1}{2}[N^+, N^-] \tag{31.3}$$

$$\frac{dN^{\pm}}{dr} = \pm [\phi, N^{\pm}] \tag{31.4}$$

Notice that with the identification of $\phi = N_3$ and $N^{\pm} = N_1 \pm iN_3$ these equations reduce to a rather familiar looking set of differential equations,

$$i\frac{dN_i}{dr} = [N_j, N_k]$$

where i, j and k are in cyclic order, resembling the Euler equations for rigid body rotation.

There is a further condition from spherical symmetry (Eqs. (30.8,9)) that

$$[t_3, \phi] = 0 \tag{31.5}$$

and

$$[t_3, N^{\pm}] = \pm N^{\pm} \tag{31.6}$$

To interpret this we introduce the concept of grading. An element x of an algebra is said to have grade n if

$$[t_3, x] = nx. \tag{31.7}$$

The grade must be half-integral as t_3 has only half-integral eigenvalues. ϕ, $N^{\pm}$ therefore have grades 0 and ± 1 respectively. Let us define operators A and B by

$$A \equiv \phi + \frac{1}{2}(N^+ - N^-)$$
$$B \equiv \frac{1}{2}(N^+ + N^-) \tag{31.8}$$

Then equations (31.5) and (31.6) give us

$$[t_3, A] = B \tag{31.9}$$

The Bogomolny equations (31.3) and (31.4) now reduce to

$$\frac{dA}{dr} = [A, B] \tag{31.10}$$

which except for the fact that r replaces it, is very close to the Lax pair equation! The constants of motion are now replaced by quantities that are independent of r, namely

$$\frac{d}{dr}(\mathrm{Tr}A^n) = 0 \tag{31.11}$$

Notice that A defined in (31.8) is not hermitian, since $N^+ - N^-$ is not a hermitian object (though ϕ is). We can, however, circumvent this difficulty by defining a new conjugation instead of the hermitian one. This is defined to be

$$A^* = e^{i\pi t_3} A^{\dagger} e^{-i\pi t_3} \tag{31.12}$$

It is quite straightforward to check that

$$A^* = A \tag{31.13}$$

$$B^* = -B \tag{31.14}$$

Returning to (31.5) and (31.6), which should in principle give us the Lie algebraic structure of ϕ and $N^{\pm}$, we realize that somehow we have to know precisely how the SU(2) generators t_3 and $t_{\pm}$ are given in terms of the generators of the gauge group. In other words we have to know how the relevant subgroup SU(2) is embedded in G. Dynkin [1965] gave the complete enumeration of all possible SU(2) subalgebras of any simple compact Lie algebra. Let us take two simple examples.

Ex. 1 SU(2): The three generators are $(\frac{\sigma_1}{2}, \frac{\sigma_2}{2}, \frac{\sigma_3}{2})$, and the SU(2) embedding is the algebra itself.

Ex. 2 SU(3): There are two possibilities:

i) Flavor group or "Isospin embedding". In the Gell-Mann basis they correspond to $\frac{1}{2}\lambda_1, \frac{1}{2}\lambda_2, \frac{1}{2}\lambda_3$ i.e.,

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

ii) "Nuclear physics embedding". In the Gell-Mann basis they correspond to λ_2, $-\lambda_5$ and λ_7,

$$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & +i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

These can be easily recognized to be the 3x3 generators of SO(3), i.e. they correspond to the spin-1 representation, whereas the isospin embedding correspond to the spin-1/2 representation. Therefore these two realizations of the SU(2) subalgebra within SU(3) are inequivalent.

These are the only two types of inequivalent SU(2) embeddings in SU(3).

As one goes to larger and larger groups the number of inequivalent embedding of SU(2) increases. There is, however, a unique one called the <u>maximal embedding</u> for every compact and simple Lie group. It is in such a situation, that the Bogomolny equation reduces to Toda molecule equation.

To find the various embeddings of SU(2) we study the different decompositions

of the adjoint multiplet into irreducible SU(2) multiplets. In SU(2) the adjoint multiplet itself is irreducible, so that is the one and only embedding. In SU(3) we have the adjoint multiplet 8 decomposing into four isospin multiplets

$$8 = 3 + 2 + 2 + 1$$

(for the meson octet these correspond to (π^+, π^0, π^-), (K^+, K^0), $(\bar{K}^0, K^-)$ and η).

It is also possible to show that with respect to the SO(3) generators T_2, $-T_5$ and T_7 the octet decomposes into two multiplets

$$8 = 5 + 3$$

The difference between the two is that in one case we have some multiplets having both half-integral and integral t values (1, 1/2, 1/2 and 0) and the other case only integral t values (2 and 1). This happens in the general situation also, i.e.,

The no. of SU(2) multiplets $\geq$ no. of integer spin multiplets.

Now, notice that any integer spin multiplet will always have one element having the t_3-value 0. This means that the corresponding generators will have grade zero. Hence the number of integer spin multiplets is always equal to the number of grade zero generators. Now since grade zero generators, by definition, commute with t_3, the number of grade zero generators will exceed the rank of the group. Thus we arrive at the important inequality.

The no. of SU(2) multiplets with respect to any embedding

$\geq$ rank of G.

For SU(3) we see that in the first case we have four and in the second case two, which is consistent with the inequality. What is important to notice is that for every group the bound can be saturated, i.e. there will be always one embedding such that the number of SU(2) multiplet will be exactly equal the rank. This is called the maximal embedding since it is the biggest possible SU(2) multiplet (and it is also the one with the smallest number of irreducible multiplets, namely the rank of the algebra).

A general prescription for obtaining the generators $t_\pm$, t_3 of the maximally embedded SU(2)'s in any arbitrary simple Lie algebra is given below.

Without any loss of generality we can consider t_3 to lie in the Cartan subalgebra, and furthermore because of the Weyl group symmetry, in the positive Weyl chamber. In other words we can take

$$t_3 = 2\delta^\vee \cdot H \tag{31.15}$$

with

$$\delta^\vee \cdot \alpha^i \geq 0 \text{ for all simple roots } \alpha^i. \tag{31.16}$$

It will be shown that for maximal embedding we must have

$$\delta^\vee = \frac{1}{2} \sum_{\alpha>0} \alpha^\vee \tag{31.17}$$

where the sum runs over all positive coroots of the algebra. To show that t_3 with the above definition of $\delta^\vee$ actually corresponds to the maximal embedding we first prove that

$$2\alpha^i \cdot \delta^\vee = 1 \text{ for all simple roots } \alpha^i \tag{31.18}$$

Proof: consider a positive root β, expanded in terms of simple roots:

$$\beta = \sum_{j=1}^{r} n_j \alpha^j, \quad n_j \geq 0$$

Its Weyl reflection $\sigma_i(\beta) = \beta - \frac{2\beta\cdot\alpha^i}{(\alpha^i)^2}\alpha^i$ is also a root and its components are n_j for $j \neq i$, while the ith component is $n_i - \frac{2\beta\cdot\alpha^i}{(\alpha^i)^2}$. $\sigma_i(\beta)$ is negative if and only if $\beta = \alpha^i$ and in that case $\sigma_i(\alpha^i) = -\alpha^i$. We thus notice that all positive roots except α^i transform into positive roots under Weyl reflection on the hyperplane perpendicular to α^i. This means

$$\sigma_\alpha^i(\delta) = \delta - \alpha^i$$

where

$$\delta = \frac{1}{2} \sum_{\beta>0} \beta .$$

On the other hand

$$\sigma_\alpha^i(\delta) = \delta - \frac{2\alpha^i\cdot\delta}{(\alpha^i)^2}\alpha^i$$

Hence we have

$$\frac{2\alpha^i\cdot\delta}{(\alpha^i)^2} = 1$$

Working in the dual space we will have $\frac{2\alpha_\vee^i\cdot\delta^\vee}{(\alpha_\vee^i)^2} = 1$, i.e. $2\alpha^i\cdot\delta^\vee = 1$. Q.E.D.

Now to show that t_3 as defined in (31.15) and (31.17) is, in fact, the 3rd component of the maximally embedded SU(2) generators, we first see that a step operator E_β for positive root β will commute with t_3 as follows:

$$\begin{aligned}[t_3, E_\beta] &= 2\delta^\vee\cdot\beta E_\beta \\ &= 2n_i(\delta^\vee\cdot\alpha^i)E_\beta \\ &= (\sum_{j=1}^{r} n_j)E_\beta .\end{aligned} \tag{31.19}$$

We thus find that with respect to t_3, all the generators of the algebra have integer grades. The positive and negative step operators have positive and negative grades respectively while the generators in the Cartan subalgebra have grade zero.

We further notice that the only generators of grade ± 1 are $E_{\pm\alpha^i}$. Hence t_+ and t_- have the form

$$t_+ = \sum_{i=1}^{r} c_i E_{+\alpha^i} \tag{31.20}$$

and

$$t_- = \sum_{i=1}^{r} c_i^* E_{-\alpha^i} \tag{31.21}$$

In order to have $[t_+, t_-] = 2t_3$ we must therefore have

$$\sum_{i,j=1}^{r} c_i c_j^* \, [E_{+\alpha^i}, E_{-\alpha^j}] = \delta^V \cdot H$$

On the other hand we know that $[E_{+\alpha^i}, E_{-\alpha^j}] = \delta_{ij} H_{\alpha^i}$. (Equation 11.6) Hence

$$\sum_{i=1}^{r} |c_i|^2 \cdot H_{\alpha^i} = \sum_{i=1}^{r} |c_i|^2 \; \frac{2}{\alpha^i} \; H_i = \Sigma \, \delta^V_i H_i \tag{31.22}$$

Since each term on the R.H.S. is known, the $|c_i|$ can be found in a straight-forward way. That $[t_3, t_\pm] = \pm t_\pm$ is already guaranteed.

Thus we have found that t_3 along with a certain combination of height 1 step operators (i.e. for simple roots) and its conjugate form the maximally embedded SU(2) subalgebra. That it is the maximal embedded is seen from the following two facts.

a) None of the generators of the Lie algebra has half-integers grade.

b) Therefore each multiplet will have a grade zero generator. Since there are r grade zero generators, the number of multiplets is r.

Returning to the spherically symmetric monopole solution we find that because of (31.5) the most general form for ψ is (since it has grade zero)

$$\psi = \sum_{\alpha=1}^{r} \psi_\alpha H_\alpha \tag{31.23}$$

Similarly since $N^\pm$ have grade ± 1, we can write

$$N^+ = \sum_{\alpha=1}^{r} f_\alpha E_\alpha \tag{31.24}$$

$$N^- = \sum_{\alpha=1}^{r} f_\alpha^* E_{-\alpha} \tag{31.25}$$

ψ_α and f_α can both depend on radius.

Substituting these in (31.3) and (31.4), we obtain

$$\frac{d\psi_\alpha}{dr} = \frac{1}{2}|f_\alpha|^2 \quad (\text{since } [E_\alpha, E_{-\beta}] = \delta_{\alpha\beta} H_\alpha) \tag{31.26}$$

and

$$\frac{df_\alpha}{dr} = \sum_\beta f_\alpha K_{\alpha\beta} \psi_\beta \quad (\text{since } [H_\beta, E_\alpha] = E_\alpha K_{\alpha\beta}) \tag{31.27}$$

$$\frac{df_\alpha^*}{dr} = \sum_\beta f_\alpha^* K_{\alpha\beta} \psi_\beta \tag{31.28}$$

(31.27) and (31.28) can be rewritten as

$$\frac{d}{dr}(\ln f_\alpha) = \sum_\beta K_{\alpha\beta}\psi_\beta = \frac{d}{dr}(\ln f_\alpha^*)$$

Hence

$$\frac{d}{dr}\ln\left(\frac{f_\alpha}{f_\alpha^*}\right) = 0 \tag{31.29}$$

indicating that the phases of f_α are independent of r. Such constant phase factors can be gauged away by constant gauge rotations in the original Bogomolny equation.

If we now define

$$\rho_\alpha = \ln|f_\alpha|^2 \tag{31.30}$$

then

$$\frac{d^2\rho_\alpha}{dr^2} = \frac{d}{dr}\left(\frac{d}{dr}\ln f_\alpha + \frac{d}{dr}\ln f_\alpha^*\right)$$

$$= \frac{d}{dr}(2K_{\alpha\beta}\psi_\beta)$$

$$= 2K_{\alpha\beta}\frac{d\psi_\beta}{dr} = K_{\alpha\beta}|f_\beta|^2 = K_{\alpha\beta}e^{\rho_\beta}$$

Thus we observe that we can solve for f_α and ψ_α and hence the gauge fields and the Higgs fields in terms of ρ_α which satisfies [Leznov and Saveliev 1978]

$$\frac{d^2\rho_\alpha}{dr^2} = K_{\alpha\beta}\, e^{\rho_\beta} \tag{31.31}$$

This is almost the Toda molecule equation that we solved before, except for the difference in sign. In spite of this difference, the equation is exactly soluble by the Lax-pair method. There is however another crucial difference. While in the Toda molecule case timelike t runs $-\infty < t < \infty$ and to solve one needs to give the initial values, in the case of the monopole spacelike r runs $0 < r < \infty$ and one has to use the fact that the fields should behave in a particular way at $r = 0$ and at $r = \infty$ in order that the energy be finite. Thus the relevant boundary conditions are two point ones, in fact of a rather subtle type. These have been solved explicitly for all simple Lie groups. [Ganoulis et al. 1982] and some explicit formulae for ϕ_n are given in my Poiana Brasov lecture notes [Olive 1982a].

Other embeddings of SU(2) in the gauge group lead to other spherically symmetric monopole solutions but no systematic discussion yet exists. Some correspond to generalizations of the Toda molecule equations and a particularly interesting example has been discussed by E. Weinberg [1982b]. He has also advanced convincing arguments [1982a] that only solutions corresponding to particularly simple embeddings, called root embeddings and possessing the lowest possible U(1) magnetic charge should correspond to elementary monopole states.

The solutions above for the maximal SU(2) embeddings tend to possess a magnetic charge which is purely U(1) and with larger values than Weinberg's elementary ones. It is therefore likely that these solutions should be interpreted as accidental superpositions of elementary monopoles.

Another way of checking this besides applying Weinberg's arguments would be to use Nahm's adaptation of the ADHM instanton construction to find solutions contiguous to the spherically symmetric ones describing slightly

separated monopoles. [Nahm 1982] In principle Nahm's method provide the most powerful tool for constructing multimonopole solution for any gauge group but so far it has not proved possible to reconstruct the explicit form of the maximally spherically symmetric solutions above.

I must now stop at a somewhat arbitrary point in the development of an ongoing subject. I have tried to concentrate on one coherent part of the subject with which I am familiar hoping that it will serve as an introduction to all the other approaches I regretfully have not been able to describe.

Let me end by asserting my belief that the theory of self-dual monopoles is clearly only just beginning and is likely to flower into a beautiful part of theoretical physics with many links to other branches of mathematics and physics.

References

S. Adler (1969): Phys. Rev. 117, 3226

M. Atiyah, V. Drinfeld, N. Hitchin and Y. Manin (1978): Phys. Lett. 65A, 185

A. Belavin, A.M. Polyakov, A.S. Schwarz, Y. Tyupkin (1975): Phys. Lett. 59B, 85

J. Bell and R. Jackiw (1969): Il Nuovo Cimento 60A, 47

E.B. Bogomolny (1976): Sov. J. Nucl. Phys. 24, 449

O. Bogoyavlensky (1976): Comm. Math. Phys. 51, 201

R. Brandt and F. Neri (1979): Nucl. Phys. B161, 253

S. Coleman (1982) "The Magnetic Monopole Fifty Years Later": Harvard HUTP82/A032

E. Corrigan and D. Olive (1976): Nucl. Phys. B110, 237

P. Dirac (1931): Proc. R. Soc. A133, 60

R.K. Dodd and R.K. Bullough (1977): Proc. Roy. Soc. London A352, 481

V.G. Drinfeld and V. Sokolov (1981): Dokl. Akad. Nauk SSSR 258, 11; English translation: Sov. Math. Dokl. 23, 457

E.B. Dynkin (1965): Amer. Math. Soc. Translation Soc. 2, 6, 111

F.J. Dyson (1949): Phys. Rev. 75, 486

F. Englert and R. Brout (1964): Phys. Rev. Lett. 13, 321

E. Fermi, J. Pasta and S. Ulam (1958): Los Alamos Report LA-1940; Collected Papers of Enrico Fermi (Univ. of Chicago Press) Vol. II, p. 978

H. Flaschka (1974): Phys. Rev. B9, 1924

P. Forgács and N. Manton (1980): Comm. Math. Phys. 72, 15

I. Frenkel (1981): Journal of Functional Analysis 44, 259

N. Ganoulis, P. Goddard and D. Olive (1982): Nucl. Phys. B205 [FS5], 601

H. Georgi (1982): Lie Algebras in Particle Physics, Benjamin

H. Georgi and S.L. Glashow (1974): Phys. Rev. Lett. 32, 438

P. Goddard and D. Olive (1978): Rep. Prog. Phys. 41, 1357

P. Goddard and D. Olive (1981a): Nucl. Phys. B191, 511

P. Goddard and D. Olive (1981b): Nucl. Phys. B191 528

J. Goldstone (1961): Il Nuovo Cimento 19, 154

References (continued)

M. Hamermesh (1962): "Group Theory and its Application to Physical Problems" (Addison-Wesley)

S. Helgason (1978): "Differential Geometry Lie Groups and Symmetric Spaces" (Academic Press)

M. Hénon (1974): Phys. Rev. B9, 1921

P. Higgs (1964): Phys. Rev. Lett. 12, 132; Phys. Rev. Lett. 13, 508

G. 't Hooft (1974): Nucl. Phys. B79, 276

J.E. Humphrey (1972): "Introduction to Lie Algebras and Representation Theory", Springer

M. Jacob (1974): Dual Theory, North Holland

T. Kibble (1967): Phys. Rev. 155, 1557

B. Kostant (1979): Advances in Mathematics 34, 195

P. Lax (1968): Comm. Pure Appl. Math. 21, 467

A.N. Leznov and M.V. Saveliev (1978): Phys. Lett. 79B, 294 PHYSICA 3D, 62 (1981)

A.N. Leznov and M.V. Saveliev (1979): Lett. Math. Phys. 3, 489

E. Lubkin (1963): Ann. Phys. NY 23, 233

I. MacDonald (1981): Seminaire Bourbaki in Springer Lecture Notes in Math. 901, 258

N. Manton (1981): Nucl. Phys. B184, 391

M. Monastyrsky and A. Perelomov (1975): JETP Lett. 21, 43

C. Montonen and D. Olive (1977): Phys. Lett. 72B, 117

J. Moser (1975): Dynamical Systems, Theory and Application (Lecture Notes in Physics 38, Springer) p. 467

W. Nahm (1982): Monopoles in Quantum Field Theory: Edited by N. Craigie, P. Goddard and W. Nahm (World Scientific). p. 87

H. Nielsen and P. Olesen (1973): Nucl. Phys. B61, 45

D. Olive (1979): Rivista del Nuovo Cimento 2, No 8, 1

D. Olive (1981): "Relation between Grand Unified and Monopole Theories", Proceedings of 1981 Erice conference

D. Olive (1982a): "Gauge Theories; Fundamental Interactions and Rigorous Results", ed. P. Dita, V. Georgescu and R. Purice (Birkhäuser, Boston) p. 33

D. Olive (1982b): "Monopoles in Quantum Field Theory", Edited by N. Craigie, P. Goddard and W. Nahm (World Scientific), p. 157

D. Olive and N. Turok (1982): Physics Lett. 117B, 193

D. Olive and N. Turok (1983a): Nucl. Phys. B215 [FS7], 470

D. Olive and N. Turok (1983b): "Algebraic Structure of Toda Systems", Univ. of Virginia preprint

D. Olive and P. West (1982): Preprint ICTP/81/82-25

M. Olshanetsky and A. Perelomov (1979): Inventiones Math. 54, 261

M. Olshanetsky and A. Perelomov (1981): Physics Reports 71, 313

A. Polykov (1974): JETP Lett. 20, 194

M.K. Prasad and C.M. Sommerfield (1975): Phys. Rev. Lett. 35, 760

G. Racah (1964): Group Theoretical Concepts and Methods in Elementary Particle Physics, p. 1, Gordon and Breach

A. Salam (1968): Proc. 8th Nobel Symp., ed. N. Svartholm, Wiley

R. Shaw (1955): Ph.D. thesis, Univ. of Cambridge

R. Slansky (1981): Physics Reports 79, 1

M. Toda (1967): J. Phys. Soc. Japan 22, 431

M. Toda (1975): Physics Reports 18, 1

M. Toda (1981): "Theory of Non-Linear Lattices", Springer Series in Solid State Sciences 20

Y. Tyupkin, V. Fateev and A.S. Schwarz (1975): JETP Lett. 21, 41

E. Weinberg (1980): Nucl. Phys. B167, 500

E. Weinberg (1982a): Nucl. Phys. B203, 445

E. Weinberg (1982b): Phys. Lett. 119B, 151

S. Weinberg (1967): Phys. Rev. Lett. 19, 1264

H. Weyl (1935): The Structure and Representation of Continuous Groups (Princeton Institute of Advanced Study Lecture Notes)

D. Wilkinson and A. Goldhaber (1977): Phys. Rev. D16, 1221

C. Wilson (1981): Ergodic Theory and Dynamical Systems 1, 361

References (continued)

T.T. Wu and C.N. Yang (1975): Phys. Rev. D12, 3845

C.N. Yang (1970): Phys. Rev. D1, 2360

C.N. Yang and R.L. Mills (1954): Phys. Rev. 96, 191