Equações de Maxwell

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\widetilde{F}_{\mu\nu} = \frac{1}{2!}\,\varepsilon_{\mu\nu\rho\sigma}\,F^{\rho\sigma}$$

$$S = \int d^4x \left[-\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} - A_\mu\,j^\mu\right]$$

$$\partial_\mu F^{\mu\nu} = j^\nu$$

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0$$

$\longrightarrow$

$$\partial_\rho \tilde{F}_{\mu\nu} + \partial_\nu \tilde{F}_{\rho\mu} + \partial_\mu \tilde{F}_{\nu\rho} = \tilde{j}_{\rho\mu\nu}$$ ← Euler-Lagrange equations

$$\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0$$ ← Identidade de Bianchi

$$j^\lambda = \frac{1}{3!}\varepsilon^{\lambda\rho\mu\nu}\,\tilde{j}_{\rho\mu\nu}$$

Simetria de gauge: $A_\mu \to A_\mu + \partial_\mu \alpha$ $\qquad F_{\mu\nu} \to F_{\mu\nu}$

Força de Lorentz: $$\frac{d\,p^\mu}{d\,\tau} = q\,F^{\mu\nu}\,v_\nu$$

$$p^\mu = (\gamma\,m\,c\,,\,p_x\,,\,p_y\,,\,p_z) \qquad v_\mu = \gamma\,(c\,,\,-v_x\,,\,v_y\,,\,v_z) \qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\eta_{\mu\nu} = \mathrm{diag.}\,(1\,,\,-1\,,\,-1\,,\,-1)$$

2 Duality in Maxwell's theory of electromagnetism

The classical theory of electromagnetism is perhaps one of the most beautiful physical theories known. It presents a high level of symmetries and seems to be "designed" to live in four dimensional space-time. It is described by Maxwell's equations, which in the absence of charges and currents, are given in terms of the antisymmetric field tensor $F^{\mu\nu}$ as[1]

$$\partial_\mu F^{\mu\nu} = 0 , \qquad \partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{2.1}$$

where $\tilde{F}_{\mu\nu}$ is the dual of $F_{\mu\nu}$, i.e.

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \tag{2.2}$$

The first set of equations are the Euler-Lagrange equations obtained from Mawxell's Lagrangean

$$\mathcal{L} \equiv -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{2.3}$$

where the variational calculus is with respect to a vector potential A_μ, from which the field tensor is derived

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.4}$$

As a consequence of (2.4) and (2.2), the second set of Maxwell's equations are trivially satisfied and are the so-called Bianchi identities.

The eletric and magnetic fields are related to the field tensor $F^{\mu\nu}$ by

$$E^i \equiv F^{0i} , \qquad B^i \equiv \frac{1}{2} \epsilon^{ijk} F_{jk} \tag{2.5}$$

or equivalently

$$B^i \equiv \tilde{F}^{0i} , \qquad E^i \equiv -\frac{1}{2} \epsilon^{ijk} \tilde{F}_{jk} \tag{2.6}$$

The Maxwell's equations (2.1) are conformally invariant, and their versions in dimensions other than four are just Poincaré invariant. However, one of the most impressive symmetries is the duality transformation. It is valid only in four dimensions because it interchanges the field tensor and its dual, and only in that dimension they have the same rank. Writing (2.1) in complex notation

$$\partial_\mu \left(F^{\mu\nu} + i \, \tilde{F}^{\mu\nu} \right) = 0 \tag{2.7}$$

one sees it is clearly invariant under the duality transformation [17]

$$\left(F^{\mu\nu} + i \, \tilde{F}^{\mu\nu} \right) \rightarrow e^{i\theta} \left(F^{\mu\nu} + i \, \tilde{F}^{\mu\nu} \right) \tag{2.8}$$

with θ being a real constant. In terms of the eletric and magnetic fields it becomes

$$\left(E^i + i \, B^i \right) \rightarrow e^{i\theta} \left(E^i + i \, B^i \right) \tag{2.9}$$

[1] We take the signature of the Minkowski metric as $(+,-,-,-)$, and use greek letters $\mu, \nu, \ldots = 0,1,2,3$ to label space-time indices, and english letters $i, j, k \ldots = 1,2,3$ to label space indices.

The density of energy

$$\frac{1}{2} \mid F^{0i} + i\tilde{F}^{0i} \mid^2 = \frac{1}{4} \mid F^{ij} + i\tilde{F}^{ij} \mid^2 = \frac{1}{2}\left(E^2 + B^2\right) \tag{2.10}$$

and the density of momentum

$$-\frac{1}{2}\left(F^{0i} + i\tilde{F}^{0i}\right)\left(F_{ij} + i\tilde{F}_{ij}\right)^* = E \wedge B \tag{2.11}$$

are invariant under (2.8).

The Lagrangean (2.3) on the other hand, is invariant under (2.8) only for $\theta = \pi$. However, it is worth noticing that the Lagrangean is the real part of the complex quantity[2]

$$\frac{1}{2}\left(F^{\mu\nu} + i\tilde{F}^{\mu\nu}\right)^2 = F^2_{\mu\nu} + iF^{\mu\nu}\tilde{F}_{\mu\nu} \tag{2.12}$$

and its imaginary part is a total derivative, i.e.

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = 2\partial_\mu W^\mu = 12E^i B_i \tag{2.13}$$

where

$$W^\mu = \epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma \tag{2.14}$$

Therefore, it is right to say that the Lagrangean and the topological term (2.13) transform as a doublet of (2.8).

Notice that we could have a scale parameter in (2.8), i.e. $\lambda e^{i\theta}$ instead of just $e^{i\theta}$, that (2.7) would still be invariant. However, that would break the invariance of the energy and momentum. Notice that $\mid F^{\mu\nu} + i\,\tilde{F}^{\mu\nu} \mid^2 = 0$, and so, that is perhaps the only quadratic quantity invariant under (2.8) with the scale parameter included.

If we want the symmetry (2.8) to hold true in the presence of matter we encounter serious difficulties. We could naively add two types of currents to (2.7) as

$$\partial_\mu\left(F^{\mu\nu} + i\,\tilde{F}^{\mu\nu}\right) = j^\nu + i\tilde{j}^\nu \tag{2.15}$$

and impose that they transform under (2.8) as

$$\left(j^\nu + i\tilde{j}^\nu\right) \rightarrow e^{i\theta}\left(j^\nu + i\tilde{j}^\nu\right) \tag{2.16}$$

Since $\tilde{j}^\nu$ becomes a source for $\tilde{F}^{\mu\nu}$ we can not derive the field tensor from a vector potential as in (2.4). That is fine at the classical level, since all physical quantities are described in terms of the field tensor. However, at the quantum level the vector potential acquires physical importance and the implementations of such ideas, as we describe in the next section, becomes very subtle. But the great restrictions come from the experimental side. The charges associated to the currents $\tilde{j}^\nu$ are static sources for the magnetic field, and so far no experimental support has appeared for their existence.

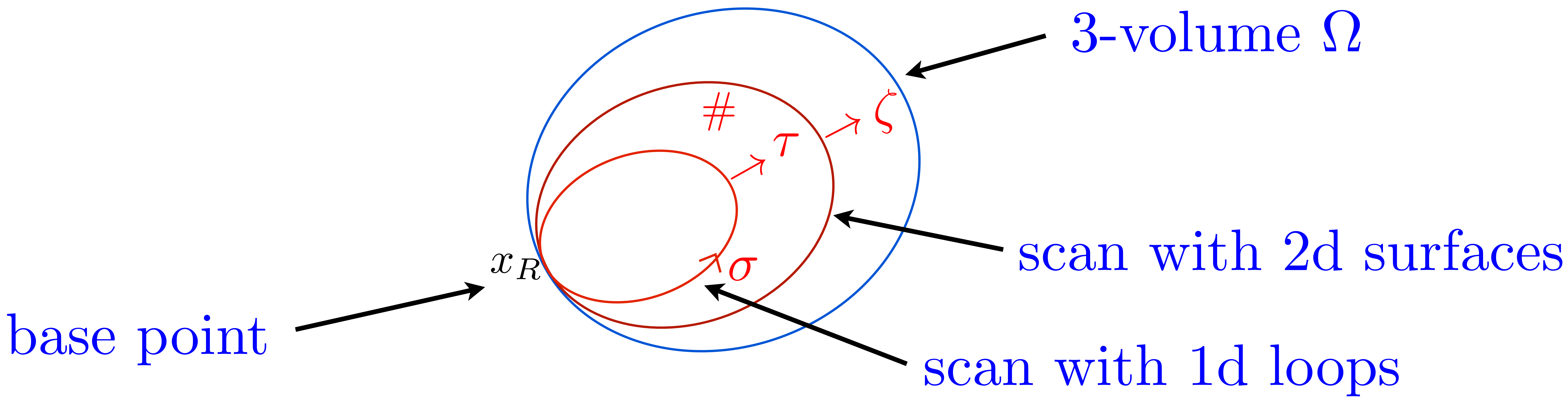
[2] Remember that $F^2_{\mu\nu} = -\tilde{F}^2_{\mu\nu}$, since $\epsilon^{\mu\nu\lambda\delta}\epsilon_{\rho\sigma\lambda\delta} = -2\left(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho\right)$

Abelian Stokes Theorem

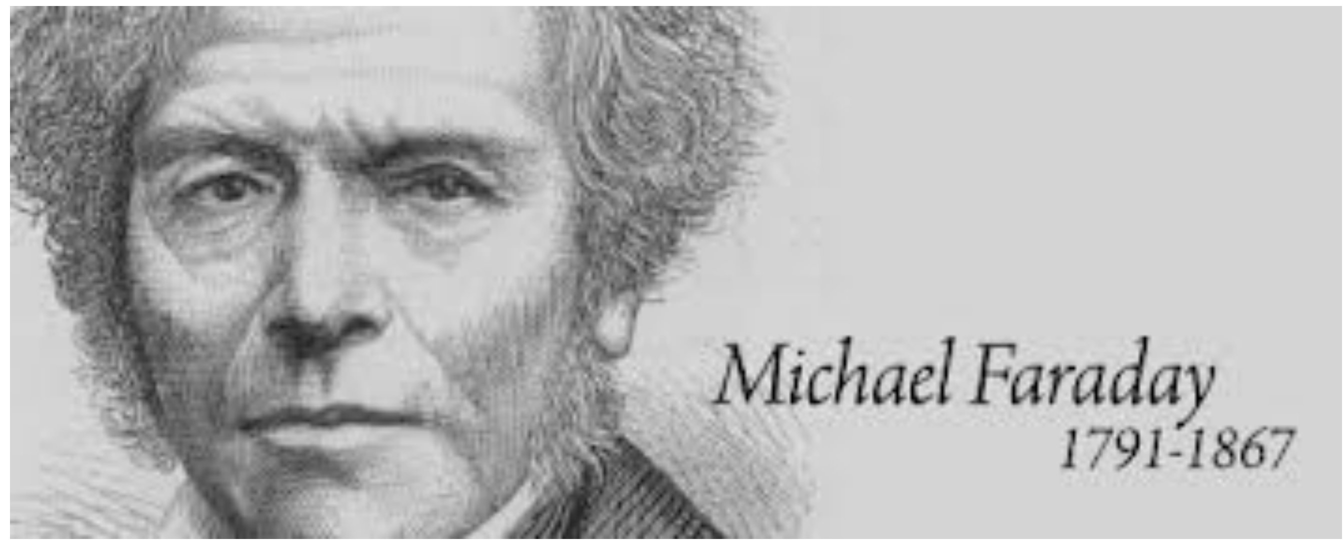
$$\int_{\partial\Omega} B = \int_{\Omega} d \wedge B$$

For an abelian two-form $B_{\mu\nu}$ and a 3-volume Ω

$$\int_{\partial\Omega} B_{\mu\nu} \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} = \int_{\Omega} \left[\partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} + \partial_\mu B_{\nu\rho}\right] \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\zeta}$$



Back to Faraday: Integral Equations



$$\partial_\mu F^{\mu\nu} = j^\nu$$

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0$$

$\longrightarrow$

$$\partial_\rho \tilde{F}_{\mu\nu} + \partial_\nu \tilde{F}_{\rho\mu} + \partial_\mu \tilde{F}_{\nu\rho} = \tilde{j}_{\rho\mu\nu}$$

$$\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0$$

$$j^\lambda = \frac{1}{3!}\varepsilon^{\lambda\rho\mu\nu}\,\tilde{j}_{\rho\mu\nu}$$

In Stokes theorem, take $B_{\mu\nu} \equiv \alpha\, F_{\mu\nu} + \beta\, \tilde{F}_{\mu\nu}$ to get

$$\int_{\partial\Omega} \left[\alpha\, F_{\mu\nu} + \beta\, \tilde{F}_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau} = \beta \int_\Omega \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta}$$

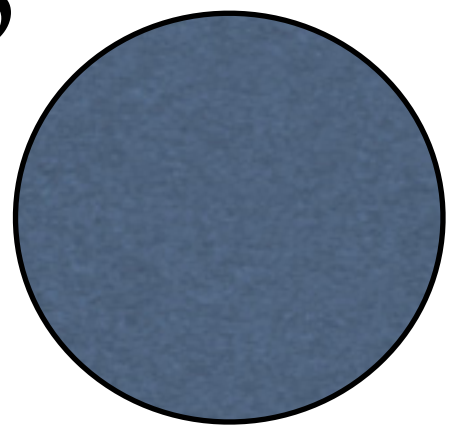
For $\alpha = 0$, $\beta = 1$, and Ω purely spatial

$$\int_{\partial\Omega} \widetilde{F}_{ij} \frac{\partial\, x^i}{\partial\, \sigma}\frac{\partial\, x^j}{\partial\, \tau}\, d\sigma\, d\tau = -\int_{\partial\Omega} \vec{E}\cdot d\vec{\Sigma} = -\int_\Omega \frac{\rho}{\varepsilon_0} = -\frac{Q}{\varepsilon_0}$$

Gauss law

Integral form of Maxwell's equations

S



Closed two dimensional surface in space-time

$$\Phi(S) = \oint_S F_{\mu\nu} \frac{\partial\, x^\mu}{\partial\, \sigma} \frac{\partial\, x^\nu}{\partial\, \tau}\, d\sigma\, d\tau$$

$$\widetilde{\Phi}(S) = \oint_S \widetilde{F}_{\mu\nu} \frac{\partial\, x^\mu}{\partial\, \sigma} \frac{\partial\, x^\nu}{\partial\, \tau}\, d\sigma\, d\tau$$

Maxwell's equations ⟷

$$\Phi(S) = 0 \qquad \beta = 0$$

$$\widetilde{\Phi}(S) = -\frac{Q}{\varepsilon_0} \qquad \alpha = 0$$

charge inside S

$$Q = -\varepsilon_0 \int_V \widetilde{J}_{\mu\nu\rho} \frac{\partial\, x^\mu}{\partial\, \sigma} \frac{\partial\, x^\nu}{\partial\, \tau} \frac{\partial\, x^\rho}{\partial\, \zeta}\, d\sigma\, d\tau d\zeta$$

$$\widetilde{J}_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma} j^\sigma$$

$S \equiv$ closed spatial surface

$$\Phi(S) = \oint_S F_{ij} \frac{\partial x^i}{\partial \sigma} \frac{\partial x^j}{\partial \tau} d\sigma\, d\tau = -c \oint_S \vec{B} \cdot d\vec{\Sigma} = 0$$

$$F_{ij} = -c\, \varepsilon_{ijk}\, B_k$$

$$d\vec{\Sigma} = \varepsilon_{ijk} \frac{\partial x^i}{\partial \sigma} \frac{\partial x^j}{\partial \tau} d\sigma\, d\tau$$

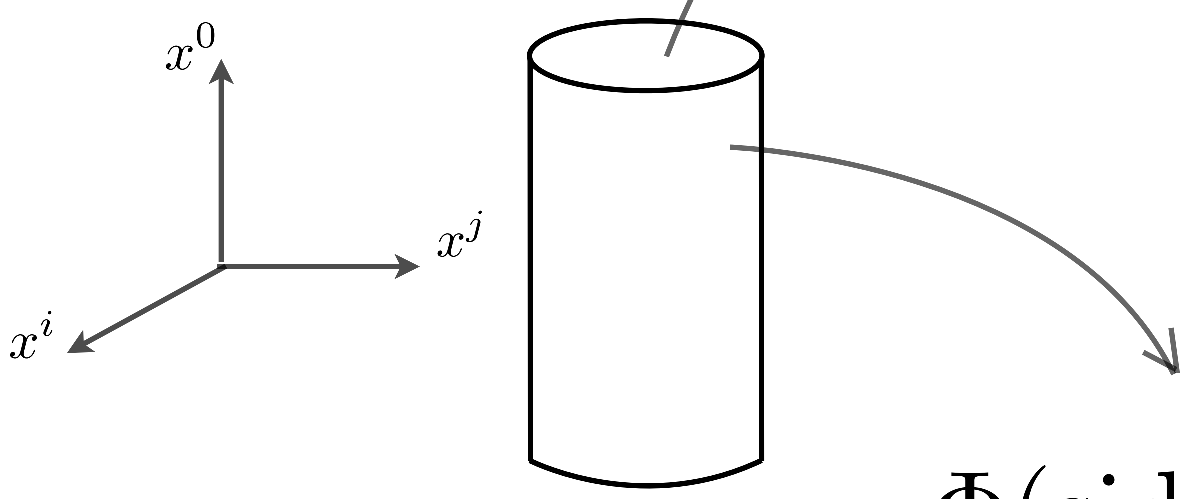
$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\widetilde{\Phi}(S) = \oint_S \widetilde{F}_{ij} \frac{\partial x^i}{\partial \sigma} \frac{\partial x^j}{\partial \tau} d\sigma\, d\tau = - \oint_S \vec{E} \cdot d\vec{\Sigma} = -\frac{Q}{\varepsilon_0}$$

$$\widetilde{F}_{ij} = -\varepsilon_{ijk}\, E_k$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

S with a time component



$$\Phi(\text{disc}) = \int_{\text{disc}} F_{ij} \frac{\partial x^i}{\partial \sigma} \frac{\partial x^j}{\partial \tau} \, d\sigma \, d\tau = -c \int_{\text{disc}} \vec{B} \cdot d\vec{\Sigma}$$

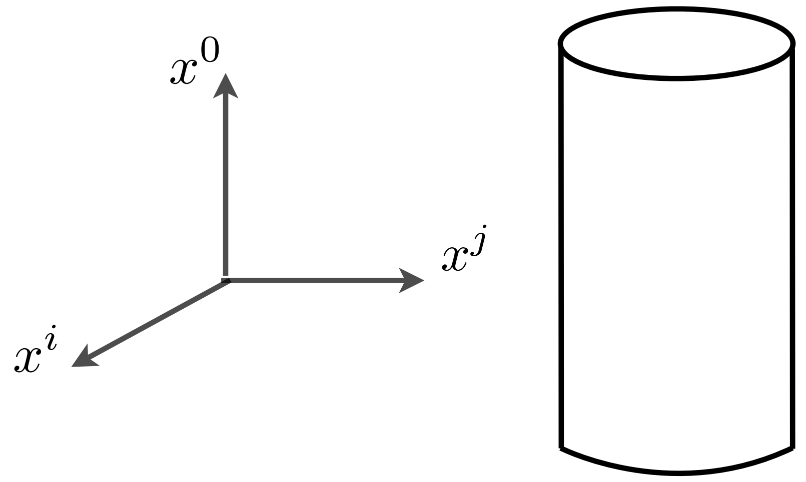
$$\Phi(\text{side}) = \int_{\text{side}} F_{k0} \frac{\partial x^k}{\partial \sigma} \frac{\partial x^0}{\partial \tau} \, d\sigma \, d\tau = - \int dx^0 \oint \vec{E} \cdot d\vec{l}$$

Then

$$\Phi(S) = 0 = -c \int_{\text{top disc}} \vec{B} \cdot d\vec{\Sigma} + c \int_{\text{botton disc}} \vec{B} \cdot d\vec{\Sigma} - \int dx^0 \oint \vec{E} \cdot d\vec{l}$$

In the limit $\delta x^0 \to 0$

$$\frac{d}{dt} \int \vec{B} \cdot d\vec{\Sigma} = - \oint \vec{E} \cdot d\vec{l} \qquad \to \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$



Analogously

$$\widetilde{\Phi}(S) = -\int_{\text{top disc}} \vec{E}\cdot d\vec{\Sigma} + \int_{\text{botton disc}} \vec{E}\cdot d\vec{\Sigma} + c\int dx^0 \oint \vec{B}\cdot d\vec{l}$$

$$= \frac{1}{c\,\varepsilon_0}\int dx^0 \int \vec{J}\cdot d\vec{\Sigma}$$

In the limit $\delta x^0 \to 0$

$$\oint \vec{B}\cdot d\vec{l} - \varepsilon_0\,\mu_0 \frac{d}{d\,t}\int \vec{E}\cdot d\vec{\Sigma} = \mu_0 \int \vec{J}\cdot d\vec{\Sigma}$$

$$\vec{\nabla}\times\vec{B} - \mu_0\,\varepsilon_0\,\frac{\partial\,\vec{E}}{\partial\,t} = \mu_0\,\vec{J}$$

Summarizing

Maxwell's eqs. are equivalent to

$$\oint_S \left[\alpha\, F_{\mu\nu} + \beta\, \widetilde{F}_{\mu\nu}\right] \frac{\partial\, x^\mu}{\partial\, \sigma}\, \frac{\partial\, x^\nu}{\partial\, \tau}\, d\sigma\, d\tau = -\beta\, \varepsilon_0 \int_V \widetilde{J}_{\mu\nu\rho}\, \frac{\partial\, x^\mu}{\partial\, \sigma}\, \frac{\partial\, x^\nu}{\partial\, \tau}\, \frac{\partial\, x^\rho}{\partial\, \zeta}\, d\sigma\, d\tau d\zeta$$

V is the volume inside S

$$\left(\widetilde{J}_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma} j^\sigma\right)$$

Maxwell's eqs. are recovered in the limit where S is infinitesimal

Formulation in SI units convention

Name	Integral equations	Differential equations
Gauss's law	$\oint\!\!\!\oint_{\partial\Omega} \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0}\iiint_\Omega \rho \,\mathrm{d}V$	$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$
Gauss's law for magnetism	$\oint\!\!\!\oint_{\partial\Omega} \mathbf{B}\cdot \mathrm{d}\mathbf{S} = 0$	$\nabla\cdot\mathbf{B} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\oint_{\partial\Sigma} \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint_\Sigma \mathbf{B}\cdot\mathrm{d}\mathbf{S}$	$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$
Ampère's circuital law (with Maxwell's addition)	$\oint_{\partial\Sigma} \mathbf{B}\cdot \mathrm{d}\boldsymbol{\ell} = \mu_0\left(\iint_\Sigma \mathbf{J}\cdot\mathrm{d}\mathbf{S} + \varepsilon_0\frac{\mathrm{d}}{\mathrm{d}t}\iint_\Sigma \mathbf{E}\cdot\mathrm{d}\mathbf{S}\right)$	$\nabla\times\mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)$

Formulation in Gaussian units convention

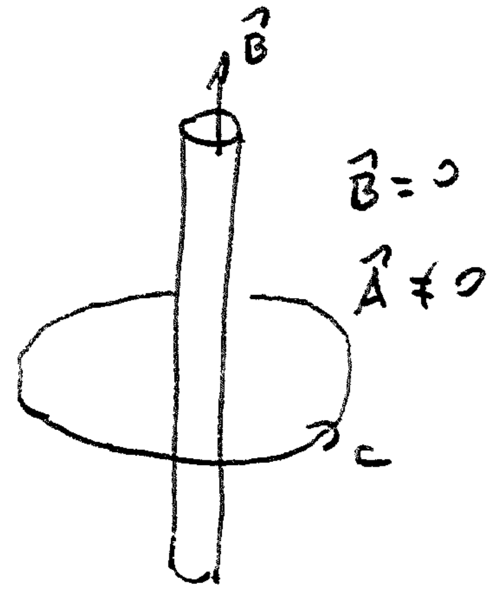
Name	Integral equations	Differential equations
Gauss's law	$\oint\!\!\!\oint_{\partial\Omega} \mathbf{E}\cdot \mathrm{d}\mathbf{S} = 4\pi\iiint_\Omega \rho \,\mathrm{d}V$	$\nabla\cdot\mathbf{E} = 4\pi\rho$
Gauss's law for magnetism	$\oint\!\!\!\oint_{\partial\Omega} \mathbf{B}\cdot \mathrm{d}\mathbf{S} = 0$	$\nabla\cdot\mathbf{B} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\oint_{\partial\Sigma} \mathbf{E}\cdot \mathrm{d}\boldsymbol{\ell} = -\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\iint_\Sigma \mathbf{B}\cdot\mathrm{d}\mathbf{S}$	$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$
Ampère's circuital law (with Maxwell's addition)	$\oint_{\partial\Sigma} \mathbf{B}\cdot \mathrm{d}\boldsymbol{\ell} = \frac{1}{c}\left(4\pi\iint_\Sigma \mathbf{J}\cdot\mathrm{d}\mathbf{S} + \frac{\mathrm{d}}{\mathrm{d}t}\iint_\Sigma \mathbf{E}\cdot\mathrm{d}\mathbf{S}\right)$	$\nabla\times\mathbf{B} = \frac{1}{c}\left(4\pi\mathbf{J} + \frac{\partial\mathbf{E}}{\partial t}\right)$

A simetria de gauge na mecânica quântica

No eletromagnetismo clássico as quantidades que tem significado físico são os campos $\vec{E}$ e $\vec{B}$. Os potenciais ϕ e $\vec{A}$ são apenas campos auxiliares

No entanto na mecânica quântica o mesmo não ocorre. Os campos ϕ e $\vec{A}$ passam a ter significado físico.

Considere um solenóide infinito



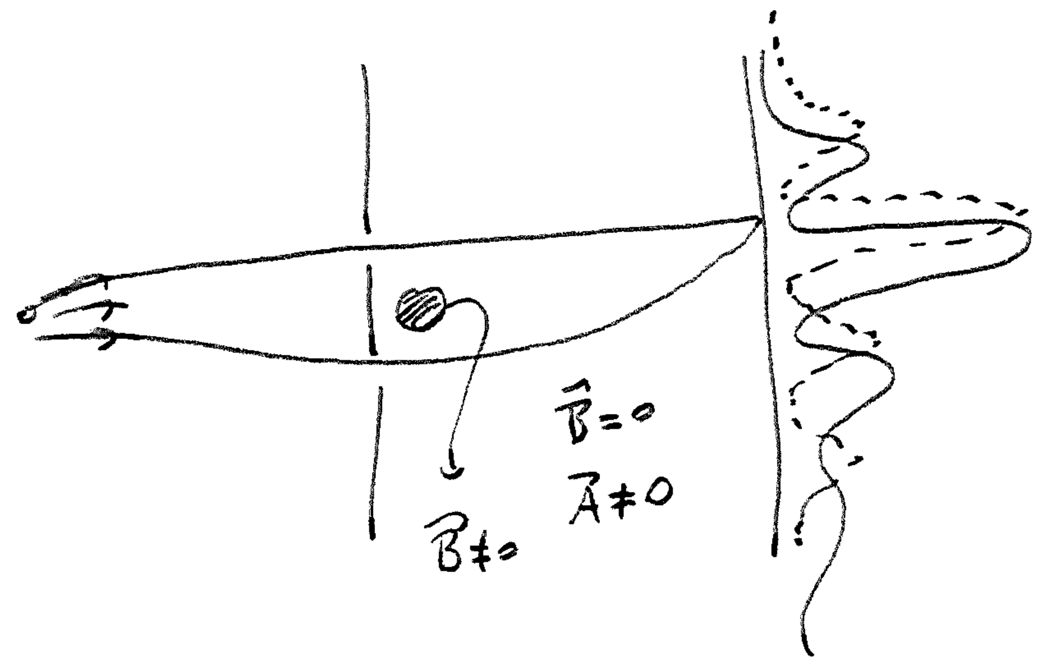
Note que no circuito c

$$\oint_c \vec{A}\cdot d\vec{s} = \int_\Sigma (\vec{\nabla}\times\vec{A})\cdot d\vec{\Sigma} = \int_\Sigma \vec{B}\cdot d\vec{\Sigma} = \text{fluxo magnético}$$

Onde Σ é uma superfície com borda C.

Como o fluxo é diferente de zero temos que ter $\vec{A}\neq 0$ fora do cilindro, apesar de ali $\vec{B}$ ser nulo.

Considere agora o famoso experimento proposto por Bohm e Aharanov em 1956 onde temos a difração de elétrons com um solenóide



A figura de difração sofre um deslocamento proporcional ao fluxo do campo magnético.

Na verdade a interação com o campo magnético de uma partícula de carga q pode ser expressa de maneira simples:

Suponha que $\langle b|a\rangle_0$ seja a amplitude de transição para ela ir de a até b livremente pela trajetória c. Então a amplitude na presença de $\vec{B}$ é:

$$\langle b|a\rangle_B = \langle b|a\rangle_0 \, e^{\frac{q}{\hbar}\int_c \vec{A}\cdot d\vec{s}}$$

No caso de um campo elétrico estático temos que a amplitude de transição de um tempo t_0 até t é modificado por

$$\langle t | t_0 \rangle_E = \langle t | t_0 \rangle \, e^{-\frac{q}{\hbar}\int_{t_0}^{t} \phi \, dt}$$

Na verdade para um campo eletromagnético genérico A_μ temos que a amplitude de transição de um ponto (a, t_0) até outro (b, t) através de uma trajetória T no espaço tempo é:

$$\langle b, t | a, t_0 \rangle_A = \langle b, t | a, t_0 \rangle \, e^{-\frac{q}{\hbar}\int_T A_\mu \cdot dx^\mu}$$

Este é o <u>princípio de gauge</u>.

Toda informação sobre a interação eletromagnética no ME está contida neste princípio

<u>Ver</u>: Lectures on Physics, Feynman
Vol II 15-5

Na verdade, este princípio não surgiu de repente. Ele evoluiu de maneira interessante, e talvez o primeiro a propô-lo foi H. Weyl em ~~[illegible]~~ 1929

Veja — O'Raifeartaigh
The Dawning of Gauge Theory.

— H. Weyl
Zeit. f. Physik 330 (1929) 56 (Electron and gravitation)

— P. A. M. Dirac
Proc. Royal Soc. London A 133 (1931)
Quantised Singularities in the Electromagnetic Field.

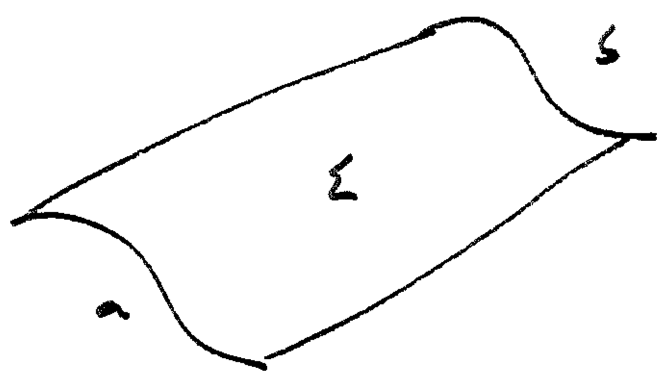
Extensões de teorias de gauge

Kalb & Ramond (PRD 9 (74) 2273),

Y. Nambu Phys. Rep 23 (76) 250 - 253

As idéias relacionadas ao princípio de gauge no eletromagnetismo podem ser extendidas para objetos que não são partículas mas sim cordas, membranas, etc.

Suponha que $\langle b|a\rangle_0$ seja a amplitude de transição para que uma corda vá de uma configuração a para b através de uma superfície Σ



Então na presença de um potencial $A_{\mu\nu}$ teremos

$$\langle b|a\rangle = \langle b|a\rangle_0 \, e^{i\frac{g}{\hbar}\int_\Sigma A_{\mu\nu}\, dx^\mu \wedge dx^\nu}$$

De ~~[illegible]~~ maneira análoga uma membrana seria

$$\langle 3|a\rangle = \langle 3|a\rangle_0 \, e^{\,i\frac{1}{\hbar}\int_{\text{Volume}} A_{\mu\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho}$$

Ou seja podemos introduzir acoplamento minimal entre objetos de dimensão d com tensores antisimétricos de rank d+1.

A ~~invariância~~ transformação de gauge seria

$$A_{\mu\nu} \to A_{\mu\nu} + \partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu$$

$$A_{\mu\nu\rho} \to A_{\mu\nu\rho} + \partial_\mu \alpha_{\nu\rho} + \partial_\rho \alpha_{\mu\nu} + \partial_\nu \alpha_{\rho\mu}$$

$$\vdots$$

$$A \to A + d\alpha$$

A: d+1 forma

α: d-forma.

Os tensores de campo seriam:

$$F_{\mu\nu\rho} = \partial_\mu A_{\nu\rho} + \partial_\rho A_{\mu\nu} + \partial_\nu A_{\rho\mu}$$

$$F_{\mu\nu\rho\sigma} = \partial_\mu A_{\nu\rho\sigma} + \text{permutações antisimétricas}$$

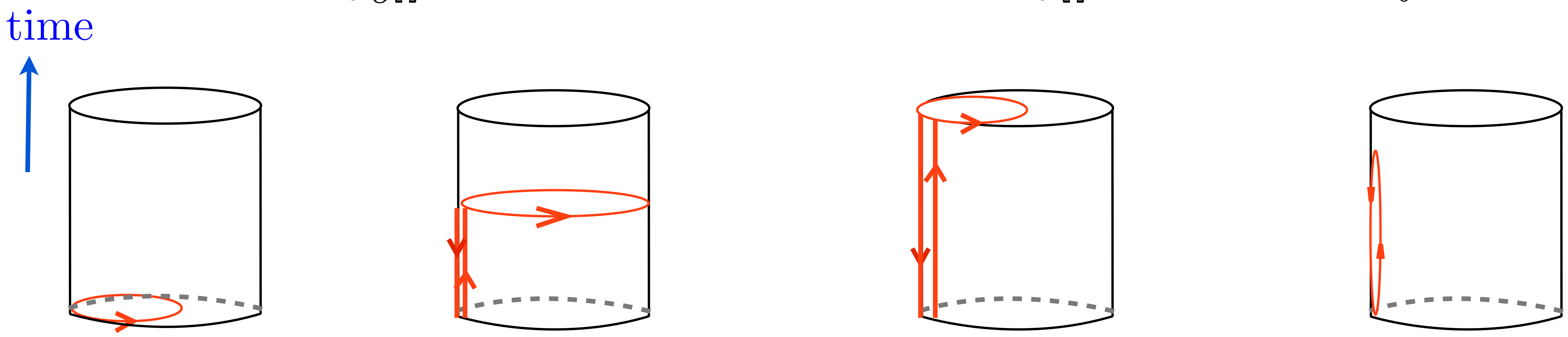
ou seja

$$F = dA$$

Integral Equations and Conservation Laws

For a 3-volume Ω without border $(\partial\Omega = 0)$

$$0 = \int_{\partial\Omega} \left[\alpha\, F_{\mu\nu} + \beta\, \tilde{F}_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau} = \beta \int_{\Omega} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta}$$



$$\int_{\Omega_0} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta} \quad \longrightarrow \quad Q(0)$$

$$\int_{S^2_\infty \times I} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta} \quad \longrightarrow \quad 0$$

$$-\int_{\Omega_t} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta} \quad \longrightarrow \quad -Q(t)$$

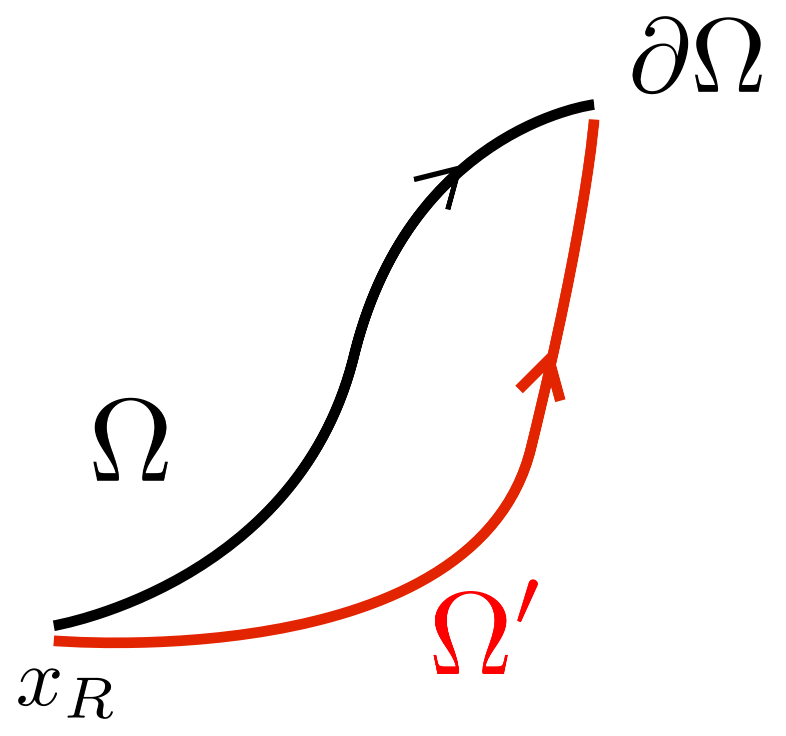
$$\int_{S^2_0 \times I} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta} \quad \longrightarrow \quad 0$$

$$\int_{\Omega} \tilde{j}_{\mu\nu\rho} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\zeta} = 0 \quad \longrightarrow \quad Q(t) = Q(0)$$

Faraday's Path Independency

$$\int_{\partial\Omega}\left[\alpha\,F_{\mu\nu}+\beta\,\tilde{F}_{\mu\nu}\right]\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\tau}=\beta\int_{\Omega}\tilde{j}_{\mu\nu\rho}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\tau}\frac{dx^{\rho}}{d\zeta}$$

$$=\beta\int_{\Omega'}\tilde{j}_{\mu\nu\rho}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\tau}\frac{dx^{\rho}}{d\zeta}$$



Ω and Ω': two 3-volumes with the same border

connection in loop space: $$\mathcal{A}\equiv\int_{\rm loop}\tilde{j}_{\mu\nu\rho}\frac{d\,x^{\mu}}{d\,\sigma}\frac{d\,x^{\nu}}{d\,\tau}\,\delta x^{\rho}$$

$$\mathcal{F}=\delta\mathcal{A}+\mathcal{A}\wedge\mathcal{A}=0$$

$$\partial_{\rho}\tilde{F}_{\mu\nu}+\partial_{\nu}\tilde{F}_{\rho\mu}+\partial_{\mu}\tilde{F}_{\nu\rho}=\tilde{j}_{\rho\mu\nu}$$

$(d\wedge\tilde{j}=0)$

abelian

The laws of Electromagnetism correspond to flat connections in loop space!!

Gauge Potentials

$$E_i = F_{0i} \qquad B_i = -\frac{1}{2\,c}\,\varepsilon_{ijk}F_{jk} \qquad j^\mu \equiv \frac{1}{\varepsilon_0}\left(\rho\,,\,-\frac{1}{c}\,J^i\right)$$

Introduce the potentials $\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$E_i = \frac{1}{c}\,\frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x^i} \rightarrow \vec{E} = \frac{1}{c}\,\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} A_0$$

$$B_i = -\frac{1}{c}\,\varepsilon_{ijk}\partial_j A_k \rightarrow \vec{B} = -\frac{1}{c}\vec{\nabla}\wedge\vec{A}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -c\,B_3 & c\,B_2 \\ -E_2 & c\,B_3 & 0 & -c\,B_1 \\ -E_3 & -c\,B_2 & c\,B_1 & 0 \end{pmatrix}$$

Lagragian $\qquad \mathcal{L} = -\frac{1}{4}\,F^2_{\mu\nu} = \frac{1}{2}\left(\vec{E}^2 - c^2\,\vec{B}^2\right)$